Double shape quantum phase transitions in the SU3-IBM: new γ-soft phase and the shape phase transition from the new γ-soft phase to the prolate shape

Authors: Prof. Xiaoshen Kang, Wang, Dr. Tao, Kang, Prof. Xiaoshen

Date: 2025-08-24T00:00:00+00:00

Abstract

Shape quantum phase transition is an important topic in nuclear structure. In this paper, we begin to study the shape quantum phase transition in the SU3-IBM. In this new proposed model, spherical-like spectra was found to resolve the spherical nucleus puzzle, which is a new γ-soft rotational mode. In this paper, the shape phase transition along the new γ-soft line is first discussed, and then the neighbouring case at the prolate side is also studied. We find that double shape phase transitions occur along a single parameter path. The new γ-softness is really a shape phase and the shape phase transition from the new γ-soft phase to the prolate shape is found. The experimental support is also found and ^{108}Pd may be the critical nucleus.

Full Text

Preamble

Double Shape Quantum Phase Transitions in the SU3-IBM: New γ-Soft Phase and the Shape Phase Transition from the New γ-Soft Phase to the Prolate Shape

De-hao Zhao,[1] Ying-xin Wu,[2] Li Gong,[1],* Ze-yu Yin,[3] Xiao-shen Kang,[1],† and Tao Wang[3],‡

[1]School of Physics, Liaoning University, Shenyang 110036, People's Republic of China
[2]School of Physics, Liaoning Normal University, Dalian 116029, People's Republic of China
[3]College of Physics, Tonghua Normal University, Tonghua 134000, People's Republic of China

Shape quantum phase transitions represent an important topic in nuclear structure theory. In this paper, we investigate shape quantum phase transitions within the SU3-IBM framework. In this newly proposed model, spherical-like spectra were discovered that resolve the spherical nucleus puzzle, revealing a new γ-soft rotational mode. Here, we first discuss the shape phase transition along the new γ-soft line, and subsequently examine the neighboring case on the prolate side. We find that double shape phase transitions occur along a single parameter path. The new γ-softness constitutes a genuine shape phase, and we identify a shape phase transition from this new γ-soft phase to the prolate shape. Experimental support for this phenomenon is also found, with ^{108}Pd potentially serving as the critical nucleus.

Keywords: SU(3)-IBM, Quantum phase transitions, Spherical nucleus puzzle

INTRODUCTION

Phase transitions are ubiquitous in nature [1, 2]. A familiar example occurs under standard atmospheric pressure: as temperature increases, ice transforms into water and then water vapor. If the atmospheric pressure is raised to a certain level, water and water vapor become indistinguishable. In atomic nuclei, nuclear shape can change as the number of protons or neutrons varies, leading to shape quantum phase transitions [3–15]. Since the control parameter in nuclei is discrete and finite, identifying these phase transitions becomes particularly intriguing [16].

Fifty years ago, Arima and Iachello proposed the interacting boson model (IBM), an influential algebraic model for describing the collective behavior of nucleons. In its simplest formulation, only s (L = 0) and d (L = 2) bosons are considered, and the Hamiltonian possesses U(6) symmetry. The model exhibits four dynamical symmetry limits (see Fig. 1 Figure 1: see original paper left): (1) the U(5) symmetry limit describes spherical shapes and their vibrations, (2) the SU(3) symmetry limit describes prolate shapes and their rotations, (3) the O(6) symmetry limit describes γ-soft rotations, and (4) the SU(3) symmetry limit describes oblate shapes and their rotations.

This simple model can describe shape phase transitions from spherical shapes to various quadrupole deformations or among different deformed shapes (see Fig. 1(a) left) [3–15]. In these studies, along a single parameter path, the nuclear shape changes only from one form to another. After 2000, an important class of shape phase transitions has attracted attention and generated controversy: the prolate-oblate shape phase transition [19, 20].

In the conventional IBM, the prolate-oblate shape phase transition is described by moving from the SU(3) symmetry limit to the SU(3) symmetry limit, with the O(6) symmetry limit serving merely as the first-order phase transition critical point [19]. This implies that O(6) γ-softness is not a shape phase. In this description, the spectra of prolate and oblate shapes are identical, which is not observed in realistic nuclei. In [21], for nuclei in the Hf-Hg region, the energy ratio $E_{4}/_{2} = E(4^{+}{}_{1})/E(2^{+}{}_{1})$ is 3.33 for prolate shapes versus 2.55 for oblate shapes ($E_4/_2$ is independent of the boson number N). Thus, this mirror symmetry appears not to exist.

Around 2020, an extension of the interacting boson model incorporating SU(3) higher-order interactions (SU3-IBM) was proposed by one of the authors (T. Wang) [22, 23], which combines ideas from the conventional IBM with the SU(3) correspondence of rigid triaxial shapes [24–28]. In this new model, the role of SU(3) symmetry is elevated to a new level, dominating all quadrupole deformations of nuclei (see Fig. 1(a) right). The model contains only the U(5) symmetry limit and the SU(3) symmetry limit. In the SU(3) symmetry limit, higher-order interactions are required. The SU(3) second-order Casimir operator $\hat{C}_2$[SU(3)] can describe prolate shapes, while the SU(3) third-order Casimir operator $\hat{C}_3$[SU(3)] can describe oblate shapes—a significant departure from the SU(3) description in the conventional IBM. These two interactions, together with the square of the SU(3) second-order Casimir operator $\hat{C}_2{}^2$[SU(3)], can describe any rigid triaxial shape.

The SU3-IBM has been successfully applied to explain the B(E2) anomaly [22, 29–38], the Cd puzzle [23, 39, 40], the prolate-oblate asymmetric shape phase transition [41–43], γ-softness in ^{196}Pt at an improved level [44, 45], E(5)-like spectra in ^{82}Kr [46], rigid triaxiality in ^{166}Er [47], and the boson number odd-even effect in $^{196-204}$Hg [48]. Collectively, these results demonstrate that the SU3-IBM provides a better description of collective behavior in nuclei. Consequently, investigating shape phase transitions within the SU3-IBM framework becomes increasingly important.

In the SU3-IBM, to resolve the Cd puzzle in Cd nuclei and other nuclei previously considered spherical [49], a spherical-like nucleus was proposed [22]. This represents a new collective excitation mode that has recently been verified in realistic nuclei [39, 40]. This new shape was not predicted by previous nuclear theories. Furthermore, the prolate-oblate asymmetric shape phase transition in the Hf-Hg region can be better described by the SU3-IBM [43]. The first transition occurs from the spherical shape to the new γ-soft rotation, and the second from the new γ-soft mode to the prolate shape. Here, the new γ-softness represents a genuine shape phase, distinct from the O(6) critical γ-softness. These results appear more realistic and highly significant.

II. HAMILTONIAN

The simplest Hamiltonian for describing shape phase transitions related to the new γ-softness in the SU3-IBM is given by [23, 43]:

$$\hat{H} = c[(1-\eta)\hat{n}_d + \eta(-\hat{C}_2[SU(3)]\hat{C}_3[SU(3)]$$

where and are two control parameters and c is the energy scale parameter, with 0 ≤ ≤ 1 and ≥ 0. If = 0, the Hamiltonian describes the spherical shape. If = 1 and = 0, it describes the prolate shape. These two cases correspond to those in the conventional IBM [15]. If = 1 and varies, the Hamiltonian describes the prolate-oblate shape phase transition [42], which is a finite-N effect.

Fig. 1(a) left shows the phase diagram of the conventional IBM in the large-N limit. Above the solid green line, spherical shapes exist, while below it, deformed shapes exist. The deformed region is divided by the solid blue line, which is part of the connection between the U(5) symmetry limit and the O(6) symmetry limit. The left side of the blue line corresponds to prolate shapes, while the right side corresponds to oblate shapes. The blue line represents the critical line between prolate and oblate shapes. The intersection of the green and blue lines is a triple point where spherical, prolate, and oblate shapes coexist [50, 51]. Clearly, O(6) γ-softness is not a shape phase. Furthermore, the IBM with up to two-body interactions cannot describe rigid triaxial shapes [52].

The phase diagram of $\hat{H}$ in the large-N limit was first discussed in [41] and is shown in Fig. 1(a) right (or Fig. 14 [Figure 14: see original paper] in [41]). The key difference between the SU3-IBM and the conventional IBM is the use of the SU(3) third-order Casimir operator $\hat{C}$_3[SU(3)] instead of the SU(3) symmetry limit to describe oblate nuclei. Above the solid green line, spherical shapes exist. Below the solid green line, deformed shapes exist. In the deformed region, the phase diagram becomes more complex than in the conventional IBM [15]. Between the SU(3) prolate and SU(3) oblate sides, there exists a SU(3) degenerate point (the blue point). On the SU(3) prolate side of this degenerate point, the ground state shape is always prolate, while on the SU(3) oblate side, it is always oblate. Thus, the shape changes abruptly across the SU(3) degenerate point. This finite-N first-order shape phase transition was studied in [42]. Connected to the SU(3) degenerate point, in the middle of the deformed region, there exists a shallow narrow region with rigid triaxial shapes, shown as the region between the two blue lines in Fig. 1(a) right. The left side of this narrow region corresponds to prolate shapes, while the right side corresponds to oblate shapes. Therefore, along the dashed blue line in Fig. 1(a) right, the shape changes from prolate to oblate via a narrow region with rigid triaxiality [41]. Although the rigid triaxial region is narrow and shallow, rigid triaxial shapes do exist. The intersection between the green line and the two blue lines is a fourfold point.

In [23], one of the authors (T. Wang) discovered an important result: for finite-N, the rigid triaxial shape becomes a new γ-soft rotational mode. This new γ-softness is a shape phase, not a critical phenomenon. Along the dashed blue line in Fig. 1(a) right, the shape changes from prolate to the new γ-soft phase, and then to oblate, which was used to describe the prolate-oblate asymmetric shape phase transition in the Hf-Hg region [43].

In [23], the solid blue line in Fig. 1(b) represents a critical line between the prolate shape and the new γ-soft rotation in the deformed region, along which the ${4^+}_2$, ${2^+}_2$, and ${6^+}_3$ states are degenerate. However, for large-N, this critical line is actually a curve that bends to the left. Thus, for large-N, the critical line cannot be described by the solid blue line [44]. To the right of the degenerate line, there exists another line along which the $4^+{}_2$ states are also degenerate [44]. Between these two degenerate lines for the $4^+{}_2$ states, it was previously supposed that the new γ-softness exists; however, this paper shows that this new γ-soft region may be larger than expected. Importantly, we find that double shape quantum phase transitions can be observed along a single parameter path.

In this paper, we study the shape phase transition along the solid blue line without emphasizing the degenerate line, which is difficult to discuss. The SU(3) degenerate point is at $_0$ = 3N/(2N+3) and the red point is at 0.9 $_0$. Based on previous analysis, the red point represents the prolate shape. This is very interesting. For the solid red line from the U(5) symmetry limit to the red point, one might intuitively expect the shape to change directly from spherical to prolate. However, through detailed numerical calculations, we find that this shape transition is not direct but proceeds through the new γ-softness region, and double shape quantum phase transitions can occur. The right portion of the solid blue line is not discussed in this paper and will be examined in a future work investigating the scope of the new γ-soft region [53].

It is important to emphasize why the O(6) γ-softness in the conventional IBM does not match actual γ-soft nuclei. In realistic nuclei, many nuclei exist in the γ-soft region, such as Os, Pt, Xe, and Ba nuclei [54], making it difficult to believe that this is merely the O(6)-softness in the conventional IBM—a critical point of shape phase transition. In the SU3-IBM, such a conceptual conflict does not exist. We believe that the shape phase transitions described by the SU3-IBM provide an accurate description of realistic shape phase transitions in nuclei. This point has been preliminarily confirmed by the prolate-oblate asymmetric shape phase transition in the Hf-Hg region [43].

For understanding the B(E2) anomaly, B(E2) values are essential. The E2 operator is defined as $\hat{T}$(E2) = q$\hat{Q}$, where q is the boson effective charge. We discuss the evolution of B(E2; $2^+{}_1 \rightarrow 0^+{}_1$), B(E2; $4^+{}_1 \rightarrow 2^+{}_1$), B(E2; $0^+{}_2 \rightarrow 2^+{}_1$), and B(E2; $2^+{}_2 \rightarrow 2^+{}_1$) values.

III. DOUBLE SHAPE QUANTUM PHASE TRANSITION

Shape quantum phase transitions are first manifested through the energy evolution of the ground state. While not directly observable, this quantity is very useful for understanding shape quantum phase transitions. Fig. 2(a) shows the energy evolution of the ground state of $\hat{H}$ along the solid blue line in Fig. 1(b) for N = 10 (dashed blue line) and N = 60 (solid blue line). Clearly, around = 0.2 (denoted by the left dashed line), a shape phase transition from spherical to the new γ-soft phase occurs. Between = 0.2 and = 1, the new γ-softness exists. It should be noted that the SU(3) degenerate point is not γ-soft.

Fig. 2(a) also shows the energy evolution of the ground state of $\hat{H}$ along the solid red line in Fig. 1(b) for N = 10 (dashed red line) and N = 60 (solid red line). For N = 60, a new shape phase transition appears around = 0.5 (denoted by the middle dashed line). The portion of the solid red line deviating from the solid blue line is prominent, showing a steeper descent. Between = 0 and = 0.2, the spherical shape exists, while between = 0.5 and = 1, the prolate shape exists. Obviously, between these two shapes, the new γ-softness exists. When changes from $_0$ to 0.9$_0$, the new γ-soft region shrinks but still exists.

A key point is that between = 0.2 and = 0.5, the red and blue lines are nearly degenerate, so when changes from $_0$ to 0.9$_0$, the energies of the new γ-softness remain nearly the same and do not decrease. This implies that the new γ-softness is truly a shape phase. Thus, along the left solid red line, double shape quantum phase transitions can occur. This cannot happen for shape phase transitions along the solid blue line or in the conventional IBM.

Some details require further elaboration. For N = 10, the solid blue line in Fig. 1(b) is nearly the critical line between the prolate shape and the new γ-soft region, so to the left of the solid blue line, it appears to be only the prolate shape region. This implies that the new γ-soft region is larger than expected, which will be further clarified in [53]. For N = 35, the critical line curves to the left, making the new shape phase transition more prominent.

In the conventional IBM, a similar result for ground state energy evolution along the solid blue line can be obtained along the evolution path from the U(5) symmetry limit to the O(6) symmetry limit. If the O(6) symmetry limit were the prolate-oblate critical point, the connected line in the deformed region between the U(5) symmetry limit and the O(6) symmetry limit would be a prolate-oblate critical line. When deviating from this critical line, the energies in the deformed region decrease, and double shape phase transitions cannot be observed [15].

The mean value of the d-boson number in the ground state, $\bar{n}$_d, is also important. Fig. 2(b) shows the evolution of $\bar{n}$_d along the solid blue line in Fig. 1(b) for N = 10 (dashed blue line) and N = 60 (solid blue line). The phase transition behavior from spherical to new γ-soft across = 0.2 is clearly visible. Fig. 2(b) also shows the $\bar{n}$_d evolution along the solid red line in Fig. 1(b) for N = 10 (dashed red line) and N = 60 (solid red line). The double shape quantum phase transitions are also evident. When > 0.5, the two red lines deviate significantly from the two blue lines. Importantly, between = 0.2 and = 0.5, the red and blue lines are nearly degenerate for both N = 10 and N = 60, confirming that the new γ-soft phase truly exists.

We now discuss observable quantities such as excited state energies, B(E2) values, and the electric quadrupole moment of the $2^+{}_1$ state. Previous discussions help confirm that double shape quantum phase transitions do exist, and these observable quantities can help identify them experimentally.

We first study the shape phase transition along the solid blue line in Fig. 1(b) from the U(5) symmetry limit to the SU(3) degenerate point. This study was previously performed in [23] for N = 7. Here, the evolution of partial low-lying states for N = 10 is shown in Fig. 3(a). The $4^{+}{}_{2}$ and $2^{+}{}_{2}$ states are nearly degenerate. At $= 0.5$, spherical-like spectra appear, where the energy of the $0^{+}{}_{3}$ state is nearly twice that of the $0^{+}{}_{2}$ state. The spherical-like spectra were recently confirmed in ^{106}Pd [40], demonstrating that the shape phase transition discussed in this paper can be found in Pd nuclei.

Fig. 3(b) shows the evolution of B(E2) values for B(E2; $2^{+}{}_{1} \rightarrow 0^{+}{}_{1}$), B(E2; $4^{+}{}_{1} \rightarrow 2^{+}{}_{1}$), B(E2; $0^{+}{}_{2} \rightarrow 2^{+}{}_{1}$), and B(E2; $2^{+}{}_{2} \rightarrow 2^{+}{}_{1}$) along the solid blue line in Fig. 1(b). The results are similar to evolutions from the U(5) symmetry limit to the O(6) symmetry limit.

Fig. 4(a) presents the evolution of partial low-lying states for N = 35 along the solid blue line in Fig. 1(b), representing new results. The shape phase transition from spherical to the new γ-soft rotation becomes more prominent. When > 0.2, the $4^{+}{}_{2}$ and $2^{+}{}_{2}$ states begin to separate because the degenerate line curves to the left. Additionally, the level anticrossing of the $0^{+}{}_{4}$ and $0^{+}{}_{3}$ states becomes more pronounced [37, 55, 56]. Fig. 4(b) shows the evolution of B(E2) values for B(E2; $2^{+}{}_{1} \rightarrow 0^{+}{}_{1}$), B(E2; $4^{+}{}_{1} \rightarrow 2^{+}{}_{1}$), B(E2; $0^{+}{}_{2} \rightarrow 2^{+}{}_{1}$), and B(E2; $2^{+}{}_{2} \rightarrow 2^{+}{}_{1}$) along the solid blue line in Fig. 1(b). The shape phase transition becomes clearer. As N increases, the values of B(E2; $2^{+}{}_{2} \rightarrow 2^{+}{}_{1}$) become smaller.

We now discuss the double shape phase transitions along the solid red line in Fig. 1(b). Fig. 5(a) shows the evolution of partial low-lying states for N = 10. Between $= 0.2$ and $= 0.5$, the $4^{+}{}_{2}$ and $2^{+}{}_{2}$ states are nearly degenerate, and the spectra resemble spherical-like spectra, indicating that this is the new γ-soft phase. When > 0.5, the shape is clearly prolate. Fig. 6(a) shows the case for N = 35, where the shape phase transition from γ-soft rotation to prolate shape becomes very prominent. Thus, $= 0.9 {}_{0}$ and $= 0.5$ represents the phase transition critical point.

In the conventional IBM, O(6) γ-softness is merely the shape phase transition critical point between prolate and oblate shapes, not a shape phase itself. No shape phase transition exists from the O(6) symmetry limit (γ-soft rotation) to the SU(3) symmetry limit (prolate shape). In the SU3-IBM, the new γ-softness is a shape phase, and the shape phase transition from the new γ-soft phase to the prolate shape truly exists.

Fig. 5(b) and Fig. 6(b) present the evolution of B(E2) values for B(E2; $2^{+}{}_{1} \rightarrow 0^{+}{}_{1}$), B(E2; $4^{+}{}_{1} \rightarrow 2^{+}{}_{1}$), B(E2; $0^{+}{}_{2} \rightarrow 2^{+}{}_{1}$), and B(E2; $2^{+}{}_{2} \rightarrow 2^{+}{}_{1}$) for N = 10 and N = 35. The double shape phase transitions are clearly visible. Across $= 0.5$, the value of B(E2; $0^{+}{}_{2} \rightarrow 2^{+}{}_{1}$) first increases and then decreases, a behavior not seen in Fig. 3(b) and Fig. 4(b).

In [39], the first evidence confirming the existence of sphere-like spectra was found through the anomalous evolution trend of electric quadrupole moments of the first $2^{+}{}_{1}$ states ($Q_{2}{}^{+}$) in Cd nuclei. We now further investigate this interesting phenomenon. Fig. 7 [Figure 7: see original paper] shows the evolution of $Q_{2}{}^{+}$ values along the solid blue line (blue lines in Fig. 7) and the solid red line (red lines in Fig. 7) for N = 7 (dotted), N = 10 (dashed), and N = 15 (solid). Along the solid red line in Fig. 1(b), the double shape phase transitions can be clearly observed. The key result is the different evolution trends along the two parameter paths. The blue path shows anomalous behavior: as N increases, the value evolves toward the oblate side. The red path shows the opposite trend: as N increases, its magnitude also increases when ≥ 0.5.

Finally, we search for experimental evidence supporting the shape phase transition from the new γ-soft phase to the prolate shape. Fig. 8(a) [Figure 8: see original paper] shows the different evolution trends of $Q_2{}^+$ values for $^{108-114}$Cd and $^{104-110}$Pd. This was first observed in [39] and can be regarded as the first strong support for the existence of spherical-like spectra. The discussions in this paper further support this conclusion and provide indirect experimental support for the shape phase transition. Direct support can be seen in Fig. 8(b). In our previous analysis, we observed that from the new γ-soft phase to the prolate shape, the value of B(E2; $0^+{}_2 \rightarrow 2^+{}_1$) first increases and then decreases. Here we define β = B(E2; $0^+{}_2 \rightarrow 2^+{}_1$)/B(E2; $2^+{}_1 \rightarrow 0^+{}_1$). Fig. 8(b) presents the β evolution for $^{104-110}$Pd, which clearly aligns with theoretical predictions.

Thus, ^{104}Pd and ^{106}Pd are two typical new γ-soft nuclei [40], and ^{108}Pd may be a critical nucleus at the transition from the new γ-soft phase to the prolate shape. However, for finite-N, spherical-like spectra and critical spectra may be difficult to distinguish, which can help us further discuss those nuclei that appear spherical.

It should be noted that these double shape quantum phase transitions cannot be verified directly. In the last two decades, the experimental discovery that nuclei previously thought to be spherical cannot be confirmed as truly spherical represents a breakthrough in nuclear structure [49]. If spherical nuclei are absent, it becomes difficult to confirm the shape phase transition from spherical to the new γ-soft phase. In our discussion, we also found that spherical nuclei and the critical nucleus at $= 0.2$ are difficult to distinguish, which can help us further investigate those nuclei that appear spherical.

IV. CONCLUSION

Based on the existence of sphere-like spectra [23, 39, 40], we have further investigated the related shape quantum phase transitions. In this paper, we have reached several new conclusions. First, the new γ-softness is a shape phase, fundamentally different from previous O(6)-softness which represents only a critical point. Second, we have discovered double quantum phase transitions along a single parameter path. We confirm that a shape phase transition indeed occurs from the new γ-soft phase to the prolate shape, and we find experimental evidence that ^{108}Pd may be a critical nucleus, which will be studied in detail later. In a future paper, we will present the scope of the new γ-soft region in the SU3-IBM for finite-N and discuss the oblate side [53].

REFERENCES

[1] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, Oxford, 1971)
[2] S. Sachdev, *Quantum Phase Transitions*, second edition (Cambridge University Press, Cambridge, U.K. 2011).
[3] R. F. Casten, Nat. Phys. **2**, 811 (2006).
[4] R. F. Casten and E. A. McCutchan, J. Phys. G: Nucl. Part. Phys. **34**, R285 (2007).
[5] D. Bonatsos and E. A. McCutchan, Nucl. Phys. News **19**, 13 (2009).
[6] R. F. Casten, Prog. Part. Nucl. Phys. **62**, 183 (2009).
[7] P. Cejnar and J. Jolie, Prog. Part. Nucl. Phys. **62**, 210 (2009).
[8] P. Cejnar, J. Jolie and R. F. Casten, Rev. Mod. Phys. **82**, 2155 (2010).
[9] R. V. Jolos and E. A. Kolganova, Phys.-Usp. **64**, 325 (2021).
[10] L. Fortunato, Prog. Part. Nucl. Phys. **121**, 103891 (2021).
[11] P. Cejnar, P. Stránský, M. Macek and M. Kloc, J. Phys. A: Math. Theor. **54**, 133001 (2021).
[12] P. Cejnar and J. Jolie, Phys. Rev. C **61**, 6237 (2000).
[13] P. Cejnar, S. Heinze, and J. Jolie, Phys. Rev. C **68**, 034326 (2003).
[14] F. Iachello and N.V. Zamfir, Phys. Rev. Lett. **92**, 212501 (2004).
[15] F. Pan, T. Wang, Y. S. Huo, and J. P. Draayer, J. Phys. G: Nucl. Part. Phys. **35**: 125105 (2008).
[16] R. F. Casten, Dimitri Kusnezov and N. V. Zamfir, Phys. Rev. Lett. **82**, 5000 (1999).
[17] F. Iachello and A. Arima, *The Interacting Boson Model*, (Cambridge University Press, Cambridge, 1987).
[18] A. Arima and F. Iachello, Phys. Rev. Lett. **35**, 1069 (1975).
[19] J. Jolie, R. F. Casten, P. von Brentano, and V. Werner, Phys. Rev. Lett. **87**, 162501 (2001).
[20] D. Bonatsos, A. Martinou, S. K. Peroulis, T. J. Mertzimekis and N. Minkov, Phys. Scr. **99**, 062003 (2024).
[21] J. Jolie and A. Linnemann, Phys. Rev. C **68**, 031301(R) (2003).
[22] T. Wang, EPL **129**, 52001 (2020).
[23] T. Wang, Chin. Phys. C **46**, 074101 (2022).
[24] G. Vanden Berghe, H. E. De Meyer and P. Van Isacker, Phys. Rev. C **32**, 1049 (1985).
[25] Y. Leschber and J. P. Draayer, Phys. Lett. B **190**, 1 (1987).
[26] O. Castaños, J. P. Draayer and Y. Leschber, Com. Phys. Comm. **52**, 71 (1988).
[27] Y. F. Smirnov, N. A. Smirnova and P. Van Isacker, Phys. Rev. C **61**, 041302(R) (2000).
[28] V. K. B. Kota, *SU(3) Symmetry in Atomic Nuclei*, (Springer Nature, Singapore,2020).
[29] Y. Zhang, F. Pan, L. R. Dai and J. P. Draayer, Phys. Rev. C **90**, 044310 (2014).
[30] Y. Zhang, Y. W. He, D. Karlsson, C. Qi, F. Pan and J. P. Draayer, Phys. Lett. B **834**, 137443 (2022).
[31] T. Wang, Phys. Rev. C **107**, 064303 (2023).

[32] Y. Zhang, S. N. Wang, F. Pan, C. Qi and J. P. Draayer, Phys. Rev. C **110**, 024303 (2024).
[33] F. Pan, Y. Zhang, Y. X. Wu, L. R. Dai and J. P. Draayer, Phys. Rev. C **110**, 054324 (2024).
[34] W. Teng, Y. Zhang and C. Qi, Chin. Phys. C **49**, 014102 (2025).
[35] Y. Zhang and W. Teng, Phys. Rev. C **111**, 014324 (2025).
[36] Y. X. Cheng, D. H. Zhao, Y. Y. Shao, T. Wang and X. S. Kang, arXiv:2503.00882.
[37] T. Wang, Y. X. Cheng, D. K. Li, X. S. Kang, S. C. Jin, T. Wang, Z. Q. Zhang, C. G. Zhang and Z. X. Zhang, arXiv:2503.22100.
[38] T. Wang, Y. X. Cheng, D. K. Li, X. S. Kang, S. C. Jin, T. Wang, Z. Q. Zhang, C. G. Zhang and Z. X. Zhang, in preparation.
[39] T. Wang, X. Chen and Y. Zhang, Chin. Phys. C **49**, 014107 (2025).
[40] T. Wang, arXiv:2501.10925v2.
[41] L. Fortunato, C. E. Alonso, J. M. Arias, J. E. García-Ramos and A. Vitturi, Phys. Rev. C **84**, 014326 (2011).
[42] Y. Zhang, F. Pan, Y. X. Liu, Y. A. Luo and J. P. Draayer, Phys. Rev. C **85**, 064312 (2012).
[43] T. Wang, B. C. He, D. K. Li and C. X. Zhou, Phys. Rev. C **78**, 064322 (2023).
[44] T. Wang, B. C. He, C. X. Zhou, D. K. Li and L. Fortunato, Chin. Phys. C **48**, 094102 (2024).
[45] C. X. Zhou, T. Wang, D. K. Li, B. C. He and L. Fortunato, in preparation.
[46] C. X. Zhou and T. Wang, Phys. Rev. C **108**, 024309 (2023).
[47] C. X. Zhou and T. Wang, in preparation.
[48] T. Wang, C. X. Zhou and L. Fortunato, arXiv: 2412.14881v2.
[49] P. E. Garrett, J. L. Wood and S. W. Yates, Phys. Scr. **93**, 063001 (2018).
[50] J. Jolie, P. Cejnar, R. F. Casten, S. Heinze, A. Linnemann and V. Werner, Phys. Rev. Lett. **89**, 182502 (2002).
[51] D. Warner, Nature **420**, 614 (2002).
[52] P. Van Isacker and J. Q. Chen, Phys. Rev. C **24**, 684 (1981).
[53] T. Wang, D. H. Zhao and X. S. Kang, in preparation.
[54] ENSDF: https://www.nndc.bnl.gov/ensdf/
[55] N. V. Zamfir, P. von Brentano, R. F. Casten and J. Jolie, Phys. Rev. C **66**, 021304(R) (2002).
[56] J. M. Arias, J. Dukelsky and J. E. García-Ramos, Phys. Rev. Lett. **91**, 162502 (2003).

Note: Figure translations are in progress. See original paper for figures.