Spectral Form Function with Applications in Beam Physics

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Abstract

To describe longitudinal fine structure such as microbunching within a particle beam, a classical approach is to define a bunching factor as the Fourier transform of the particle longitudinal density distribution. Such a 1D definition of the bunching factor can be generalized to a 6D spectral form function (SFF) to describe more complicated structures in phase space. The complex SFF provides another complete description of the beam in the spectral domain and can offer complementary and valuable insights into beam dynamics studies, which typically invoke the real particle density distribution. The basic properties and Fokker-Planck equation of the SFF are presented, along with its solution in a general coupled linear lattice. Example applications of the SFF in electron storage ring physics and laser-induced microbunching are presented.

Full Text

Preamble

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Abstract

To describe longitudinal fine structure such as microbunching within a particle beam, the classical approach defines a bunching factor as the Fourier transform of the particle longitudinal density distribution. This one-dimensional definition can be generalized to a six-dimensional spectral form function (SFF) to describe more complicated structures in phase space [1]. The complex SFF provides a

complete description of the beam in the spectral domain and offers complementary and valuable insight into beam dynamics studies, which typically invoke the real particle density distribution.

This paper presents the basic properties and Fokker-Planck equation of the SFF, along with its solution in a general coupled linear lattice. Example applications in electron storage ring physics and laser-induced microbunching are discussed.

Introduction

Microbunching enables laser-like radiation generation from charged particle beams and has been one of the main driving forces advancing accelerator light sources over the past decades, with free-electron lasers being the most prominent example.

While we typically focus on the longitudinal coordinate of particles when quantifying the degree of microbunching—since radiation from a relativistic beam is dominantly in the forward direction—strictly speaking, all six particle phase-space coordinates can impact coherent radiation [2]. One can even create novel 6D structures in phase space for various purposes, such as tailoring radiation properties or controlling collective beam dynamics. One example is the creation of helical microbunching for light generation with orbital angular momentum [3]. The classical 1D definition of bunching factor is clearly insufficient for many applications, which motivates our investigation of this generalized spectral form function (SFF). However, we recognize that the potential applications of SFF can be much broader than this original motivation.

The 6D particle state vector is defined as $X \equiv (x, \dots, \delta)^T$, and the 6D spectral vector as $K \equiv (k_x, \dots, k_\delta)$. The normalized charge density function $\psi(X)$ satisfies $\int \psi(X)dX = 1$ and $\psi(X) \geq 0$. The SFF is then defined as:

$$F(K) \equiv \int \psi(X)e^{-iKX}dX.$$

Linear symplectic dynamics in an accelerator is dictated by a quadratic Hamiltonian $H = \frac{1}{2}X^THX$, where $H = H^T$. The Hamiltonian equations in matrix form are $\frac{dX}{ds} = SHX$, with S being the symplectic form. The evolution of the particle state vector from initial point s_i to final point s_f can be described by a symplectic transfer matrix according to $X(s_f) = R(s_f, s_i)X(s_i)$, with $R(s_f, s_i) = e^{\int_{s_i}^{s_f} SHds'}$ if H is s-independent. Correspondingly, the transfer matrix for K is $K(s_f) = K(s_i)R^{-1}(s_f, s_i)$. From the continuity equation and Hamiltonian equations follows the Liouville equation:

$$\frac{\partial \psi}{\partial s} + \nabla_X \cdot (\psi \dot{X}) = 0 \quad \text{where} \quad \nabla_X \equiv \left(\frac{\partial}{\partial x}, \dots, \frac{\partial}{\partial \delta}\right).$$

The corresponding equation in spectral domain is:

$$\frac{\partial F}{\partial s} + [F, H] = 0, \quad \text{with} \quad H_K = -\frac{1}{2} KSHSK^T,$$

from which follows:

$$\psi(X, s_f) = \psi(R^{-1}(s_f, s_i)X, s_i), \quad F(K, s_f) = F(KR(s_f, s_i), s_i).$$

We recognize that the work presented in this section has been obtained previously by Yampolsky [1].

Fokker-Planck Equation

Now let us add non-symplectic processes such as damping and diffusion. We simplify the discussion by assuming that the damping coefficients are independent of the particle state vector, as is the case for radiation damping. The equation of motion becomes:

$$\frac{dX}{ds} = (SH + B)X + \xi(s),$$

where $\xi(s)$ is a stochastic process satisfying $\int \xi_i p(\xi_i) d\xi_i = 0$, $\int p(\xi) d\xi = 1$, and $\int \xi_i(s)\xi_j(s')p(\xi_i(s), \xi_j(s'))d\xi_i d\xi_j = D_{ij}\delta(s - s')$, with $p(\xi)$ being the probability distribution function of ξ. We assume the noise is Gaussian white noise. In the above equation, B is responsible for deterministic damping or antidamping, and ξ for diffusion. Note that we assume diffusion is a continuous-diffusion process, rather than a jump-diffusion process whose rigorous description requires the Kramers-Moyal expansion [4]. Quantum excitation, for example, is more accurately modeled by a jump-diffusion process.

Denoting $C \equiv SH + B$, and noting that $\mathrm{Tr}(SH) = 0$, we can derive the Fokker-Planck equation for $\psi(X)$:

$$\frac{\partial \psi}{\partial s} + \nabla_X \cdot (\psi CX) = \frac{1}{2} \frac{\partial^2}{\partial X_i \partial X_j} (\psi D_{ij}).$$

Since $\psi(X)$ and $F(K)$ form a Fourier transform pair:

$$\psi(X) = \frac{1}{(2\pi)^6} \int F(K) e^{iKX} dK,$$

the corresponding equation in the spectral domain is:

$$\frac{\partial F}{\partial s} - (KC)(\nabla_K F) = -\frac{1}{2} KDK^T F, \quad \text{where} \quad \nabla_K \equiv \left(\frac{\partial}{\partial k_x}, \dots, \frac{\partial}{\partial k_\delta} \right).$$

From the right-hand side of this equation, it is clear that diffusion has a stronger impact on high-frequency bunching, i.e., finer structures in phase space.

Note that if there is no diffusion in phase space ($D = 0$), we have:

$$\frac{\partial \psi}{\partial s} + (\nabla_X \psi) \cdot (CX) = -\mathrm{Tr}(C)\psi = -\mathrm{Tr}(B)\psi,$$

and:

$$\frac{\partial F}{\partial s} - (KC)(\nabla_K F) = 0.$$

Thus, information about fine structure in phase space can only be destroyed by diffusion or stochastic processes. Deterministic linear transport with damping or antidamping can only rotate, shrink, or expand the structure.

Solution of the Fokker-Planck Equation

We now solve the Fokker-Planck equation. First, we assume the accelerator lattice is piecewise constant, and we solve the equation in each piece where C is a constant matrix.

We diagonal

Note: Figure translations are in progress. See original paper for figures.