Positive Definiteness and Stability of Interval Tensors

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Abstract

In this paper, we focus on the positive definiteness and Hurwitz stability of interval tensors. First, we introduce auxiliary tensors $\mathcal{A}^z$ and establish equivalent conditions for the positive (semi-)definiteness of interval tensors. That is, an interval tensor is positive definite if and only if all $\mathcal{A}^z$ are positive (semi-)definite. For Hurwitz stability, it is revealed that the stability of the symmetric interval tensor $\mathcal{A}_s^I$ implies the stability of the interval tensor $\mathcal{A}^I$, and the stability of symmetric interval tensors is equivalent to that of auxiliary tensors $\tilde{\mathcal{A}}^z$. Finally, taking 4th order 3-dimensional interval tensors as examples, the specific sufficient conditions are established for their positive (semi-)definiteness.

Full Text

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Abstract

This paper investigates the positive definiteness and Hurwitz stability of interval tensors. First, we introduce auxiliary tensors $\mathcal{A}_z$ and establish equivalent conditions for the positive (semi-)definiteness of interval tensors. Specifically, an interval tensor is positive definite if and only if all $\mathcal{A}_z$ are positive (semi-)definite. For Hurwitz stability, we reveal that the stability of symmetric interval tensors $\mathcal{A}_I^s$ can deduce the stability of the interval tensor $\mathcal{A}_I$, and the stability of symmetric interval tensors is equivalent to that of auxiliary tensors $\tilde{\mathcal{A}}_z$. Finally, taking 4th-order 3-dimensional interval tensors as examples, we construct specific sufficient conditions for their positive (semi-)definiteness.

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1 Introduction

Uncertainty analysis has always been a core issue in scientific and engineering computations. Interval analysis, introduced by Moore in the 1960s, has proven to be an effective tool for handling uncertainty and has yielded fruitful results in the theory of interval matrices [1–6]. By representing matrix elements as intervals, interval matrices provide a mathematical framework for characterizing uncertainties in system parameters, playing a key role in many areas such as stability analysis of dynamic systems and engineering error estimation [7–10].

However, with the advancement of scientific research, many practical problems have extended from linear systems to multilinear systems. Tensors, as higher-dimensional generalizations of matrices, have demonstrated strong descriptive capabilities in fields such as data science, quantum physics, and complex networks [11–21]. Interval tensor theory is not only a natural extension of interval matrix analysis theory but also an urgent need for addressing high-dimensional uncertainty problems.

Positive definiteness and stability are important concepts in linear algebra and dynamical systems theory, and numerous relevant studies have been conducted [22–27]. Rohn [27] systematically established criteria for determining positive definiteness, Hurwitz stability, and Schur stability of interval matrices, and proved that an interval matrix possesses a certain property if and only if a finite number of its vertex matrices satisfy that property.

It is well-known that the positive definiteness of tensors is equivalent to all H-eigenvalues (or Z-eigenvalues) being positive (non-negative) [28]. The sign of the minimum eigenvalue can verify the positive definiteness of tensors and can also be used to check the stability of multilinear systems [28–38]. For interval tensors, Beheshti et al. [39] extended some classes of interval matrices to classes of interval tensors and studied their properties and characterizations. Cui and Zhang [40] established bounds for H-eigenvalues of even-order real symmetric interval tensors or nonnegative interval tensors. Rahmati and Tawhid [41] generalized the row-property of a set of matrices to the slice-property of a set of tensors, pointing out that the slice-positive definiteness of a convex tensor set is equivalent to that of its extreme point set (Lemma 3.1). However, for mth-order n-dimensional interval tensors, which form a type of convex tensor set, the number of extreme points is 2^{n^m}, leading to exponential growth in computational complexity.

Regarding Hurwitz stability, Bozorgmanesh et al. [42] established an equivalence relationship between the stability of symmetric interval tensors and the positive definiteness of negative interval tensors, and provided an estimation range for the real parts of E-eigenvalues by characterizing a tensor counterpart of Bendixson's theorem, analogous to its matrix formulation. However, this conclusion only applies to symmetric interval tensors. Other studies on interval tensors can be found in [43–45]. Additionally, the relationship between the stability of asymmetric interval tensors and their real symmetricized tensors has not been fully clarified, leaving a theoretical gap.

To address these issues, based on existing theory of positive definiteness and stability of interval tensors and combined with specific example analysis, this paper conducts the following core work. First, we introduce the auxiliary index set $Y = \{z \in \mathbb{R}^n \mid |z_j| = 1\}$, define the specific tensor $\mathcal{A}_z = \mathcal{A}_c - \Delta \times_1 T_z \times_2 T_z \times \cdots \times_m T_z$ (where T_z is a diagonal matrix), and prove that when m is even, the positive definiteness of the interval tensor $\mathcal{A}_I$ can be determined by verifying only 2^{n-1} tensors, reducing the computational complexity from 2^{n^m} to 2^{n-1}. In addition, we establish the equivalence relationship between the positive definiteness of the interval tensor $\mathcal{A}_I$ and its symmetric interval tensor $\mathcal{A}_I^s$, and prove that if $\mathcal{A}_I^s$ is Hurwitz stable, then $\mathcal{A}_I$ must be Hurwitz stable. Finally, taking 4th-order 3-dimensional interval tensors as the research object, we obtain multiple sets of directly applicable sufficient conditions for positive definiteness, providing specific tools for the performance analysis of low-order and low-dimensional uncertain systems in engineering.

2 Preliminaries

In this section, we present the necessary definitions and notations related to tensors and interval tensors.

We establish the following notation conventions. Vectors will be denoted as $\{x, y, \cdots\}$, matrices as $\{A, B, \cdots\}$, and tensors as $\{\mathcal{A}, \mathcal{B}, \cdots\}$. The set of all mth-order n-dimensional real tensors is denoted as $T_{m,n}$, and the entries of an mth-order n-dimensional tensor $\mathcal{A}$ are denoted by $(a_{i_1 i_2 \cdots i_m})$, where $i_j \in [n] = \{1, 2, \cdots, n\}$. Entries $a_{ii\cdots i}$ for $i \in [n]$ are called diagonal entries, and others are called off-diagonal entries. The inequalities $x \le (<) y$, $A \le (<) B$, and $\mathcal{A} \le (<) \mathcal{B}$ are understood componentwise, and we define the absolute value as the tensor $|\mathcal{A}| = (|a_{i_1 i_2 \cdots i_m}|)$.

For any tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in T_{m,n}$, if its entries $a_{i_1 i_2 \cdots i_m}$ remain unchanged under any permutation of their indices, i.e., for every $\sigma \in S_m$ (where S_m denotes the symmetric group of m elements), $a_{i_1 i_2 \cdots i_m} = a_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_m)}$, then $\mathcal{A}$ is termed a symmetric tensor. The set of all mth-order n-dimensional real symmetric tensors is denoted as $S_{m,n}$.

For an mth-degree homogeneous polynomial of n variables, $f_{\mathcal{A}}(x) = \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m} = \mathcal{A}x^m$, where its coefficient tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in T_{m,n}$, there exists a unique symmetric tensor $\mathcal{A}_s = ((a_s)_{i_1 i_2 \cdots i_m}) \in S_{m,n}$ such that $f_{\mathcal{A}}(x) \equiv \sum_{i_1,\cdots,i_m=1}^{n} (a_s)_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m}$, which is the symmetrization of $\mathcal{A}$, and $(a_s)_{i_1 i_2\cdots i_m} = \frac{1}{m!}\sum_{\sigma\in S_m} a_{\sigma(i_1)\cdots\sigma(i_m)}$.

Let $\underline{\mathcal{A}} = (\underline{a}_{i_1 i_2\cdots i_m})$, $\overline{\mathcal{A}} = (\overline{a}_{i_1 i_2\cdots i_m})$ be mth-order n-dimensional real tensors with $\underline{\mathcal{A}} \le \overline{\mathcal{A}}$. The set of tensors $\mathcal{A}_I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}] = \{\mathcal{A} : \underline{\mathcal{A}} \le \mathcal{A} \le \overline{\mathcal{A}}\}$ is called an interval tensor. Denote $\mathcal{A}_c = (a^c_{i_1 i_2\cdots i_m}) = \frac{\underline{\mathcal{A}}+\overline{\mathcal{A}}}{2}$ and $\Delta = (\delta_{i_1 i_2\cdots i_m}) = \frac{\overline{\mathcal{A}}-\underline{\mathcal{A}}}{2}$, then $\mathcal{A}_I = [\mathcal{A}_c - \Delta, \mathcal{A}_c + \Delta]$.

Clearly, $\Delta \ge \mathcal{O}$ is a nonnegative tensor (i.e., all entries $\delta_{i_1 i_2\cdots i_m}$ are nonnegative), where $\mathcal{O}$ is the tensor in $T_{m,n}$ with all entries zero. $\mathcal{A}_I$ is said to be symmetric if both $\mathcal{A}_c$ and Δ are symmetric. With each interval tensor $\mathcal{A}_I = [\mathcal{A}_c - \Delta, \mathcal{A}_c + \Delta]$, we associate the symmetric interval tensor $\mathcal{A}^s_I = [\mathcal{A}^s_c - \Delta^s, \mathcal{A}^s_c + \Delta^s]$.

Remark 2.1. If $\mathcal{A} \in \mathcal{A}_I$, then $\mathcal{A}_s \in \mathcal{A}^s_I$, and $\mathcal{A}_I$ is symmetric if and only if $\mathcal{A}_I = \mathcal{A}^s_I$.

For $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$, $x \in \mathbb{R}^n$, $P = (p_{ij}) \in \mathbb{R}^{n\times n}$, and $k \in [m]$, $\mathcal{A} \times_1 P \times_2 P \times_3 \cdots \times_m P$ is a tensor in $T_{m,n}$ defined by $(\mathcal{A} \times_1 P \times_2 P \times_3 \cdots \times_m P)_{i_1 i_2\cdots i_m} = \sum_{j_1,j_2,\cdots,j_m=1}^{n} a_{j_1 j_2\cdots j_m} p_{i_1 j_1} \cdots p_{i_m j_m}$.

And $\mathcal{A}x^{m-1}$ is a vector in $\mathbb{R}^n$ with the ith component $(\mathcal{A}x^{m-1})_i := \sum_{i_2,\cdots,i_m=1}^{n} a_{i i_2\cdots i_m} x_{i_2} \cdots x_{i_m}$ for $1 \le i \le n$.

Definition 2.1. [28, 41, 42] Let m be even, $\mathcal{A}, \mathcal{A}_c, \Delta \in T_{m,n}$ and $\Delta \ge \mathcal{O}$, $\mathcal{A}_I = [\mathcal{A}_c - \Delta, \mathcal{A}_c + \Delta]$. - (a) $\mathcal{A}$ is called positive semi-definite if $\mathcal{A}x^m \ge 0$ for each $x \in \mathbb{R}^n$ $\{0\}$. $\mathcal{A}_I$ is called positive semi-definite if each $\mathcal{A} \in \mathcal{A}_I$ is positive semi-definite. - (b) $\mathcal{A}$ is called positive definite if $\mathcal{A}x^m > 0$ for each $x \in \mathbb{R}^n$ $\{0\}$. $\mathcal{A}_I$ is called positive definite if each $\mathcal{A} \in \mathcal{A}_I$ is positive definite.

Definition 2.2. [28] Let $\mathcal{A} \in T_{m,n}$, $\lambda \in \mathbb{C}$ and $x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{C}^n$. λ is called an eigenvalue and x an eigenvector corresponding to λ if $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$ where $x^{[m-1]} = (x_1^{m-1}, \cdots, x_n^{m-1})^\top$. Moreover, when $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, λ is called an H-eigenvalue and x an H-eigenvector corresponding to λ.

Definition 2.3. [28] Let $\mathcal{A} \in T_{m,n}$, $\lambda \in \mathbb{C}$ and $x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{C}^n$. λ is called an E-eigenvalue and x an E-eigenvector corresponding to λ if

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^\top x = 1. \end{cases}$$

Moreover, when $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, λ is called a Z-eigenvalue and x a Z-eigenvector corresponding to λ.

We denote by $\lambda^H_{\min}(\mathcal{A})$ ($\lambda^Z_{\min}(\mathcal{A})$) and $\lambda^H_{\max}(\mathcal{A})$ ($\lambda^Z_{\max}(\mathcal{A})$) the minimum and maximum H(Z)-eigenvalue of $\mathcal{A}$, respectively (obviously, $\lambda^H_{\min}(-\mathcal{A}) = -\lambda^H_{\max}(\mathcal{A})$, $\lambda^Z_{\min}(-\mathcal{A}) = -\lambda^Z_{\max}(\mathcal{A})$). The H-spectral radius is denoted by $\rho_H(\mathcal{A})$.

Lemma 2.1. [28] Let $\mathcal{A} \in S_{m,n}$ and m be even. - (a) $\mathcal{A}$ is positive (semi-)definite if and only if all of its H-eigenvalues are positive (nonnegative), which means $\lambda_{\min}^{H}(\mathcal{A}) > (\geq)0$. - (b) $\mathcal{A}$ is positive (semi-)definite if and only if all of its Z-eigenvalues are positive (nonnegative), which means $\lambda_{\min}^{Z}(\mathcal{A}) > (\geq)0$.

Lemma 2.2. [38] Let $\mathcal{A} \in S_{m,n}$. Then the eigenvalues of $\mathcal{A}$ lie in the union of n disks in $\mathbb{C}$. These n disks have the diagonal entries of $\mathcal{A}$ as their centers and the sums of the absolute values of the off-diagonal entries as their radii.

3 Positive Definiteness of Interval Tensors

Rahmati and Tawhid showed that a compact convex set C of tensors has the slice-positive-(semi-)definite property if and only if the set X of extreme points of C has the slice-positive-(semi-)definite property. Here, C has the slice-positive-(semi-)definite property if every tensor in $\hat{C}$ is a positive-(semi-)definite tensor, where the slice-completion $\hat{C}$ is the set of all tensors that have one slice in common with a tensor in C [41]. A point $x \in C$ is called an extreme point of C if whenever $x = (1-k)y + kz$ for some $y, z \in C$ and $k \in (0,1)$, we must have $x = y = z$; see [46]. In the following, we always assume m is even, as this is only meaningful for even-order tensors.

Lemma 3.1. [41] Suppose that C is a compact, convex set in $T_{m,n}$. Let X be the set of all extreme points of C. Then C has the slice-positive-(semi-)definite property if and only if X has the slice-positive-(semi-)definite property.

If $C = \mathcal{A}_I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$, then the set X of all extreme points of C is $E = \{\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in T_{m,n} : a_{i_1 i_2 \cdots i_m} = \underline{a}_{i_1 i_2 \cdots i_m} \text{ or } a_{i_1 i_2 \cdots i_m} = \overline{a}_{i_1 i_2 \cdots i_m}, i_1, i_2, \cdots, i_m \in [n]\}$. Therefore, the following result can be obtained from Lemma 3.1.

Corollary 3.1. Let $\mathcal{A}_I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$ be an interval tensor in $T_{m,n}$. Then E is positive (semi-)definite if and only if C is positive (semi-)definite.

Based on this result, verifying the positive definiteness of an interval tensor $\mathcal{A}_I$ in $T_{m,n}$ requires checking the positive definiteness of 2^{n^m} tensors, as the cardinality of E is 2^{n^m}. In what follows, we will improve this result, reducing the number of tensors that need verification to 2^{n-1}.

First, we define an auxiliary index set $Y = \{z \in \mathbb{R}^n \mid |z_j| = 1 \text{ for } j = 1, \cdots, n\}$, which has cardinality 2^n. For each $z \in Y$, we denote by T_z the $n \times n$ diagonal matrix whose diagonal entries come from the vector z. Next, for each $z \in Y$, we introduce the tensor $\mathcal{A}_z$ via the expression

$$\mathcal{A}_z = (a^z_{i_1 i_2 \cdots i_m}) = \mathcal{A}_c - \Delta \times_1 T_z \times_2 T_z \times_3 \cdots \times_m T_z.$$

For each $i_j \in [n]$, $j = 1, 2, \cdots, m$, the entry $a^z_{i_1 i_2 \cdots i_m}$ of $\mathcal{A}_z$ satisfies

$$a^z_{i_1 i_2 \cdots i_m} = \begin{cases} a^c_{i_1 i_2 \cdots i_m} - \delta_{i_1 i_2 \cdots i_m} & \text{if } z_{i_1} z_{i_2} \cdots z_{i_m} = 1, \\ a^c_{i_1 i_2 \cdots i_m} + \delta_{i_1 i_2 \cdots i_m} & \text{if } z_{i_1} z_{i_2} \cdots z_{i_m} = -1. \end{cases}$$

Thus, $\mathcal{A}_z$ belongs to $\mathcal{A}_I$ for every $z \in Y$. When m is even, since $\mathcal{A}_{-z} = \mathcal{A}_z$, the number of distinct tensors $\mathcal{A}_z$ is at most 2^{n-1} (and equal to 2^{n-1} if $\Delta > 0$).

If $\mathcal{A}_I$ is symmetric, each $\mathcal{A}_z$ is symmetric as well. These tensors $\mathcal{A}_z$, $z \in Y$, will be employed to characterize the positive (semi-)definiteness of an interval tensor through finite procedures.

Let us introduce a function $g : T_{m,n} \to \mathbb{R}$ defined for a tensor $\mathcal{A} \in T_{m,n}$ by

$$g(\mathcal{A}) = \min_{\|x\|_2=1} \mathcal{A}x^m.$$

Obviously, g is well-defined. In the following theorem, we summarize the basic properties of g that will be used in the proofs of the main theorems in subsequent sections.

Theorem 3.1. The function g has the following properties: - (i) For each $\mathcal{A} \in T_{m,n}$, $g(\mathcal{A}) = g(\mathcal{A}_s)$. - (ii) For each $\mathcal{A} \in S_{m,n}$, $g(\mathcal{A}) = \lambda^H_{\min}(\mathcal{A})$. - (iii) For each $\mathcal{A}, \mathcal{D} \in T_{m,n}$, $|g(\mathcal{A}+\mathcal{D}) - g(\mathcal{A})| \le \rho_H(\mathcal{D}_s)$. - (iv) g is continuous in $T_{m,n}$. - (v) For each interval tensor $\mathcal{A}_I$, $\min\{g(\mathcal{A}) : \mathcal{A} \in \mathcal{A}_I\} = \min\{g(\mathcal{A}_z) : z \in Y\}$. - (vi) For each interval tensor $\mathcal{A}_I$, $\min\{g(\mathcal{A}) : \mathcal{A} \in \mathcal{A}_I\} = \min\{g(\mathcal{A}) : \mathcal{A} \in \mathcal{A}^s_I\}$.

Proof. (i) This follows directly from the definition of symmetrization. (ii) See the proof of Theorem 2.2 in Ref. [38]. (iii) For each $\mathcal{A}, \mathcal{D} \in T_{m,n}$, from the definition it follows that $g(\mathcal{A}+\mathcal{D}) \ge g(\mathcal{A}) + g(\mathcal{D})$ and $g(\mathcal{A}) = g((\mathcal{A}+\mathcal{D}) + (-\mathcal{D})) \ge g(\mathcal{A}+\mathcal{D}) + g(-\mathcal{D})$, which implies $|g(\mathcal{A}+\mathcal{D}) - g(\mathcal{A})| \le \max\{|g(\mathcal{D})|, |g(-\mathcal{D})|\} = \max\{|g(\mathcal{D}_s)|, |g(-\mathcal{D}_s)|\} = \max\{|\lambda^H_{\min}(\mathcal{D}_s)|, |\lambda^H_{\max}(\mathcal{D}_s)|\} = \rho_H(\mathcal{D}_s)$. (iv) According to (iii), $|g(\mathcal{A}+\mathcal{D}) - g(\mathcal{A})| \le \rho_H(\mathcal{D}_s)$ for each $\mathcal{A}, \mathcal{D} \in T_{m,n}$. By Lemma 2.2, one can prove that g is continuous in $T_{m,n}$. (v) On the one hand, for each interval tensor $\mathcal{A}_I$, clearly $\min\{g(\mathcal{A}) : \mathcal{A} \in \mathcal{A}_I\} \le \min\{g(\mathcal{A}_z) : z \in Y\}$ since $\mathcal{A}_z \in \mathcal{A}_I$. On the other hand, for each $\mathcal{A} \in \mathcal{A}_I$ and $x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n$, we have

$$|x_i(\mathcal{A}x^{m-1})_i - x_i(\mathcal{A}_c x^{m-1})_i| = |x_i((\mathcal{A} - \mathcal{A}_c)x^{m-1})_i| \le |x_i|(\Delta|x|^{m-1})_i.$$

Let $z = \operatorname{sgn}(x) = (\operatorname{sgn}(x_1), \operatorname{sgn}(x_2), \cdots, \operatorname{sgn}(x_n))^\top$, where $\operatorname{sgn}(x_i) = 1$ if $x_i \ge 0$ and -1 if $x_i < 0$. Then

$$x_i(\mathcal{A}x^{m-1})_i \ge x_i(\mathcal{A}_c x^{m-1})_i - |x_i|(\Delta|x|^{m-1})_i = x_i(\mathcal{A}_z x^{m-1})_i.$$

Hence, $g(\mathcal{A}) \ge \min\{g(\mathcal{A}_z) : z \in Y\}$ holds for each $\mathcal{A} \in \mathcal{A}_I$, which implies $\min\{g(\mathcal{A}) : \mathcal{A} \in \mathcal{A}_I\} \ge \min\{g(\mathcal{A}_z) : z \in Y\}$. The assertion follows. (vi) For each $z \in Y$, denote by $\tilde{\mathcal{A}}_z = (\tilde{a}^z_{i_1 i_2 \cdots i_m})$ the tensor $\mathcal{A}_z$ for $\mathcal{A}^s_I$, i.e., $\tilde{\mathcal{A}}_z = \mathcal{A}^s_c - \Delta^s \times_1 T_z \times_2 T_z \times_3 \cdots \times_m T_z$, where

$$\tilde{a}^z_{i_1 i_2 \cdots i_m} = \frac{1}{m!} \sum_{\sigma \in S_m} [a^c_{\sigma(i_1)\cdots\sigma(i_m)} - \delta_{\sigma(i_1)\cdots\sigma(i_m)} \operatorname{sgn}(z_{i_1} z_{i_2} \cdots z_{i_m})] = (a_s)^z_{i_1 i_2 \cdots i_m}.$$

Thus $\tilde{\mathcal{A}}_z = (\mathcal{A}_z)_s$ for each $z \in Y$. Then employing (i), we obtain $g(\mathcal{A}_z) = g(\mathcal{A}^s_z)$. Hence, assertion (v) implies that the minimum values of g over $\mathcal{A}_I$ and $\mathcal{A}^s_I$ are equal.

As a consequence of Theorem 3.1, we obtain the following characterization.

Theorem 3.2. Let $\mathcal{A}_I$ be an interval tensor in $T_{m,n}$. Then the following assertions are equivalent: - (a) $\mathcal{A}_I$ is positive (semi-)definite. - (b) $\mathcal{A}_I^s$ is positive (semi-)definite. - (c) $\mathcal{A}_z$ is positive (semi-)definite for each $z \in Y$.

Proof. By Lemma 2.1, $\mathcal{A}_I$ is positive (semi-)definite if and only if $\min\{g(\mathcal{A}) : \mathcal{A} \in \mathcal{A}_I\} > (\geq) 0$ holds. Then by (vi) of Theorem 3.1, (a) and (b) are equivalent. From assertion (v) of Theorem 3.1, (a) and (c) are equivalent.

4 Stability of Interval Tensors

A tensor $\mathcal{A} \in T_{m,n}$ is called Hurwitz stable if the real part of any of its E-eigenvalues satisfies $\mathrm{Re}(\lambda) < 0$; in other words, all its eigenvalues lie in the open left half of the complex plane. An interval tensor $\mathcal{A}_I$ in $T_{m,n}$ is said to be Hurwitz stable if each $\mathcal{A} \in \mathcal{A}_I$ is Hurwitz stable. Bozorgmanesh et al. showed that for every E-eigenvalue λ of a tensor $\mathcal{A} \in T_{m,n}$, we have $\lambda^Z_{\min}(\mathcal{A}_s) \leq \mathrm{Re}(\lambda) \leq \lambda^Z_{\max}(\mathcal{A}_s)$, which is analogous to the well-known Bendixson's theorem in the matrix case [42, 47]. Utilizing this, the relationship between Hurwitz stability and positive definiteness of interval tensors has been obtained.

Lemma 4.1. [42] Let $\mathcal{A}_I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$ be a symmetric interval tensor in $T_{m,n}$. $\mathcal{A}_I$ is Hurwitz stable if and only if $-\mathcal{A}_I = [-\overline{\mathcal{A}}, -\underline{\mathcal{A}}]$ is positive definite.

Based on this lemma, similar to (c) of Theorem 3.2, we can derive another equivalent condition for Hurwitz stability of interval tensors. This equivalent condition merely requires verifying the stability of a finite number of tensors.

Unlike the previous section, where we used the tensors $\mathcal{A}_z = \mathcal{A}_c - \Delta \times_1 T_z \times_2 T_z \times_3 \cdots \times_m T_z$, in this section we characterize Hurwitz stability by means of tensors

$$\tilde{\mathcal{A}}_z = (\tilde{a}^z_{i_1 i_2 \cdots i_m}) = \mathcal{A}_c + \Delta \times_1 T_z \times_2 T_z \times_3 \cdots \times_m T_z, \quad z \in Y.$$

It is evident that $\tilde{\mathcal{A}}_z$ belongs to $\mathcal{A}_I$, and all $\tilde{\mathcal{A}}_z$ tensors are symmetric when the interval tensor $\mathcal{A}_I$ itself is symmetric.

Theorem 4.1. Let $\mathcal{A}_I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$ be a symmetric interval tensor in $T_{m,n}$. Then $\mathcal{A}_I$ is Hurwitz stable if and only if for each $z \in Y$, $\tilde{\mathcal{A}}_z$ is Hurwitz stable.

Proof. The necessity is obvious because $\tilde{\mathcal{A}}_z \in \mathcal{A}_I$ for each $z \in Y$.

To prove sufficiency, let $z \in Y$. Since $\tilde{\mathcal{A}}_z$ is Hurwitz stable and symmetric, all Z-eigenvalues of $\tilde{\mathcal{A}}_z$ are negative. Therefore, all Z-eigenvalues of the symmetric tensor $-\tilde{\mathcal{A}}_z = -\mathcal{A}_c - \Delta \times_1 T_z \times_2 T_z \times_3 \cdots \times_m T_z$ are positive, and by Lemma 2.1, $-\tilde{\mathcal{A}}_z$ is positive definite. However, $-\tilde{\mathcal{A}}_z$ corresponds exactly to the tensor $\mathcal{A}_z$ associated with the interval tensor $-\mathcal{A}_I$. Therefore, based on assertion (c) of Theorem 3.2, $-\mathcal{A}_I$ is positive definite. By virtue of Lemma 4.1, it follows that $\mathcal{A}_I$ is Hurwitz stable.

Theorem 3.2 established that the positive (semi-)definiteness of a general interval tensor can be equivalently characterized via its associated symmetric interval tensor $\mathcal{A}_I^s$. Regrettably, this favorable property does not extend to Hurwitz stability; only one direction of the implication holds true.

Theorem 4.2. If $\mathcal{A}_I^s$ is Hurwitz stable, then $\mathcal{A}_I$ is also Hurwitz stable.

Proof. Let λ be an E-eigenvalue of a tensor $\mathcal{A} \in \mathcal{A}_I$. Then by inequality (4) we have $\mathrm{Re}(\lambda) \leq \lambda_{\max}^Z(\mathcal{A}_s) < 0$ because the symmetric tensor $\mathcal{A}_s$ belongs to $\mathcal{A}_I^s$ and thus has all E-eigenvalues negative. This proves that $\mathcal{A}_I$ is Hurwitz stable.

The converse does not hold, as one may consider the example given in [27] regarding interval matrices, i.e., the case where $m = 2$.

5 Some Positive Definite 4th-Order 3-Dimensional Interval Tensors

Let $\underline{\mathcal{A}} = (\underline{a}_{ijkl})$, $\overline{\mathcal{A}} = (\overline{a}_{ijkl}) \in S_{4,3}$, and $\mathcal{A}_I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$, where $i, j, k, l \in [3]$.

In this section, we only consider the symmetric interval tensor $\mathcal{A}_I$. From the results of the previous section, it follows that to prove the positive (semi-)definiteness of the 4th-order 3-dimensional interval tensor $\mathcal{A}_I$, it suffices to verify the positive (semi-)definiteness of four specific tensors $\mathcal{A}_{z_1}, \mathcal{A}_{z_2}, \mathcal{A}_{z_3}$, and $\mathcal{A}_{z_4}$, where $z_1 = (1,1,1)^\top$, $z_2 = (1,1,-1)^\top$, $z_3 = (1,-1,1)^\top$, $z_4 = (-1,1,1)^\top \in Y = \{z \in \mathbb{R}^3 \mid |z_j| = 1 \text{ for } j = 1,2,3\}$. By leveraging this insight, we derive several sufficient conditions for the positive (semi-)definiteness of 4th-order 3-dimensional interval tensors in this section. Furthermore, we can obtain some sufficient conditions for the positive (semi-)definiteness of $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in S_{4,3}$.

For tensor $\mathcal{A} = (a_{ijkl}) \in S_{4,3}$, its corresponding homogeneous polynomial is

$$\begin{aligned} f_{\mathcal{A}}(x) = {} & a_{1111}x_1^4 + a_{2222}x_2^4 + a_{3333}x_3^4 + 6(a_{1122}x_1^2x_2^2 + a_{1133}x_1^2x_3^2 + a_{2233}x_2^2x_3^2) \\ & + 4(a_{1112}x_1^3x_2 + a_{1113}x_1^3x_3 + a_{1222}x_1x_2^3 + a_{1333}x_1x_3^3 + a_{2223}x_2^3x_3 + a_{2333}x_2x_3^3) \\ & + 12(a_{1123}x_1^2x_2x_3 + a_{1223}x_1x_2^2x_3 + a_{1233}x_1x_2x_3^2), \end{aligned}$$

where $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$. In the subsequent discussion, we only need to focus on the scenario where $a_{iiii} = \underline{a}_{iiii}$ and $a_{iijj} = \underline{a}_{iijj}$, i.e., $\delta_{iiii} = \delta_{iijj} = 0$ for all $i, j \in [3]$, $i \neq j$. This is because the entries in question here all correspond to the coefficients of the square terms.

Theorem 5.1. For a 4th-order 3-dimensional real interval tensor $\mathcal{A}_I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$, where $\underline{a}_{1111} = \underline{a}_{2222} = \underline{a}_{3333} = 0$, $\mathcal{A}_I$ is positive semi-definite if $\underline{a}_{iiij} = \overline{a}_{iiij} = 0$ for all $i, j \in [3]$, $i \neq j$, $\underline{a}_{1123} = -1$, $\overline{a}_{1123} = 1$, $\underline{a}_{1223} = \underline{a}_{1233} = \overline{a}_{1223} = \overline{a}_{1233} = 0$, $\underline{a}_{1122}, \underline{a}_{1133} \geq 1$, and $\underline{a}_{2233} \geq 0$.

Proof. Let $a_{iiii} = \underline{a}_{iiii} = 0$ for all $i \in [3]$, $a_{1122} = a_{1133} = \underline{a}_{1122} = \underline{a}_{1133} = 1$, and $a_{2233} = \underline{a}_{2233} = 0$. Then $a_{iiij}^c = 0$ for all $i, j \in [3]$, $i \neq j$, $a_{1122}^c = a_{1133}^c = 1$, $a_{1223}^c = a_{1233}^c = a_{1123}^c = a_{2233}^c = a_{iiii}^c = 0$, $\delta_{iiii} = \delta_{iijj} = \delta_{iiij} = \delta_{1233} = \delta_{1223} = 0$ for all $i, j \in [3]$, and $\delta_{1123} = 1$.

Since for $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \quad \{0\}$,

$$f_{\mathcal{A}_{z_1}}(x) = f_{\mathcal{A}_{z_4}}(x) = -12x_1^2x_2x_3 + 6(x_1^2x_2^2 + x_1^2x_3^2) = 6(x_1x_2 - x_1x_3)^2,$$
$$f_{\mathcal{A}_{z_2}}(x) = f_{\mathcal{A}_{z_3}}(x) = 12x_1^2x_2x_3 + 6(x_1^2x_2^2 + x_1^2x_3^2) = 6(x_1x_2 + x_1x_3)^2,$$

$\mathcal{A}_z$ is positive semi-definite for each $z \in Y$. According to Theorem 3.2, $\mathcal{A}_I$ is positive semi-definite.

Corollary 5.1. Let $\mathcal{A} = (a_{ijkl}) \in S_{4,3}$ and $a_{iiii} \geq 0$ for all $i \in \{1, 2, 3\}$. Then $\mathcal{A}$ is positive semi-definite if $a_{iiij} = a_{1223} = a_{1233} = 0$, $a_{iijj} \geq 0$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$, $a_{1122} \geq |a_{1123}|$, and $a_{1133} \geq |a_{1123}|$. Moreover, if $a_{iiii} > 0$ for all $i \in \{1, 2, 3\}$, then $\mathcal{A}$ is positive definite if $a_{iiij} = a_{1223} = a_{1233} = 0$, $a_{iijj} \geq 0$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$, $a_{1122} \geq |a_{1123}|$, and $a_{1133} \geq |a_{1123}|$.

Theorem 5.2. For a 4th-order 3-dimensional real interval tensor $\mathcal{A}_I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$, where $\underline{a}_{1111} = \underline{a}_{2222} = 0$ and $\underline{a}_{3333} = 1$, $\mathcal{A}_I$ is positive semi-definite if $\underline{a}_{1112} = \underline{a}_{1113} = \underline{a}_{1222} = \underline{a}_{2223} = \underline{a}_{1333} = \overline{a}_{1112} = \overline{a}_{1113} = \overline{a}_{1222} = \overline{a}_{2223} = \overline{a}_{1333} = 0$, $\underline{a}_{2333} = -1$, $\overline{a}_{2333} = 1$, and one of the following conditions is satisfied: - (a) $\underline{a}_{iijk} = \overline{a}_{iijk} = 0$ for all $i, j, k \in [3]$, $i \neq j$, $i \neq k$, $j \neq k$, $\underline{a}_{1122}, \underline{a}_{1133} \geq 0$, and $\underline{a}_{2233} \geq 2$. - (b) $\underline{a}_{1123} = -1$, $\overline{a}_{1123} = 1$, $\underline{a}_{1223} = \underline{a}_{1233} = \overline{a}_{1223} = \overline{a}_{1233} = 0$, and $\underline{a}_{1122}, \underline{a}_{1133} \geq 1$ and $\underline{a}_{2233} \geq 2$.

Proof. Let $a_{1111} = \underline{a}_{1111} = a_{2222} = \underline{a}_{2222} = 0$, and $a_{3333} = \underline{a}_{3333} = 1$.

(a) Take $a_{1122} = a_{1133} = \underline{a}_{1122} = \underline{a}_{1133} = 0$ and $a_{2233} = \underline{a}_{2233} = 2$. Then $a^c_{iiii} = a^c_{iijj} = 0$ for all $i, j \in [3]$, $i \neq j$, $a^c_{3333} = 1$, $a^c_{iijk} = 0$ for all $i, j, k \in [3]$, $i \neq j$, $i \neq k$, $j \neq k$, $\delta_{iiii} = \delta_{iijj} = \delta_{iijk} = \delta_{1112} = \delta_{1222} = \delta_{2223} = \delta_{1333} = \delta_{1113} = 0$ for all $i, j \in [3]$, $i \neq j$, and $\delta_{2333} = 1$.

Since for $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \quad \{0\}$,

$$f_{\mathcal{A}_{z_1}}(x) = f_{\mathcal{A}_{z_4}}(x) = x_3^4 - 4x_2x_3^3 + 12x_2^2x_3^2 = (2x_2x_3 - x_3^2)^2 + 8x_2^2x_3^2,$$
$$f_{\mathcal{A}_{z_2}}(x) = f_{\mathcal{A}_{z_3}}(x) = x_3^4 + 4x_2x_3^3 + 12x_2^2x_3^2 = (2x_2x_3 + x_3^2)^2 + 8x_2^2x_3^2,$$

$\mathcal{A}_z$ is positive semi-definite for each $z \in Y$. According to Theorem 3.2, $\mathcal{A}_I$ is positive semi-definite.

(b) Take $a_{1122} = a_{1133} = \underline{a}_{1122} = \underline{a}_{1133} = 1$ and $a_{2233} = \underline{a}_{2233} = 2$. Then $a^c_{iiii} = a^c_{iijj} = 0$ for all $i, j \in [3]$, $i \neq j$, $a^c_{3333} = 1$, $a^c_{1122} = a^c_{1133} = 1$, $a^c_{iijk} = 0$ for all $i, j, k \in [3]$, $i \neq j$, $i \neq k$, $j \neq k$, $\delta_{iiii} = \delta_{iijj} = \delta_{1123} = \delta_{1223} = \delta_{1233} = \delta_{1222} = \delta_{2223} = \delta_{1333} = \delta_{1113} = 0$ for all $i, j \in [3]$, $i \neq j$, and $\delta_{1123} = \delta_{2333} = 1$.

Since for $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \quad \{0\}$,

$$\begin{aligned} f_{\mathcal{A}_{z_1}}(x) = f_{\mathcal{A}_{z_4}}(x) &= x_3^4 - 4x_2x_3^3 - 12x_1^2x_2x_3 + 6(x_1^2x_2^2 + x_1^2x_3^2 + 2x_2^2x_3^2) \\ &= (2x_2x_3 - x_3^2)^2 + 6(x_1x_2 - x_1x_3)^2 + 8x_2^2x_3^2, \\ f_{\mathcal{A}_{z_2}}(x) = f_{\mathcal{A}_{z_3}}(x) &= x_3^4 + 4x_2x_3^3 + 12x_1^2x_2x_3 + 6(x_1^2x_2^2 + x_1^2x_3^2 + 2x_2^2x_3^2) \\ &= (2x_2x_3 + x_3^2)^2 + 6(x_1x_2 + x_1x_3)^2 + 8x_2^2x_3^2, \end{aligned}$$

$\mathcal{A}_z$ is positive semi-definite for each $z \in Y$. According to Theorem 3.2, $\mathcal{A}_I$ is positive semi-definite.

Corollary 5.2. Let $\mathcal{A} = (a_{ijkl}) \in S_{4,3}$, where $a_{1111}, a_{2222} \geq 0$ and $a_{3333} \geq 1$. Then $\mathcal{A}$ is positive semi-definite if $a_{1112} = a_{1222} = a_{2223} = a_{1333} = a_{1113} = 0$, $a_{iijj} \geq 0$ for $i, j \in [3]$, $i \neq j$, $a_{1223} = a_{1233} = 0$, $a_{3333} \geq |a_{2333}|$, $a_{2233} \geq \frac{2}{3}|a_{2333}|$, and one of the following conditions is satisfied: - (a) $a_{1123} = 0$. - (b) $a_{1122} \geq |a_{2333}|$, $a_{1133} \geq |a_{2333}|$, $a_{3333} \geq |a_{1123}|$, $a_{1122} \geq |a_{1123}|$, $a_{1133} \geq |a_{1123}|$, and $a_{2233} \geq \frac{2}{3}|a_{1123}|$.

Moreover, if $a_{1111}, a_{2222} > 0$ and $a_{3333} \geq 1$, then $\mathcal{A}$ is positive definite if condition (a) or (b) is satisfied.

Theorem 5.3. For a 4th-order 3-dimensional real interval tensor $\mathcal{A}_I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$, where $\underline{a}_{1111} = 0$ and $\underline{a}_{2222} = \underline{a}_{3333} = 1$, $\mathcal{A}_I$ is positive semi-definite if $\underline{a}_{1112} = \underline{a}_{1113} = \overline{a}_{1112} = \overline{a}_{1113} = 0$, and one of the following conditions is satisfied: - (a) $\underline{a}_{2223} = \underline{a}_{2333} = \underline{a}_{1123} = -1$, $\overline{a}_{2223} = \overline{a}_{2333} = \overline{a}_{1123} = 1$, $\underline{a}_{1222} = \underline{a}_{1333} = \overline{a}_{1222} = \overline{a}_{1333} = 0$, $\underline{a}_{1223} = \underline{a}_{1233} = \overline{a}_{1223} = \overline{a}_{1233} = 0$, and $\underline{a}_{iijj} \geq 1$ for all $i, j \in [3]$, $i \neq j$. - (b) $\underline{a}_{2333} = -1$, $\overline{a}_{2333} = 1$, $\underline{a}_{1222} = \underline{a}_{2223} = \underline{a}_{1333} = \overline{a}_{1222} = \overline{a}_{2223} = \overline{a}_{1333} = 0$, $\underline{a}_{1223} = -1$, $\overline{a}_{1223} = 1$, $\underline{a}_{1123} = \underline{a}_{1233} = \overline{a}_{1123} = \overline{a}_{1233} = 0$, $\underline{a}_{1133}, \underline{a}_{2233} \geq 1$, and $\underline{a}_{1122} \geq 2$.

Proof. Let $a_{1111} = \underline{a}_{1111} = 0$, and $a_{2222} = \underline{a}_{2222} = a_{3333} = \underline{a}_{3333} = 1$.

(a) Take $a_{iijj} = \underline{a}_{iijj} = 1$ for all $i, j \in [3]$, $i \neq j$. Then $a^c_{iiii} = 0$, $a^c_{iijj} = 1$ for all $i, j \in [3]$, $i \neq j$, $a^c_{3333} = 1$, $a^c_{iijk} = 0$ for all $i, j, k \in [3]$, $i \neq j$, $i \neq k$, $j \neq k$, $\delta_{iiii} = \delta_{iijj} = \delta_{1112} = \delta_{1222} = \delta_{1333} = \delta_{1113} = \delta_{1223} = \delta_{1233} = 0$ for all $i, j \in [3]$, $i \neq j$, and $\delta_{2223} = \delta_{2333} = \delta_{1123} = 1$.

Since for $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \quad \{0\}$,

$$\begin{aligned} f_{\mathcal{A}_{z_1}}(x) = f_{\mathcal{A}_{z_4}}(x) &= x_2^4 + x_3^4 - 4(x_2^3x_3 - 2x_2^2x_3^2 + x_2x_3^3) - 12x_1^2x_2x_3 + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) \\ &= (x_2^2 - 2x_2x_3 + x_3^2)^2 + 6(x_1x_2 - x_1x_3)^2, \\ f_{\mathcal{A}_{z_2}}(x) = f_{\mathcal{A}_{z_3}}(x) &= x_2^4 + x_3^4 + 4(x_2^3x_3 + 2x_2^2x_3^2 + x_2x_3^3) + 12x_1^2x_2x_3 + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) \\ &= (x_2^2 + 2x_2x_3 + x_3^2)^2 + 6(x_1x_2 + x_1x_3)^2, \end{aligned}$$

$\mathcal{A}_z$ is positive semi-definite for each $z \in Y$. According to Theorem 3.2, $\mathcal{A}_I$ is positive semi-definite.

(b) Take $a_{1133} = a_{2233} = \underline{a}_{1133} = \underline{a}_{2233} = 1$ and $a_{1122} = \underline{a}_{1122} = 2$. Then $a^c_{iiii} = 0$, $a^c_{iijj} = 1$ for all $i, j \in [3]$, $i \neq j$, $a^c_{3333} = 1$, $a^c_{iijk} = 0$ for all $i, j, k \in [3]$, $i \neq j$, $i \neq k$, $j \neq k$, $\delta_{iiii} = \delta_{iijj} = \delta_{1112} = \delta_{1222} = \delta_{2223} = \delta_{1333} = \delta_{1113} = \delta_{1123} = \delta_{1233} = 0$ for all $i, j \in [3]$, $i \neq j$, and $\delta_{2333} = \delta_{1223} = 1$.

Since for $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \quad \{0\}$,

$$\begin{aligned} f_{\mathcal{A}_{z_1}}(x) &= x_2^4 + x_3^4 - 4x_2x_3^3 - 12x_1x_2^2x_3 + 6\left(\frac{x_1^2}{2} + x_2^2 + x_3^2\right)^2 \\ &= (x_2^2 - 2x_1x_3)^2 + (x_3^2 - 2x_2x_3 + 2x_1x_2)^2 + 2(x_1x_3 - x_2x_3)^2, \\ f_{\mathcal{A}_{z_2}}(x) &= x_2^4 + x_3^4 + 4x_2x_3^3 + 12x_1x_2^2x_3 + 6\left(\frac{x_1^2}{2} + x_2^2 + x_3^2\right)^2 \\ &= (x_2^2 + 2x_1x_3)^2 + (x_3^2 + 2x_2x_3 + 2x_1x_2)^2 + 2(x_1x_3 - x_2x_3)^2, \\ f_{\mathcal{A}_{z_3}}(x) &= x_2^4 + x_3^4 + 4x_2x_3^3 - 12x_1x_2^2x_3 + 6\left(\frac{x_1^2}{2} + x_2^2 + x_3^2\right)^2 \\ &= (x_2^2 - 2x_1x_3)^2 + (x_3^2 + 2x_2x_3 - 2x_1x_2)^2 + 2(x_1x_3 + x_2x_3)^2, \\ f_{\mathcal{A}_{z_4}}(x) &= x_2^4 + x_3^4 - 4x_2x_3^3 + 12x_1x_2^2x_3 + 6\left(\frac{x_1^2}{2} + x_2^2 + x_3^2\right)^2 \\ &= (x_2^2 + 2x_1x_3)^2 + (x_3^2 - 2x_2x_3 - 2x_1x_2)^2 + 2(x_1x_3 + x_2x_3)^2, \end{aligned}$$

$\mathcal{A}_z$ is positive semi-definite for each $z \in Y$. According to Theorem 3.2, $\mathcal{A}_I$ is positive semi-definite.

Corollary 5.3. Let $\mathcal{A} = (a_{ijkl}) \in S_{4,3}$, where $a_{1111} \geq 0$ and $a_{2222}, a_{3333} \geq 1$. Then $\mathcal{A}$ is positive semi-definite if $a_{1112} = a_{1222} = a_{1333} = a_{1113} = 0$, and one of the following conditions is satisfied: - (a) $a_{iijj} \geq |a_{2333}|$, $a_{iijj} \geq |a_{2223}|$, $a_{iijj} \geq |a_{1123}|$ for $i, j \in [3]$, $i \neq j$, and $a_{1223} = a_{1233} = 0$. - (b) $a_{1122} \geq \frac{2}{3}|a_{2333}|$, $a_{1122} \geq \frac{2}{3}|a_{1223}|$, $a_{1133} \geq |a_{2333}|$, $a_{1133} \geq |a_{1223}|$, $a_{2233} \geq |a_{2333}|$, $a_{2233} \geq \frac{2}{3}|a_{1223}|$, and $a_{1123} = a_{1233} = a_{2223} = 0$.

Moreover, if $a_{1111} > 0$ and $a_{2222}, a_{3333} \geq 1$, then $\mathcal{A}$ is positive definite if $a_{1112} = a_{1222} = a_{1333} = a_{1113} = 0$ and condition (a) or (b) is satisfied.

Theorem 5.4. For a 4th-order 3-dimensional real interval tensor $\mathcal{A}_I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$, where $\underline{a}_{1111} = \underline{a}_{2222} = \underline{a}_{3333} = 1$, $\mathcal{A}_I$ is positive definite if $\underline{a}_{1222} = \underline{a}_{2333} = \underline{a}_{1113} = \overline{a}_{1222} = \overline{a}_{2333} = \overline{a}_{1113} = 0$, $\underline{a}_{1112} = \underline{a}_{2223} = -1$, $\overline{a}_{1112} = \overline{a}_{2223} = 1$, and one of the following conditions is satisfied: - (a) $a_{iijj} = \frac{2}{3}$, $\underline{a}_{iijk} = \overline{a}_{iijk} = 0$ for all $i, j, k \in [3]$, $i \neq j$, $i \neq k$, $j \neq k$, and $\underline{a}_{1333} = -1$, $\overline{a}_{1333} = 1$. - (b) $a_{iijj} = 1$ for all $i, j \in [3]$, $i \neq j$, $\underline{a}_{1123} = \underline{a}_{1223} = \overline{a}_{1123} = \overline{a}_{1223} = 0$, $\underline{a}_{1233} = -1$, $\overline{a}_{1233} = 1$, and $\underline{a}_{1333} = \overline{a}_{1333} = 0$.

Proof. Let $a_{iiii} = \underline{a}_{iiii} = 1$ for all $i \in [3]$.

(a) Take $a_{iijj} = \underline{a}_{iijj} = \frac{2}{3}$ for all $i, j \in [3]$, $i \neq j$. Then $a^c_{iiii} = 1$, $a^c_{iijj} = \frac{2}{3}$ for all $i, j \in [3]$, $i \neq j$, $a^c_{iijk} = 0$ for all $i, j, k \in [3]$, $i \neq j$, $i \neq k$, $j \neq k$, $\delta_{iiii} = \delta_{iijj} = \delta_{1222} = \delta_{2333} = \delta_{1113} = \delta_{iijk} = 0$ for all $i, j \in [3]$, $i \neq j$, and $\delta_{1112} = \delta_{2223} = \delta_{1333} = 1$.

Since for $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \quad \{0\}$,

$$
\begin{aligned}
f_{\mathcal{A}_{z_1}}(x) &= x_1^4 + x_2^4 + x_3^4 - 4(x_1^3x_2 - 2x_1^2x_2^2 + x_1x_2^3) - 4(x_2^3x_3 - 2x_2^2x_3^2 + x_2x_3^3) \\
&\quad - 4(x_1^3x_3 - 2x_1^2x_3^2 + x_1x_3^3) + 4(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2) \\
&= (x_1^2 - 2x_1x_2)^2 + (x_2^2 - 2x_2x_3)^2 + (x_3^2 - 2x_1x_3)^2, \\
f_{\mathcal{A}_{z_2}}(x) &= x_1^4 + x_2^4 + x_3^4 - 4(x_1^3x_2 - 2x_1^2x_2^2 + x_1x_2^3) + 4(x_2^3x_3 + 2x_2^2x_3^2 + x_2x_3^3) \\
&\quad - 4(x_1^3x_3 - 2x_1^2x_3^2 + x_1x_3^3) - 4(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2) \\
&= (x_1^2 - 2x_1x_2)^2 + (x_2^2 + 2x_2x_3)^2 + (x_3^2 - 2x_1x_3)^2, \\
f_{\mathcal{A}_{z_3}}(x) &= x_1^4 + x_2^4 + x_3^4 + 4(x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3) - 4(x_2^3x_3 - 2x_2^2x_3^2 + x_2x_3^3) \\
&\quad - 4(x_1^3x_3 - 2x_1^2x_3^2 + x_1x_3^3) - 4(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2) \\
&= (x_1^2 + 2x_1x_2)^2 + (x_2^2 - 2x_2x_3)^2 + (x_3^2 - 2x_1x_3)^2, \\
f_{\mathcal{A}_{z_4}}(x) &= x_1^4 + x_2^4 + x_3^4 + 4(x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3) + 4(x_2^3x_3 + 2x_2^2x_3^2 + x_2x_3^3) \\
&\quad - 4(x_1^3x_3 - 2x_1^2x_3^2 + x_1x_3^3) + 4(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2) \\
&= (x_1^2 + 2x_1x_2)^2 + (x_2^2 + 2x_2x_3)^2 + (x_3^2 - 2x_1x_3)^2,
\end{aligned}
$$

$\mathcal{A}_z$ is positive definite for each $z \in Y$. According to Theorem 3.2, $\mathcal{A}_I$ is positive definite.

(b) Take $a_{iijj} = \underline{a}_{iijj} = 1$ for all $i, j \in [3]$, $i \neq j$. Then $a^c_{iiii} = a^c_{iijj} = 1$ for all $i, j \in [3]$, $i \neq j$, $a^c_{iijk} = 0$ for all $i, j, k \in [3]$, $i \neq j$, $i \neq k$, $j \neq k$, $\delta_{iiii} = \delta_{iijj} = \delta_{1222} = \delta_{2333} = \delta_{1113} = \delta_{1333} = \delta_{1123} = \delta_{1223} = 0$ for all $i, j \in [3]$, $i \neq j$, and $\delta_{1112} = \delta_{2223} = \delta_{1233} = 1$.

Since for $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \quad \{0\}$,

$$
\begin{aligned}
f_{\mathcal{A}_{z_1}}(x) &= x_1^4 + x_2^4 + x_3^4 - 4(x_1^3x_2 - 2x_1^2x_2^2 + x_1x_2^3) - 4(x_2^3x_3 - 2x_2^2x_3^2 + x_2x_3^3) \\
&\quad - 12x_1x_2x_3^2 + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) \\
&= (x_1^2 - 2x_1x_2 + x_2^2)^2 + (x_2^2 - 2x_2x_3 + 2x_1x_3)^2 + 2(x_2x_3 - x_1x_2)^2, \\
f_{\mathcal{A}_{z_2}}(x) &= x_1^4 + x_2^4 + x_3^4 - 4(x_1^3x_2 - 2x_1^2x_2^2 + x_1x_2^3) - 4(x_2^3x_3 - 2x_2^2x_3^2 + x_2x_3^3) \\
&\quad + 12x_1x_2x_3^2 + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) \\
&= (x_1^2 - 2x_1x_2 + x_2^2)^2 + (x_2^2 + 2x_2x_3 - 2x_1x_3)^2 + 2(x_2x_3 + x_1x_2)^2, \\
f_{\mathcal{A}_{z_3}}(x) &= x_1^4 + x_2^4 + x_3^4 + 4(x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3) + 4(x_2^3x_3 + 2x_2^2x_3^2 + x_2x_3^3) \\
&\quad + 12x_1x_2x_3^2 + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) \\
&= (x_1^2 + 2x_1x_2 + x_2^2)^2 + (x_2^2 + 2x_2x_3 + 2x_1x_3)^2 + 2(x_2x_3 - x_1x_2)^2, \\
f_{\mathcal{A}_{z_4}}(x) &= x_1^4 + x_2^4 + x_3^4 + 4(x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3) - 4(-x_2^3x_3 + 2x_2^2x_3^2 - x_2x_3^3) \\
&\quad + 12x_1x_2x_3^2 + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) \\
&= (x_1^2 + 2x_1x_2 + x_2^2)^2 + (x_2^2 - 2x_2x_3 - 2x_1x_3)^2 + 2(x_2x_3 + x_1x_2)^2,
\end{aligned}
$$

$\mathcal{A}_z$ is positive definite for each $z \in Y$. According to Theorem 3.2, $\mathcal{A}_I$ is positive definite.

Corollary 5.4. Let $\mathcal{A} = (a_{ijkl}) \in S_{4,3}$, where $a_{1111}, a_{2222}, a_{3333} \geq 1$. Then $\mathcal{A}$ is positive definite if $a_{1222} = a_{2333} = a_{1113} = 0$, and one of the following conditions is satisfied: - (a) $a_{iiii} \geq |a_{1112}|$, $a_{iiii} \geq |a_{2223}|$, $a_{iiii} \geq |a_{1333}|$, $a_{iijj} \geq \frac{2}{3}|a_{1333}|$, and $a_{iijk} = 0$ for $i,j,k \in [3]$, $i \neq j$, $i \neq k$, $j \neq k$. - (b) $a_{iiii} \geq |a_{1112}|$, $a_{iiii} \geq |a_{2223}|$, $a_{iiii} \geq |a_{1123}|$, $a_{iijj} \geq |a_{1112}|$, $a_{iijj} \geq |a_{2223}|$, $a_{iijj} \geq |a_{1123}|$ for $i,j \in [3]$, $i \neq j$, and $a_{1113} = a_{1223} = a_{1233} = 0$.

6 Conclusions

This paper studies the positive definiteness and Hurwitz stability of interval tensors, clarifying that the positive definiteness of an interval tensor is equivalent to that of its symmetric interval tensor and auxiliary tensors $\mathcal{A}_z$. It also reveals that the stability of a symmetric interval tensor implies the stability of the interval tensor, and the stability of the symmetric interval tensor is equivalent to that of auxiliary tensors $\tilde{\mathcal{A}}_z$. In addition, sufficient conditions for the positive definiteness of 4th-order 3-dimensional interval tensors are provided.

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