Representation Uniqueness of Generalized Triangular Fuzzy Numbers and Generalized Trapezoidal Fuzzy Numbers

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Abstract

In this paper, we introduce the generalized triangular fuzzy numbers and the generalized trapezoidal fuzzy numbers. Then it is shown the representation uniqueness of the generalized triangular fuzzy numbers and the representation uniqueness of the generalized trapezoidal fuzzy numbers. As corollaries of these conclusions, we have conclusions on the representation unique-ness of the triangular fuzzy numbers and on the representation uniqueness of the trapezoidal fuzzy numbers. Furthermore, we present several equivalent forms of some conclusions of the representation uniqueness. We also give some relationship among the triangular fuzzy numbers, the trapezoidal fuzzy numbers, the generalized triangular fuzzy numbers, and the generalized trapezoidal fuzzy numbers.

Full Text

Preamble

Representation Uniqueness of Generalized Triangular Fuzzy Numbers and Generalized Trapezoidal Fuzzy Numbers

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Abstract

In this paper, we introduce generalized triangular fuzzy numbers and generalized trapezoidal fuzzy numbers. We then establish the representation uniqueness of generalized triangular fuzzy numbers and generalized trapezoidal fuzzy numbers. As corollaries of these results, we obtain conclusions regarding the representation uniqueness of triangular fuzzy numbers and trapezoidal fuzzy numbers. Furthermore, we present several equivalent forms of some conclusions concerning representation uniqueness and establish relationships among triangular fuzzy numbers, trapezoidal fuzzy numbers, generalized triangular fuzzy numbers, and generalized trapezoidal fuzzy numbers.

Keywords: triangular fuzzy numbers; trapezoidal fuzzy numbers; representation uniqueness; generalized triangular fuzzy numbers; generalized trapezoidal fuzzy numbers

1. Introduction

Let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{R}^m$ the m-dimensional Euclidean space. We write $\mathbb{R}^1$ as $\mathbb{R}$. Usually, symbols (a, b, c, d) with $a, b, c, d \in \mathbb{R}$ represent elements in $\mathbb{R}^4$, and symbols (a, b, c) with $a, b, c \in \mathbb{R}$ represent elements in $\mathbb{R}^3$.

In this paper, for each $a, b, c, d \in \mathbb{R}$, we use $[a, b, c, d]$ instead of (a, b, c, d) to represent the corresponding element in $\mathbb{R}^4$, and use $[a, b, c]$ instead of (a, b, c) to represent the corresponding element in $\mathbb{R}^3$.

We denote by T the set $\{[a, b, c, d] \in \mathbb{R}^4 : a \leq b \leq c \leq d\}$ and by T_0 the set $\{[a, b, c, d] \in \mathbb{R}^4 : a < b \leq c < d\}$. Clearly $T_0 \subset T$.

We denote by G the set $\{[a, b, c] \in \mathbb{R}^3 : a \leq b \leq c\}$ and by G_0 the set $\{[a, b, c] \in \mathbb{R}^3 : a < b < c\}$. Clearly $G_0 \subset G$.

In theoretical research and practical applications, triangular fuzzy numbers and trapezoidal fuzzy numbers are frequently used fuzzy sets [**?**, **?**]. Each triangular fuzzy number can be represented as (a, b, c) with $[a, b, c] \in G_0$. Each trapezoidal fuzzy number can be represented as (a, b, c, d) with $[a, b, c, d] \in T_0$.

Naturally, we ask: For each triangular fuzzy number u, does there exist a unique $[a, b, c] \in G_0$ such that $u = (a, b, c)$? For each trapezoidal fuzzy number u, does there exist a unique $[a, b, c, d] \in T_0$ such that $u = (a, b, c, d)$? If these representation uniqueness questions can be answered affirmatively, it would greatly facilitate the analysis and discussion of triangular and trapezoidal fuzzy numbers.

In this paper, we introduce the concepts of generalized triangular fuzzy numbers and generalized trapezoidal fuzzy numbers, which generalize triangular and trapezoidal fuzzy numbers, respectively. The symbols Tag, Tap, Trag, and Trap denote the sets of triangular fuzzy numbers, trapezoidal fuzzy numbers, generalized triangular fuzzy numbers, and generalized trapezoidal fuzzy numbers, respectively.

We show that for each generalized triangular fuzzy number u, there exists a unique $[a, b, c] \in G$ such that $u = (a, b, c)$, and that for each generalized trapezoidal fuzzy number u, there exists a unique $[a, b, c, d] \in T$ such that $u = (a, b, c, d)$. As corollaries, we affirmatively answer the representation uniqueness questions for triangular and trapezoidal fuzzy numbers, providing enhanced versions of these results.

Furthermore, we present several equivalent forms of some conclusions regarding representation uniqueness. Based on these equivalent forms and fundamental properties of Tag, Tap, Trag, and Trap, we establish relationships among these four families of fuzzy sets. We also present properties of Tag, Tap, Trag, and Trap without relying on representation uniqueness results. The conclusions of this paper contribute to research on triangular and trapezoidal fuzzy numbers.

The remainder of this paper is organized as follows. Section 2 reviews fundamental concepts related to fuzzy sets, triangular fuzzy numbers, and trapezoidal fuzzy numbers, and establishes basic properties of the latter two. In Section 3, we introduce generalized triangular and trapezoidal fuzzy numbers and examine properties of the four families Tag, Trag, Tap, and Trap. Section 4 presents representation uniqueness results for Tag, Trag, Tap, and Trap, provides several equivalent forms of some conclusions, and establishes relationships among these sets. Section 5 explores relationships between Tag and Tap, and between Trag and Trap. Finally, Section 6 concludes the paper.

2. Fuzzy Sets, Triangular Fuzzy Numbers, and Trapezoidal Fuzzy Numbers

This section reviews fundamental concepts related to fuzzy sets, triangular fuzzy numbers, and trapezoidal fuzzy numbers, and establishes basic properties of the latter two. For fuzzy theory and applications, we refer readers to \cite{1–13}.

Let Y be a nonempty set. The symbol $\mathcal{P}(Y)$ denotes the power set of Y, i.e., the set of all subsets of Y. The symbol $\mathcal{F}(Y)$ denotes the set of all fuzzy sets in Y, i.e., functions from Y to $[0,1]$. Given $u \in \mathcal{F}(Y)$ and $\alpha \in (0,1]$, the α-cut $[u]_\alpha$ of u is defined by $[u]_\alpha := \{x \in Y : u(x) \geq \alpha\}$.

Let Y be a topological space. The symbol $C(Y)$ denotes the set of all nonempty closed subsets of Y, and $K(Y)$ denotes the set of all nonempty compact subsets of Y. For $u \in \mathcal{F}(Y)$, the 0-cut $[u]_0$ of u is defined by $[u]_0 := \overline{\{x \in Y : u(x) > 0\}}$, where $\overline{S}$ denotes the topological closure of S in Y. The set $[u]_0$ is called the support of u, also denoted by $\operatorname{supp} u$.

Properties of distances on fuzzy sets are discussed in \cite{14–17}.

Definition 2.1. We use Tag to denote the set of all triangular fuzzy numbers:

$$\text{Tag} := \{(a,b,c) : [a,b,c] \in G_0\},$$

where for any $[a,b,c] \in G_0$, the triangular fuzzy number (a,b,c) is defined as the fuzzy set $u \in \mathcal{F}(\mathbb{R})$ given by

$$u(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x \leq b, \\ \frac{c-x}{c-b}, & \text{if } b \leq x \leq c, \\ 0, & \text{if } x \in \mathbb{R} \quad [a,c]. \end{cases}$$

Definition 2.2. We use Tap to denote the set of all trapezoidal fuzzy numbers:

$$\text{Tap} := \{(a,b,c,d) : [a,b,c,d] \in T_0\},$$

where for any $[a,b,c,d] \in T_0$, the trapezoidal fuzzy number (a,b,c,d) is defined as the fuzzy set $u \in \mathcal{F}(\mathbb{R})$ given by

$$u(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \le x \le b, \\ 1, & \text{if } b \le x \le c, \\ \frac{d-x}{d-c}, & \text{if } c \le x \le d, \\ 0, & \text{if } x \in \mathbb{R} \quad [a,d]. \end{cases}$$

Remark 2.3. (i) (a) Tag $= \{(a,b,c) : [a,b,c] \in G_0\}$. (b) Both the statement "if $u \in$ Tag, then there exists $[a,b,c] \in G_0$ satisfying $u = (a,b,c)$" and its converse are true. ($u \in \{(a,b,c) : [a,b,c] \in G_0\}$ means there exists $[a,b,c] \in G_0$ such that $u = (a,b,c)$.) Clearly (a) (b) and (b) (a). Since (a) is known, (b) holds. (ii) (a) Tap $= \{(a,b,c,d) : [a,b,c,d] \in T_0\}$. (b) Both the statement "if $u \in$ Tap, then there exists $[a,b,c,d] \in T_0$ satisfying $u = (a,b,c,d)$" and its converse are true. ($u \in \{(a,b,c,d) : [a,b,c,d] \in T_0\}$ means there exists $[a,b,c,d] \in T_0$ such that $u = (a,b,c,d)$.) Clearly (a) (b) and (b) (a). Since (a) is known, (b) holds.

The statements in this remark are quite obvious and will be used without citation. If combining a statement (c) with a statement in this remark yields a statement (d), we will say (c) implies (d).

We say that two fuzzy sets are equal if they have the same membership function.

Remark 2.4. (i) For each $a,b,c \in \mathbb{R}$, $(a,b,c) \in$ Tag if and only if $(a,b,b,c) \in$ Tap. (ii) Each triangular fuzzy number (a,b,c) is the trapezoidal fuzzy number (a,b,b,c). (iii) Tag $\subseteq$ Tap. (iv) We can also define Tag based on Tap as follows:

Tag is the set of all triangular fuzzy numbers. Tag $:= \{(a,b,c) : [a,b,c] \in G_0\}$, where for any $[a,b,c] \in G_0$, the triangular fuzzy number (a,b,c) is defined as the (a,b,b,c) in Tap.

(v) Each trapezoidal fuzzy number (a,b,c,d) with $b = c$ is the triangular fuzzy number (a,b,d).

We provide a routine proof of (i). Since Tag $= \{(a',b',c') : [a',b',c'] \in G_0\}$, we have (i1) for each $a,b,c \in \mathbb{R}$, $(a,b,c) \in \text{Tag} \iff [a,b,c] \in G_0$. Since Tap $= \{(a',b',c',d') : [a',b',c',d'] \in T_0\}$, we have (i2) for each $a,b,c \in \mathbb{R}$, $(a,b,b,c) \in \text{Tap} \iff [a,b,b,c] \in T_0$. Clearly, for each $a,b,c \in \mathbb{R}$, $[a,b,c] \in G_0$ if and only if $[a,b,b,c] \in T_0$. By (i1) and (i2), (i) holds.

Given $(a,b,c) \in$ Tag, by (i) we have $(a,b,b,c) \in$ Tap. By Definitions 2.1 and 2.2, (a,b,c) and (a,b,b,c) have the same membership function. Thus $(a,b,c) = (a,b,b,c)$, so (ii) holds. (iii) and (iv) follow immediately from (ii).

We show (v). Given $(a,b,c,d) \in$ Tap with $b = c$, by (i) we have $(a,b,d) \in$ Tag (see also (I) below). Then by (ii), $(a,b,d) = (a,b,b,d)$. So (v) holds.

We believe it is acceptable not to point out contents such as those mentioned in clauses (I) and (II), because they are easy to see. (I) $(a, b, c) \in$ Tag or $(a, b, b, c) \in$ Tap implies $a, b, c \in \mathbb{R}$. So (i) can be stated as: $(a, b, c) \in$ Tag if and only if $(a, b, b, c) \in$ Tap. (II) (ii) implies that if $(a, b, c) \in$ Tag then $(a, b, b, c) \in$ Tap. (v) implies that if $(a, b, b, d) \in$ Tap then $(a, b, d) \in$ Tag. So (ii) and (v) holding implies (i) holds. The above proof of (v) indicates that (i) and (ii) holding implies (v) holds. Below we show that (i) and (v) holding implies (ii) holds.

Given $(a, b, c) \in$ Tag, by (i) we have $(a, b, b, c) \in$ Tap (see also (I)). Then by (v), $(a, b, b, c) = (a, b, c)$. So (ii) holds.

3. Generalized Triangular Fuzzy Numbers and Generalized Trapezoidal Fuzzy Numbers

This section introduces generalized triangular fuzzy numbers and generalized trapezoidal fuzzy numbers, and presents properties of the four families of fuzzy sets Tag, Trag, Tap, and Trap, including relationships among them.

Definition 3.1. We use Trag to denote the set of all generalized triangular fuzzy numbers:

$$\text{Trag} := \{(a, b, c) : [a, b, c] \in G\},$$

where for any $[a, b, c] \in G$, the generalized triangular fuzzy number (a, b, c) is defined as the fuzzy set $u \in \mathcal{F}(\mathbb{R})$ in the following way: u is the triangular fuzzy number (a, b, c) when $a < b < c$;

$$u(x) = \begin{cases} \frac{c-x}{c-b}, & \text{if } b \leq x \leq c, \\ 0, & \text{if } x \in \mathbb{R}\ \ [b, c], \end{cases} \quad \text{when } a = b < c;$$

$$u(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x \leq b, \\ 0, & \text{if } x \in \mathbb{R}\ \ [a, b], \end{cases} \quad \text{when } a < b = c;$$

$$u(x) = \begin{cases} 1, & \text{if } x = b, \\ 0, & \text{if } x \in \mathbb{R}\ \ \{b\}, \end{cases} \quad \text{when } a = b = c.$$

We note that (a) Tag $= \{(a, b, c) : [a, b, c] \in G_0\}$, Trag $= \{(a, b, c) : [a, b, c] \in G\}$, and $G_0 \subset G$. In Definition 3.1, for any $[a, b, c] \in G$, we define (a, b, c) in four different cases. The case when $a < b < c$ can be written as the case when $[a, b, c] \in G_0$ because for any $[a, b, c] \in G$, $a < b < c$ if and only if $[a, b, c] \in G_0$. (In fact, for any $[a, b, c] \in \mathbb{R}^3$, $a < b < c \iff [a, b, c] \in G_0$.) Thus we have (b) for each $[a, b, c] \in G_0$, by Definition 3.1, $(a, b, c) \in$ Trag is just the $(a, b, c) \in$ Tag. So by (a) and (b), the concept of generalized triangular fuzzy numbers is a generalization of the concept of triangular fuzzy numbers. Hence Tag $\subseteq$ Trag.

Definition 3.2. We use Trap to denote the set of all generalized trapezoidal fuzzy numbers:

$$\text{Trap} := \{(a, b, c, d) : [a, b, c, d] \in T\},$$

where for any $[a,b,c,d] \in T$, the generalized trapezoidal fuzzy number (a,b,c,d) is defined as the fuzzy set $u \in \mathcal{F}(\mathbb{R})$ in the following way: u is the trapezoidal fuzzy number (a,b,c,d) when $a < b \le c < d$;

$$u(x) = \begin{cases} 1, & \text{if } b \le x \le c, \\ \frac{d-x}{d-c}, & \text{if } c \le x \le d, \\ 0, & \text{if } x \in \mathbb{R} \quad [b,d], \end{cases} \qquad \text{when } a = b \le c < d;$$

$$u(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \le x \le b, \\ 1, & \text{if } b \le x \le c, \\ 0, & \text{if } x \in \mathbb{R} \quad [a,c], \end{cases} \qquad \text{when } a < b \le c = d;$$

$$u(x) = \begin{cases} 1, & \text{if } b \le x \le c, \\ 0, & \text{if } x \in \mathbb{R} \quad [b,c], \end{cases} \qquad \text{when } a = b \le c = d.$$

We note that (a) Tap $= \{(a,b,c,d) : [a,b,c,d] \in T_0\}$, Trap $= \{(a,b,c,d) : [a,b,c,d] \in T\}$, and $T_0 \subset T$. In Definition 3.2, for any $[a,b,c,d] \in T$, we define (a,b,c,d) in four different cases. The case when $a < b \le c < d$ can be written as the case when $[a,b,c,d] \in T_0$ because for any $[a,b,c,d] \in T$, $a < b \le c < d$ if and only if $[a,b,c,d] \in T_0$. (In fact, for any $[a,b,c,d] \in \mathbb{R}^4$, $a < b \le c < d \iff [a,b,c,d] \in T_0$.) Thus we have (b) for each $[a,b,c,d] \in T_0$, by Definition 3.2, $(a,b,c,d) \in$ Trap is just the $(a,b,c,d) \in$ Tap. So the concept of generalized trapezoidal fuzzy numbers is a generalization of the concept of trapezoidal fuzzy numbers. Hence Tap $\subseteq$ Trap.

Remark 3.3. (i) (a) Trag $= \{(a,b,c) : [a,b,c] \in G\}$. (b) Both the statement "if $u \in$ Trag, then there exists $[a,b,c] \in G$ satisfying $u = (a,b,c)$" and its converse are true. ($u \in \{(a,b,c) : [a,b,c] \in G\}$ means there exists $[a,b,c] \in G$ such that $u = (a,b,c)$.) Clearly (a) (b) and (b) (a). Since (a) is known, (b) holds. (ii) (a) Trap $= \{(a,b,c,d) : [a,b,c,d] \in T\}$. (b) Both the statement "if $u \in$ Trap, then there exists $[a,b,c,d] \in T$ satisfying $u = (a,b,c,d)$" and its converse are true. ($u \in \{(a,b,c,d) : [a,b,c,d] \in T\}$ means there exists $[a,b,c,d] \in T$ such that $u = (a,b,c,d)$.) Clearly (a) (b) and (b) (a). Since (a) is known, (b) holds.

The statements in this remark are quite obvious and will be used without citation. If combining a statement (c) with a statement in this remark yields a statement (d), we will say (c) implies (d).

Remark 3.4. (i) For each $a,b,c \in \mathbb{R}$, $(a,b,c) \in$ Trag if and only if $(a,b,b,c) \in$ Trap. (ii) Each generalized triangular fuzzy number (a,b,c) is the generalized trapezoidal fuzzy number (a,b,b,c). (iii) Trag $\subseteq$ Trap. (iv) We can also define Trag based on Trap as follows:

Trag is the set of all generalized triangular fuzzy numbers. Trag $:= \{(a,b,c) : [a,b,c] \in G\}$, where for any $[a,b,c] \in G$, the generalized triangular fuzzy number (a,b,c) is defined to be the (a,b,b,c) in Trap.

(v) Each generalized trapezoidal fuzzy number (a, b, c, d) with $b = c$ is the generalized triangular fuzzy number (a, b, d).

We provide a routine proof of (i). Since Trag $= \{(a', b', c') : [a', b', c'] \in G\}$, we have (i1) for each $a, b, c \in \mathbb{R}$, $(a, b, c) \in$ Trag $\iff$ $[a, b, c] \in G$. Since Trap $= \{(a', b', c', d') : [a', b', c', d'] \in T\}$, we have (i2) for each $a, b, c \in \mathbb{R}$, $(a, b, b, c) \in$ Trap $\iff$ $[a, b, b, c] \in T$. Clearly, for each $a, b, c \in \mathbb{R}$, $[a, b, c] \in G$ if and only if $[a, b, b, c] \in T$. By (i1) and (i2), (i) holds.

Given $(a, b, c) \in$ Trag, by (i) we have $(a, b, b, c) \in$ Trap. By Definitions 3.1 and 3.2, (a, b, c) and (a, b, b, c) have the same membership function in the four cases $a < b < c$, $a = b < c$, $a < b = c$, and $a = b = c$. Thus $(a, b, c) = (a, b, b, c)$, so (ii) holds. (iii) and (iv) follow immediately from (ii).

We show (v). Given $(a, b, c, d) \in$ Trap with $b = c$, by (i) we have $(a, b, d) \in$ Trag (see also (I) below). Then by (ii), $(a, b, d) = (a, b, b, d)$. So (v) holds.

We believe it is acceptable not to point out contents such as those mentioned in clauses (I) and (II), because they are easy to see. (I) $(a, b, c) \in$ Trag or $(a, b, b, c) \in$ Trap implies $a, b, c \in \mathbb{R}$. So (i) can be stated as: $(a, b, c) \in$ Trag if and only if $(a, b, b, c) \in$ Trap. (II) (ii) implies that if $(a, b, c) \in$ Trag then $(a, b, b, c) \in$ Trap. (v) implies that if $(a, b, b, d) \in$ Trap then $(a, b, d) \in$ Trag. So (ii) and (v) holding implies (i) holds. The above proof of (v) indicates that (i) and (ii) holding implies (v) holds. Below we show that (i) and (v) holding implies (ii) holds.

Given $(a, b, c) \in$ Trag, by (i) we have $(a, b, b, c) \in$ Trap (see also (I)). Then by (v), $(a, b, b, c) = (a, b, c)$. So (ii) holds.

4. Representation Uniqueness of Generalized Triangular and Generalized Trapezoidal Fuzzy Numbers

This section establishes representation uniqueness for Tag, Trag, Tap, and Trap. We provide several equivalent forms of some conclusions and establish relationships among these sets.

For any $[a, b, c, d]$ and $[a_1, b_1, c_1, d_1]$ in $\mathbb{R}^4$, $[a, b, c, d] = [a_1, b_1, c_1, d_1]$ means $a = a_1$, $b = b_1$, $c = c_1$, and $d = d_1$. For any (a, b, c, d) and (a_1, b_1, c_1, d_1) in Trap, $(a, b, c, d) = (a_1, b_1, c_1, d_1)$ means (a, b, c, d) and (a_1, b_1, c_1, d_1) are the same fuzzy set.

For any $[a, b, c]$ and $[a_1, b_1, c_1]$ in $\mathbb{R}^3$, $[a, b, c] = [a_1, b_1, c_1]$ means $a = a_1$, $b = b_1$, and $c = c_1$. For any (a, b, c) and (a_1, b_1, c_1) in Trag, $(a, b, c) = (a_1, b_1, c_1)$ means (a, b, c) and (a_1, b_1, c_1) are the same fuzzy set.

Theorem 4.1(ii) gives the representation uniqueness of generalized trapezoidal fuzzy numbers. Theorem 4.1(iv) gives the representation uniqueness of trapezoidal fuzzy numbers.

Theorem 4.1. (i) Let $u = (a, b, c, d) \in$ Trap. Then $[u]_0 = [a, d]$ and $[u]_1 = [b, c]$. (ii) Let (a, b, c, d) and $(a_1, b_1, c_1, d_1) \in$ Trap. Then $(a, b, c, d) = (a_1, b_1, c_1, d_1)$ if and only if $[a, b, c, d] = [a_1, b_1, c_1, d_1]$. (iii) Let $u = (a, b, c, d) \in$ Tap. Then $[u]_0 = [a, d]$ and $[u]_1 = [b, c]$. (iv) Let (a, b, c, d) and $(a_1, b_1, c_1, d_1) \in$ Tap. Then $(a, b, c, d) = (a_1, b_1, c_1, d_1)$ if and only if $[a, b, c, d] = [a_1, b_1, c_1, d_1]$.

Proof. By Definition 3.2 and straightforward calculations, we obtain (i). (One way to perform these calculations is based on observing the graphs of the membership functions of (a, b, c, d) in the four cases $a < b \le c < d$, $a = b \le c < d$, $a < b \le c = d$, and $a = b \le c = d$.) Now we show (ii). If $[a, b, c, d] = [a_1, b_1, c_1, d_1]$, i.e., $a = a_1$, $b = b_1$, $c = c_1$, and $d = d_1$, then by Definition 3.2, $(a, b, c, d) = (a_1, b_1, c_1, d_1)$.

Suppose $(a, b, c, d) = (a_1, b_1, c_1, d_1)$. Then $[(a, b, c, d)]_1 = [(a_1, b_1, c_1, d_1)]_1$ and $[(a, b, c, d)]_0 = [(a_1, b_1, c_1, d_1)]_0$. By (i), this means $[b, c] = [b_1, c_1]$ and $[a, d] = [a_1, d_1]$, which is equivalent to $a = a_1$, $b = b_1$, $c = c_1$, and $d = d_1$; that is, $[a, b, c, d] = [a_1, b_1, c_1, d_1]$. So (ii) is proved.

Since Tap is a subset of Trap, (iii) follows immediately from (i), and (iv) follows immediately from (ii). (iii) is easy and should be known.

Proposition 4.2(ii) gives the representation uniqueness of generalized triangular fuzzy numbers. Proposition 4.2(iv) gives the representation uniqueness of triangular fuzzy numbers.

Proposition 4.2. (i) Let $u = (a, b, c) \in$ Trag. Then $[u]_0 = [a, c]$ and $[u]_1 = \{b\}$. (ii) Let (a, b, c) and $(a_1, b_1, c_1) \in$ Trag. Then $(a, b, c) = (a_1, b_1, c_1)$ if and only if $[a, b, c] = [a_1, b_1, c_1]$. (iii) Let $u = (a, b, c) \in$ Tag. Then $[u]_0 = [a, c]$ and $[u]_1 = \{b\}$. (iv) Let (a, b, c) and $(a_1, b_1, c_1) \in$ Tag. Then $(a, b, c) = (a_1, b_1, c_1)$ if and only if $[a, b, c] = [a_1, b_1, c_1]$.

Proof. By Definition 3.1 and straightforward calculations, we obtain (i). (One way to perform these calculations is based on observing the graphs of the membership functions of (a, b, c) in the four cases $a < b < c$, $a = b < c$, $a < b = c$, and $a = b = c$.) Now we show (ii). If $[a, b, c] = [a_1, b_1, c_1]$, i.e., $a = a_1$, $b = b_1$, and $c = c_1$, then by Definition 3.1, $(a, b, c) = (a_1, b_1, c_1)$.

Suppose $(a, b, c) = (a_1, b_1, c_1)$. Then $[(a, b, c)]_1 = [(a_1, b_1, c_1)]_1$ and $[(a, b, c)]_0 = [(a_1, b_1, c_1)]_0$. By (i), this means $\{b\} = \{b_1\}$ and $[a, c] = [a_1, c_1]$, which is equivalent to $a = a_1$, $b = b_1$, and $c = c_1$; that is, $[a, b, c] = [a_1, b_1, c_1]$. So (ii) is proved.

Since Tag is a subset of Trag, (iii) follows immediately from (i), and (iv) follows immediately from (ii). (iii) is easy and should be known.

The above proofs of Theorem 4.1 and Proposition 4.2 are similar.

Remark 4.3. Proposition 4.2 is a corollary of Theorem 4.1. This is because for $k = i, ii, iii, iv$, Proposition 4.2(k) is a corollary of Theorem 4.1(k). "Theorem 4.1(ii) Proposition 4.2(ii)." Assume Theorem 4.1(ii) holds. Let $(a, b, c) \in$ Trag.

Then $[(a,b,c)]_0 = [(a,b,b,c)]_0 = [a,c]$ (by Remark 3.4(ii), (a,b,c) is the (a,b,b,c) in Trap, so the first equality holds; by Theorem 4.1(i), the second equality holds), and $[(a,b,c)]_1 = [(a,b,b,c)]_1 = [b,b] = \{b\}$ (by Remark 3.4(ii), (a,b,c) is the (a,b,b,c) in Trap, so the first equality holds; by Theorem 4.1(i), the second equality holds). So Proposition 4.2(i) holds. "Theorem 4.1(ii) Proposition 4.2(ii)." Assume Theorem 4.1(ii) holds. Let (a,b,c) and $(a_1,b_1,c_1) \in$ Trag. By Remark 3.4(ii), (a,b,c) is the (a,b,b,c) in Trap and (a_1,b_1,c_1) is the (a_1,b_1,b_1,c_1) in Trap. Then $(a,b,c) = (a_1,b_1,c_1)$ means $(a,b,b,c) = (a_1,b_1,b_1,c_1)$. By Theorem 4.1(ii), $(a,b,b,c) = (a_1,b_1,b_1,c_1)$ means $[a,b,b,c] = [a_1,b_1,b_1,c_1]$. Clearly $[a,b,b,c] = [a_1,b_1,b_1,c_1]$ means $[a,b,c] = [a_1,b_1,c_1]$ (see also (I) below). From this, it follows that $(a,b,c) = (a_1,b_1,c_1)$ if and only if $[a,b,c] = [a_1,b_1,c_1]$. So Proposition 4.2(ii) holds. "Theorem 4.1(iii) Proposition 4.2(iii)." Assume Theorem 4.1(iii) holds. Let $(a,b,c) \in$ Tag. Then $[(a,b,c)]_0 = [(a,b,b,c)]_0 = [a,c]$ (by Remark 2.4(ii), (a,b,c) is the (a,b,b,c) in Tap, so the first equality holds; by Theorem 4.1(iii), the second equality holds), and $[(a,b,c)]_1 = [(a,b,b,c)]_1 = [b,b] = \{b\}$ (by Remark 2.4(ii), (a,b,c) is the (a,b,b,c) in Tap, so the first equality holds; by Theorem 4.1(iii), the second equality holds). So Proposition 4.2(iii) holds. "Theorem 4.1(iv) Proposition 4.2(iv)." Assume Theorem 4.1(iv) holds. Let (a,b,c) and $(a_1,b_1,c_1) \in$ Tag. By Remark 2.4(ii), (a,b,c) is the (a,b,b,c) in Tap and (a_1,b_1,c_1) is the (a_1,b_1,b_1,c_1) in Tap. Then $(a,b,c) = (a_1,b_1,c_1)$ means $(a,b,b,c) = (a_1,b_1,b_1,c_1)$. By Theorem 4.1(iv), $(a,b,b,c) = (a_1,b_1,b_1,c_1)$ means $[a,b,b,c] = [a_1,b_1,b_1,c_1]$. Clearly $[a,b,b,c] = [a_1,b_1,b_1,c_1]$ means $[a,b,c] = [a_1,b_1,c_1]$ (see also (I-I) below). From this, it follows that $(a,b,c) = (a_1,b_1,c_1)$ if and only if $[a,b,c] = [a_1,b_1,c_1]$. So Proposition 4.2(iv) holds.

The contents in (I) and (II) below are easy to see. (I) (α) For each $l,m,n,l_1,m_1,n_1 \in \mathbb{R}$, the conditions (I-1) $[l,m,m,n] = [l_1,m_1,m_1,n_1]$, (I-2) $l = l_1$, $m = m_1$, and $n = n_1$, and (I-3) $[l,m,n] = [l_1,m_1,n_1]$, are equivalent. Clearly (I-1) (I-2) and (I-2) (I-3). This means (I-1) (I-2) (I-3). So (α) holds. That $a,b,c,a_1,b_1,c_1 \in \mathbb{R}$ is implicit in the fact that (a,b,c) and (a_1,b_1,c_1) are in Trag. So by (α), $[a,b,b,c] = [a_1,b_1,b_1,c_1]$ means $[a,b,c] = [a_1,b_1,c_1]$. (II) That $a,b,c,a_1,b_1,c_1 \in \mathbb{R}$ is implicit in the fact that (a,b,c) and (a_1,b_1,c_1) are in Tag. So by (α), $[a,b,b,c] = [a_1,b_1,b_1,c_1]$ means $[a,b,c] = [a_1,b_1,c_1]$.

Remark 4.4. (a) For each $u \in$ Trap, $S(u,T)$ is a singleton set, where $S(u,T) := \{[a,b,c,d] \in T : u = (a,b,c,d)\}$. (b) For each $u \in$ Trag, $S(u,G)$ is a singleton set, where $S(u,G) := \{[a,b,c] \in G : u = (a,b,c)\}$.

Clearly (a) means the following $(\bar{a})$: $(\bar{a})$ For each $u \in$ Trap, there exists $[a,b,c,d]$ which is the unique element of T satisfying $u = (a,b,c,d)$.

Clearly (b) means the following $(\bar{b})$: $(\bar{b})$ For each $u \in$ Trag, there exists $[a,b,c]$ which is the unique element of G satisfying $u = (a,b,c)$.

Consider statements (a') Theorem 4.1(ii), and (b') Proposition 4.2(ii). It is easy to see that $(a) \Leftrightarrow (a')$ and $(b) \Leftrightarrow (b')$ (see the proof below). Since (a') and (b')

hold, it follows that (a) and (b) hold; that is, $(\bar{a})$ and $(\bar{b})$ hold.

Assume (a') holds. Let $u \in$ Trap. Then there exists $[a,b,c,d] \in T$ satisfying $u = (a,b,c,d)$. If there is any $[a_1,b_1,c_1,d_1] \in T$ satisfying $u = (a_1,b_1,c_1,d_1)$, then (a,b,c,d) and (a_1,b_1,c_1,d_1) are in Trap and equal, and hence by (a'), $[a_1,b_1,c_1,d_1] = [a,b,c,d]$. So (a) holds.

Conversely, assume (a) holds. The "if" part of (a') is obvious. A routine proof of this part is given in the proof of Theorem 4.1. The "only if" part of (a') follows immediately from (a). A routine proof of this part is given as follows. Let (a,b,c,d) and (a_1,b_1,c_1,d_1) be in Trap with $(a,b,c,d) = (a_1,b_1,c_1,d_1)$. Then $[a,b,c,d]$ and $[a_1,b_1,c_1,d_1]$ are in T (in this paper, (e,f,g,h) is well-defined only when $[e,f,g,h] \in T$). Thus by (a), $[a,b,c,d] = [a_1,b_1,c_1,d_1]$. Then the "only if" part of (a') is true. Hence (a') holds.

So $(a) \Leftrightarrow (a')$. Assume (b') holds. Let $u \in$ Trag. Then there exists $[a,b,c] \in G$ satisfying $u = (a,b,c)$. If there is any $[a_1,b_1,c_1] \in G$ satisfying $u = (a_1,b_1,c_1)$, then (a,b,c) and (a_1,b_1,c_1) are in Trag and equal, and hence by (b'), $[a_1,b_1,c_1] = [a,b,c]$. So (b) holds.

Conversely, assume (b) holds. The "if" part of (b') is obvious. A routine proof of this part is given in the proof of Proposition 4.2. The "only if" part of (b') follows immediately from (b). A routine proof of this part is given as follows. Let (a,b,c) and (a_1,b_1,c_1) be in Trag with $(a,b,c) = (a_1,b_1,c_1)$. Then $[a,b,c]$ and $[a_1,b_1,c_1]$ are in G (in this paper, (e,f,g) is well-defined only when $[e,f,g] \in G$). Thus by (b), $[a,b,c] = [a_1,b_1,c_1]$. Then the "only if" part of (b') is true. Hence (b') holds.

So $(b) \Leftrightarrow (b')$. The above proofs of $(a) \Leftrightarrow (a')$ and $(b) \Leftrightarrow (b')$ are similar.

Remark 4.5. (a) For each $u \in$ Tap, $S(u,T)$ is a singleton subset of T_0, where $S(u,T) := \{[a,b,c,d] \in T : u = (a,b,c,d)\}$. (b) For each $u \in$ Tag, $S(u,G)$ is a singleton subset of G_0, where $S(u,G) := \{[a,b,c] \in G : u = (a,b,c)\}$.

Clearly (a) means the following $(\bar{a})$: $(\bar{a})$ For each $u \in$ Tap, there exists $[a,b,c,d] \in T_0$ which is the unique element of T satisfying $u = (a,b,c,d)$.

Clearly (b) means the following $(\bar{b})$: $(\bar{b})$ For each $u \in$ Tag, there exists $[a,b,c] \in G_0$ which is the unique element of G satisfying $u = (a,b,c)$.

We show $(\bar{a})$. Let $u \in$ Tap. Then there exists $[a,b,c,d] \in T_0$ satisfying $u = (a,b,c,d)$. Note that $u \in$ Trap and $[a,b,c,d] \in T$. By Remark 4.4$(\bar{a})$, this $[a,b,c,d] \in T_0$ is the unique element of T satisfying $u = (a,b,c,d)$. So $(\bar{a})$ holds.

We show $(\bar{b})$. Let $u \in$ Tag. Then there exists $[a,b,c] \in G_0$ satisfying $u = (a,b,c)$. Note that $u \in$ Trag and $[a,b,c] \in G$. By Remark 4.4$(\bar{b})$, this $[a,b,c] \in G_0$ is the unique element of G satisfying $u = (a,b,c)$. So $(\bar{b})$ holds.

Obviously $(\bar{a})$ and $(\bar{b})$ holding means (a) and (b) hold.

From the above proof of $(\bar{a})$, we can see that $(\bar{a})$ is a corollary of Remark 4.4$(\bar{a})$. This means (a) is a corollary of Remark 4.4(a).

From the above proof of $(\bar{b})$, we can see that $(\bar{b})$ is a corollary of Remark 4.4$(\bar{b})$. This means (b) is a corollary of Remark 4.4(b).

Below we show that Theorem 4.1(iv) $(\bar{a})$. Assume Theorem 4.1(iv) holds. Let $u \in$ Tap. Then there exists $[a, b, c, d] \in T_0$ satisfying $u = (a, b, c, d)$. If there is any $[a_1, b_1, c_1, d_1] \in T$ satisfying $u = (a_1, b_1, c_1, d_1)$, then $(a, b, c, d) = u = (a_1, b_1, c_1, d_1) \in$ Tap, and hence by Theorem 4.1(iv), $[a_1, b_1, c_1, d_1] = [a, b, c, d]$. Thus $(\bar{a})$ holds.

Assume $(\bar{a})$ holds. Let (a, b, c, d) and (a_1, b_1, c_1, d_1) be in Tap. If $[a, b, c, d] = [a_1, b_1, c_1, d_1]$, i.e., $a = a_1$, $b = b_1$, $c = c_1$, and $d = d_1$, then by Definition 2.2, $(a, b, c, d) = (a_1, b_1, c_1, d_1)$. Suppose $(a, b, c, d) = (a_1, b_1, c_1, d_1)$. Then $[a, b, c, d]$ and $[a_1, b_1, c_1, d_1]$ are in T (in this paper, (e, f, g, h) is well-defined only when $[e, f, g, h] \in T$). Thus by $(\bar{a})$, $[a, b, c, d] = [a_1, b_1, c_1, d_1]$. Hence Theorem 4.1(iv) holds.

So Theorem 4.1(iv) $(\bar{a})$. Below we show that Proposition 4.2(iv) $(\bar{b})$.

Assume Proposition 4.2(iv) holds. Let $u \in$ Tag. Then there exists $[a, b, c] \in G_0$ satisfying $u = (a, b, c)$. If there is any $[a_1, b_1, c_1] \in G$ satisfying $u = (a_1, b_1, c_1)$, then $(a, b, c) = u = (a_1, b_1, c_1) \in$ Tag, and hence by Proposition 4.2(iv), $[a_1, b_1, c_1] = [a, b, c]$. Thus $(\bar{b})$ holds.

Conversely, assume $(\bar{b})$ holds. Let (a, b, c) and (a_1, b_1, c_1) be in Tag. If $[a, b, c] = [a_1, b_1, c_1]$, i.e., $a = a_1$, $b = b_1$, and $c = c_1$, then by Definition 2.1, $(a, b, c) = (a_1, b_1, c_1)$. Suppose $(a, b, c) = (a_1, b_1, c_1)$. Then $[a, b, c]$ and $[a_1, b_1, c_1]$ are in G (in this paper, (e, f, g) is well-defined only when $[e, f, g] \in G$). Thus by $(\bar{b})$, $[a, b, c] = [a_1, b_1, c_1]$. Hence Proposition 4.2(iv) holds.

So Proposition 4.2(iv) $(\bar{b})$.

Remark 4.6. (a) For each $u \in$ Tap, $S(u, T_0)$ is a singleton set, where $S(u, T_0) := \{[a, b, c, d] \in T_0 : u = (a, b, c, d)\}$. (b) For each $u \in$ Tag, $S(u, G_0)$ is a singleton set, where $S(u, G_0) := \{[a, b, c] \in G_0 : u = (a, b, c)\}$.

Clearly (a) means the following $(\bar{a})$: $(\bar{a})$ For each $u \in$ Tap, there exists $[a, b, c, d]$ which is the unique element of T_0 satisfying $u = (a, b, c, d)$.

Clearly (b) means the following $(\bar{b})$: $(\bar{b})$ For each $u \in$ Tag, there exists $[a, b, c]$ which is the unique element of G_0 satisfying $u = (a, b, c)$.

Obviously, we can also state $(\bar{a})$ and $(\bar{b})$ as follows: $(\bar{a})$ For each $u \in$ Tap, there exists $[a, b, c, d] \in T_0$ which is the unique element of T_0 satisfying $u = (a, b, c, d)$. $(\bar{b})$ For each $u \in$ Tag, there exists $[a, b, c] \in G_0$ which is the unique element of G_0 satisfying $u = (a, b, c)$.

Based on these descriptions of $(\bar{a})$ and $(\bar{b})$ or directly, we can see that Remark 4.5$(\bar{a})$ implies $(\bar{a})$ and Remark 4.5$(\bar{b})$ implies $(\bar{b})$. Note that Remark 4.5$(\bar{a})$ and

Remark 4.5($\bar{b}$) are proved. So ($\bar{a}$) and ($\bar{b}$) hold. In other words, (a) and (b) hold.

From the above proof of ($\bar{a}$), we can see that ($\bar{a}$) is a corollary of Remark 4.5($\bar{a}$). This means (a) is a corollary of Remark 4.5(a).

From the above proof of ($\bar{b}$), we can see that ($\bar{b}$) is a corollary of Remark 4.5($\bar{b}$). This means (b) is a corollary of Remark 4.5(b).

Below we give a routine proof of ($\bar{a}$) $\Rightarrow$ ($\bar{b}$). Assume ($\bar{a}$) holds. Let $u \in$ Tag. Then there exists $[a, b, c] \in G_0$ satisfying $u = (a, b, c)$. Suppose there exists $[a_1, b_1, c_1] \in G_0$ satisfying $u = (a_1, b_1, c_1)$. Note that $u \in$ Tag $\subset$ Trap, that both $[a, b, b, c]$ and $[a_1, b_1, b_1, c_1]$ are in T_0 (see also (I) below), and that by Remark 2.4(ii), $(a, b, b, c) = u = (a_1, b_1, b_1, c_1)$. Thus by ($\bar{a}$), $[a, b, b, c] = [a_1, b_1, b_1, c_1]$. This means $[a, b, c] = [a_1, b_1, c_1]$ (see also (α) in Remark 4.3). So ($\bar{b}$) holds. (I) (α) (α-1) $[e, f, g] \in G_0$ if and only if $[e, f, f, g] \in T_0$; (α-2) $[e, f, g] \in G$ if and only if $[e, f, f, g] \in T$. Clearly (α) holds. Since $[a, b, c]$ and $[a_1, b_1, c_1]$ are in G_0, by (α-1), $[a, b, b, c]$ and $[a_1, b_1, b_1, c_1]$ are in T_0.

We call both Remark 4.5($\bar{a}$) and Remark 4.6($\bar{a}$) the representation uniqueness of trapezoidal fuzzy numbers, although Remark 4.5($\bar{a}$) is an enhanced version of Remark 4.6($\bar{a}$).

We call both Remark 4.5($\bar{b}$) and Remark 4.6($\bar{b}$) the representation uniqueness of triangular fuzzy numbers, although Remark 4.5($\bar{b}$) is an enhanced version of Remark 4.6($\bar{b}$).

We mention that several equivalent forms of Theorem 4.1(ii) are given in Remark 4.7.

Remark 4.7. We claim the following statements. (a) Let $(a, b, c, d) \in$ Trap and let A be a subset of T. Define $S := \{(a_1, b_1, c_1, d_1) : [a_1, b_1, c_1, d_1] \in A\}$. Then (a-1) $(a, b, c, d) \in S$ if and only if $[a, b, c, d] \in A$; (a-2) $(a, b, c, d) \notin S$ if and only if $[a, b, c, d] \notin A$. (b) Let $(a, b, c, d) \in$ Trap and let A_1 and A_2 be two subsets of T. Define $S_1 := \{(a_1, b_1, c_1, d_1) : [a_1, b_1, c_1, d_1] \in A_1\}$ and $S_2 := \{(a_2, b_2, c_2, d_2) : [a_2, b_2, c_2, d_2] \in A_2\}$. Then (b-1) $(a, b, c, d) \in S_1$ S_2 if and only if $[a, b, c, d] \in A_1$ A_2; (b-2) S_1 $S_2 = \{(f, g, h, k) : [f, g, h, k] \in A_1$ $A_2\}$. (c) Tap $\subsetneq$ Trap, (d) Trag $\subsetneq$ Trap, and (e) Tag $\subsetneq$ Tap.

First we show (a). To do this, we only need to show (a-1) as (a-1) (a-2). The "if" part of (a-1) is obvious. Suppose $(a, b, c, d) \in S$. This means there exists $[a_1, b_1, c_1, d_1] \in A$ such that $(a, b, c, d) = (a_1, b_1, c_1, d_1)$. Since (a, b, c, d) and (a_1, b_1, c_1, d_1) are in Trap (see also (I) below), by Theorem 4.1(ii), $[a, b, c, d] = [a_1, b_1, c_1, d_1]$. So $[a, b, c, d] \in A$. Thus the "only if" part of (a-1) is proved. So (a-1) holds. Hence (a) is true. (Theorem 4.1(ii) (a-1) is proved in this paragraph.)

Next we show (b). Consider (i) $(a, b, c, d) \in S_1$ S_2, (ii) $(a, b, c, d) \in S_1$ but $(a, b, c, d) \notin S_2$, (iii) $[a, b, c, d] \in A_1$ but $[a, b, c, d] \notin A_2$, and (iv) $[a, b, c, d] \in A_1$ A_2. By (a), (ii) (iii). This means (i) (iv), as (i) (ii) and (iii) (iv). So (b-1) is proved. ((a) (b-1) is proved in this paragraph.) (b-2) follows immediately from (b-1). A routine proof of (b-2) is given below.

Let $[f,g,h,k] \in A_1 \quad A_2$. Then $(f,g,h,k) \in$ Trap, and by (b-1), $(f,g,h,k) \in S_1 \ S_2$. Thus $S_1 \ S_2 \supseteq \{(f,g,h,k) : [f,g,h,k] \in A_1 \ A_2\}$. Let $(f,g,h,k) \in S_1 \ S_2$. Then $(f,g,h,k) \in$ Trap (see also (II) below), and by (b-1), $[f,g,h,k] \in A_1 \quad A_2$. Thus $S_1 \quad S_2 \subseteq \{(f,g,h,k) : [f,g,h,k] \in A_1 \quad A_2\}$. So (b-2) holds. ((b-1) (b-2) is proved in this paragraph.)

Now we show (c). Trap Tap $= \{(a,b,c,d) : [a,b,c,d] \in T\} \quad \{(a,b,c,d) : [a,b,c,d] \in T_0\} =$ (by (b-2)) $\{(a,b,c,d) : [a,b,c,d] \in T \quad T_0\} \neq \emptyset$ (clearly $T \quad T_0 \neq \emptyset$. This means the $\neq$ holds.). Thus Trap $\neq$ Tap. We have known that Tap $\subseteq$ Trap. So (c) is true.

Now we show (d). Put $T_1 := \{[a,b,c,d] : [a,b,c,d] \in T \text{ with } b = c\}$. We can see that Trag $= \{(a,b,c) : [a,b,c] \in G\} = \{(a,b,b,c) : [a,b,c] \in G\} = \{(a,b,b,c) : [a,b,b,c] \in T\} = \{(a,b,c,d) : [a,b,c,d] \in T_1\}$, where the second equality follows from Remark 3.4(ii), the other equalities are easy to see (see also (III) below). Thus Trap Trag $= \{(a,b,c,d) : [a,b,c,d] \in T\} \quad \{(a,b,c,d) : [a,b,c,d] \in T_1\} =$ (by (b-2)) $\{(a,b,c,d) : [a,b,c,d] \in T \quad T_1\} \neq \emptyset$ (clearly $T \quad T_1 \neq \emptyset$. This means the $\neq$ holds.). Thus Trap $\neq$ Trag. We have known that Trag $\subseteq$ Trap. So (d) is true.

Finally we show (e). Put $T_2 := \{[a,b,c,d] : [a,b,c,d] \in T_0 \text{ with } b = c\}$. We can see that Tag $= \{(a,b,c) : [a,b,c] \in G_0\} = \{(a,b,b,c) : [a,b,c] \in G_0\} = \{(a,b,b,c) : [a,b,b,c] \in T_0\} = \{(a,b,c,d) : [a,b,c,d] \in T_2\}$, where the second equality follows from Remark 2.4(ii), the other equalities are easy to see (see also (IV) below). Thus Tap Tag $= \{(a,b,c,d) : [a,b,c,d] \in T_0\} \quad \{(a,b,c,d) : [a,b,c,d] \in T_2\} =$ (by (b-2)) $\{(a,b,c,d) : [a,b,c,d] \in T_0 \quad T_2\} \neq \emptyset$ (clearly $T_0 \quad T_2 \neq \emptyset$. This means the $\neq$ holds.). Hence Tap $\neq$ Tag. We have known that Tag $\subseteq$ Tap. So (e) is true.

(I) Clearly $(a_1,b_1,c_1,d_1) \in$ Trap as $[a_1,b_1,c_1,d_1] \in A \subseteq T$.

Let $l,m,n,t \in \mathbb{R}$. Consider (I-1) (l,m,n,t) is well-defined, (I-2) $[l,m,n,t] \in T$, (I-3) $(l,m,n,t) \in$ Trap. In this paper, (l,m,n,t) is well-defined only when $[l,m,n,t] \in T$; that is, (I-1) (I-2). Clearly (I-2) (I-3) and (I-3) (I-1). So (I-1) (I-2) (I-3). Thus when using (a), (b) or Theorem 4.1(i)(ii), we do not need to verify that a certain (l,m,n,t) is in Trap if it is well-defined. For example, here we do not need to mention that "(a,b,c,d) and (a_1,b_1,c_1,d_1) are in Trap". This conclusion follows from the fact that (a,b,c,d) and (a_1,b_1,c_1,d_1) are well-defined. This fact is implicit in the previously given expression "$(a,b,c,d) = (a_1,b_1,c_1,d_1)$". That $(a,b,c,d) \in$ Trap is one of the prerequisites of (a).

We think that (I-1) (I-2) (I-3) can be used without citing as it is easy to see. In this paper, we don't always illustrate that there are multiple ways to prove that a certain element belongs to Trap, as we do here, because it's easy to see.

Let B be a subset of $\mathbb{R}^4$. Consider (I-4) $S_B = \{(e,f,g,h) : [e,f,g,h] \in B\}$ is well-defined, (I-5) for each $[e,f,g,h] \in B$, (e,f,g,h) is well-defined, and (I-6)

$B \subseteq T$. Clearly (I-4) (I-5) and (I-5) (I-6). This means (I-4) (I-5) (I-6). Thus, we can conclude that a certain subset D of $\mathbb{R}^4$ is included in T if $\{(e, f, g, h) : [e, f, g, h] \in D\}$ is well-defined.

(II) Clearly $S_1 \subseteq$ Trap as $A_1 \subseteq T$. So $(f, g, h, k) \in S_1 \quad S_2 \subseteq S_1 \subseteq$ Trap.

In fact $(f, g, h, k) \in$ Trap does not need to be mentioned since it follows from the fact that (f, g, h, k) is well-defined (see (I-1) (I-3) given above), which is implicit in the preceding expression "$(f, g, h, k) \in S_1 \quad S_2$".

(III) Note that for any $a, b, c \in \mathbb{R}$, $[a, b, b, c] \in T$ is equivalent to $[a, b, c] \in G$. So the third equality holds.

Put $A := \{(a, b, b, c) : [a, b, b, c] \in T\}$ and $B := \{(a, b, c, d) : [a, b, c, d] \in T_1\}$. The fourth equality means that (III-1) $A \subseteq B$; that is, for each $[a, b, b, c] \in T$, $(a, b, b, c) \in B$, and (III-2) $B \subseteq A$; that is, for each $[a, b, c, d] \in T_1$, $(a, b, c, d) \in A$. Given $[a, b, b, c] \in T$. Then $[a, b, b, c] \in T_1$ (obviously the converse is true). Thus $(a, b, b, c) \in \{(a, b, c, d) : [a, b, c, d] \in T_1\} = B$ (see (III-3) below). Hence (III-1) holds. Given $[a, b, c, d] \in T_1$. This means $b = c$ and $[a, b, c, d] \in T$. Then $(a, b, c, d) = (a, b, b, d) \in \{(l, m, m, n) : [l, m, m, n] \in T\} = A$ (see (III-4) below). Hence (III-2) holds. So the fourth equality holds.

(III-3) Conversely, suppose $(a, b, b, c) \in B$. Then $(a, b, b, c) \in$ Trap as (a, b, b, c) is well-defined. Also $T_1 \subseteq T$. Thus by (a-1), $[a, b, b, c] \in T_1$.

(III-4) Let $a, b, c, d \in \mathbb{R}$. Suppose $(a, b, c, d) = (a, b, b, d)$. Then (a, b, c, d) and (a, b, b, d) are in Trap as they are well-defined. Thus by Theorem 4.1(ii), $[a, b, c, d] = [a, b, b, d]$; that is, $b = c$. (a, b, c, d) is well-defined means $[a, b, c, d] \in T$.

(IV) Note that for any $a, b, c \in \mathbb{R}$, $[a, b, b, c] \in T_0$ is equivalent to $[a, b, c] \in G_0$. So the third equality holds.

Put $C := \{(a, b, b, c) : [a, b, b, c] \in T_0\}$ and $D := \{(a, b, c, d) : [a, b, c, d] \in T_2\}$. The fourth equality means that (IV-1) $C \subseteq D$; that is, for each $[a, b, b, c] \in T_0$, $(a, b, b, c) \in D$, and (IV-2) $D \subseteq C$; that is, for each $[a, b, c, d] \in T_2$, $(a, b, c, d) \in C$. Given $[a, b, b, c] \in T_0$. Then $[a, b, b, c] \in T_2$ (obviously the converse is true). Thus $(a, b, b, c) \in \{(a, b, c, d) : [a, b, c, d] \in T_2\} = D$. Hence (IV-1) holds. Given $[a, b, c, d] \in T_2$. This means $b = c$ and $[a, b, c, d] \in T_0$. Then $(a, b, c, d) = (a, b, b, d) \in \{(l, m, m, n) : [l, m, m, n] \in T_0\} = C$. Hence (IV-2) holds. So the fourth equality holds.

(V) (f) The six statements (a), (a-1), (a-2), (b-1), (b-2), and Theorem 4.1(ii) are equivalent.

Clearly (a-1) (a-2). So (a) (a-1) (a-2). (Obviously, the converse is true.) Theorem 4.1(ii) (a-1) and (a) (b-1) (b-2) have been shown in the above contents. To show (f) we only need to show that (b-2) Theorem 4.1(ii), a proof of which is given below.

Assume (b-2) holds. Let (a, b, c, d) and (a_1, b_1, c_1, d_1) be in Trap. Clearly if $[a, b, c, d] = [a_1, b_1, c_1, d_1]$ then $(a, b, c, d) = (a_1, b_1, c_1, d_1)$. Suppose $(a, b, c, d) = (a_1, b_1, c_1, d_1)$. Set $S_1 := \{(a_1, b_1, c_1, d_1)\}$, $S_2 := \{(a, b, c, d)\}$, $A_1 := \{[a_1, b_1, c_1, d_1]\}$ and $A_2 := \{[a, b, c, d]\}$. Clearly $S_1 = \{(e, f, g, h) : [e, f, g, h] \in A_1\}$ and $S_2 = \{(e, f, g, h) : [e, f, g, h] \in A_2\}$. We can see that $\{(e, f, g, h) : [e, f, g, h] \in A_1 \quad A_2\}$ = (by (b-2)) $S_1 \quad S_2 = \emptyset$ (see (V-1) below). So $A_1 \quad A_2 = \emptyset$; that is, $[a_1, b_1, c_1, d_1] = [a, b, c, d]$ (see (V-2) below). Thus Theorem 4.1(ii) holds. Hence (b-2) Theorem 4.1(ii). So (f) is proved.

(V-1) (a, b, c, d) and (a_1, b_1, c_1, d_1) are in Trap means that $[a, b, c, d]$ and $[a_1, b_1, c_1, d_1]$ are in T, which means that A_1 and A_2 are subsets of T. So (b-2) can be used here. We think that the fact that A_1 and A_2 are subsets of T does not need to be mentioned as it is implicit in the fact that (a, b, c, d) and (a_1, b_1, c_1, d_1) are in Trap (the contents in the first sentence of this paragraph indicate that these two facts are equivalent.).

(V-2) (α) Let B be a subset of $\mathbb{R}^4$. Put $S_B := \{(e, f, g, h) : [e, f, g, h] \in B\}$. Then $S_B = \emptyset$ if and only if $B = \emptyset$.

Clearly if $B = \emptyset$ then $S_B = \emptyset$. Suppose $S_B = \emptyset$. Then S_B is well-defined. This means $B \subseteq T$ (see (I-4) (I-6)). So if $B \neq \emptyset$, then $S_B \neq \emptyset$, which is a contradiction. Thus $B = \emptyset$. Hence (α) holds. $(a_1, b_1, c_1, d_1) \in$ Trap means $[a_1, b_1, c_1, d_1] \in T$. So $A_1 \subseteq T$. Thus $A_1 \quad A_2 \subseteq T \subseteq \mathbb{R}^4$. Thus by ($\alpha$), $\{(e, f, g, h) : [e, f, g, h] \in A_1 \quad A_2\} = \emptyset$ if and only if $A_1 \quad A_2 = \emptyset$.

It is easy to see that the fact that $A_1 \quad A_2 \subseteq \mathbb{R}^4$ is implicit in the fact that $(a_1, b_1, c_1, d_1) \in$ Trap. Also (α) can be used directly without citing as it is easy to see. So we think we can directly write $\{(e, f, g, h) : [e, f, g, h] \in A_1 \quad A_2\} = \emptyset$ if and only if $A_1 \quad A_2 = \emptyset$.

We mention that several equivalent forms of Proposition 4.2(ii) are given in Remark 4.8.

Remark 4.8. We claim the following statements. (a) Let $(a, b, c) \in$ Trag and let A be a subset of G. Define $S := \{(a_1, b_1, c_1) : [a_1, b_1, c_1] \in A\}$. Then (a-1) $(a, b, c) \in S$ if and only if $[a, b, c] \in A$; (a-2) $(a, b, c) \notin S$ if and only if $[a, b, c] \notin A$. (b) Let $(a, b, c) \in$ Trag and let A_1 and A_2 be two subsets of G. Define $S_1 := \{(a_1, b_1, c_1) : [a_1, b_1, c_1] \in A_1\}$ and $S_2 := \{(a_2, b_2, c_2) : [a_2, b_2, c_2] \in A_2\}$. Then (b-1) $(a, b, c) \in S_1 \quad S_2$ if and only if $[a, b, c] \in A_1 \quad A_2$; (b-2) $S_1 \quad S_2 = \{(f, g, h) : [f, g, h] \in A_1 \quad A_2\}$. (c) Tag $\subsetneq$ Trag.

First we show (a). To do this, we only need to show (a-1) as (a-1) (a-2). The "if" part of (a-1) is obvious. Suppose $(a, b, c) \in S$. This means there exists $[a_1, b_1, c_1] \in A$ such that $(a, b, c) = (a_1, b_1, c_1)$. Since (a, b, c) and (a_1, b_1, c_1) are in Trag (see also (I) below), by Proposition 4.2(ii), $[a, b, c] = [a_1, b_1, c_1]$. So $[a, b, c] \in A$. Thus the "only if" part of (a-1) is proved. So (a-1) holds. Hence (a) is proved. (Proposition 4.2(ii) (a-1) is proved in this paragraph.)

Next we show (b). Consider (i) $(a, b, c) \in S_1 \quad S_2$, (ii) $(a, b, c) \in S_1$ but $(a, b, c) \notin S_2$, (iii) $[a, b, c] \in A_1$ but $[a, b, c] \notin A_2$, and (iv) $[a, b, c] \in A_1 \quad A_2$. By (a),

(ii) (iii). This means (i) (iv), as (i) (ii) and (iii) (iv). So (b-1) is proved. ((a) (b-1) is proved in this paragraph.) (b-2) follows immediately from (b-1). A routine proof of (b-2) is given below.

Let $[f, g, h] \in A_1 \quad A_2$. Then $(f, g, h) \in$ Trag, and by (b-1), $(f, g, h) \in S_1 \quad S_2$. Thus $S_1 \quad S_2 \supseteq \{(f, g, h) : [f, g, h] \in A_1 \quad A_2\}$. Let $(f, g, h) \in S_1 \quad S_2$. Then $(f, g, h) \in$ Trag (see also (II) below), and by (b-1), $[f, g, h] \in A_1 \quad A_2$. Thus $S_1 \quad S_2 \subseteq \{(f, g, h) : [f, g, h] \in A_1 \quad A_2\}$. So (b-2) holds. ((b-1) (b-2) is proved in this paragraph.)

Now we show (c). Trag Tag $= \{(a, b, c) : [a, b, c] \in G\} \quad \{(a, b, c) : [a, b, c] \in G_0\} =$ (by (b-2)) $\{(a, b, c) : [a, b, c] \in G \quad G_0\} \neq \emptyset$ (clearly $G \quad G_0 \neq \emptyset$. This means the $\neq$ holds.). Thus Trag $\neq$ Tag. We have known that Tag $\subseteq$ Trag. So (c) is true.

(I) Clearly $(a_1, b_1, c_1) \in$ Trag as $[a_1, b_1, c_1] \in A \subseteq G$.

Let $l, m, n \in \mathbb{R}$. Consider (I-1) (l, m, n) is well-defined, (I-2) $[l, m, n] \in G$, (I-3) $(l, m, n) \in$ Trag. In this paper, (l, m, n) is well-defined only when $[l, m, n] \in G$; that is, (I-1) (I-2). Clearly (I-2) (I-3) and (I-3) (I-1). So (I-1) (I-2) (I-3). Thus when using (a), (b) or Proposition 4.2(i)(ii), we do not need to verify that a certain (l, m, n) is in Trag if it is well-defined. For example, here we do not need to mention that "(a, b, c) and (a_1, b_1, c_1) are in Trag". This conclusion follows from the fact that (a, b, c) and (a_1, b_1, c_1) are well-defined. This fact is implicit in the previously given expression "$(a, b, c) = (a_1, b_1, c_1)$". That $(a, b, c) \in$ Trag is one of the prerequisites of (a).

We think that the fact (I-1) (I-2) (I-3) can be used without citing as it is easy to see. In this paper, we do not always illustrate that there are multiple ways to prove that a certain element belongs to Trag, as we do here, because it is easy to see.

Let C be a subset of $\mathbb{R}^3$. Consider (I-4) $S_C = \{(l, m, n) : [l, m, n] \in C\}$ is well-defined, (I-5) for each $[l, m, n] \in C$, (l, m, n) is well-defined, and (I-6) $C \subseteq G$. Clearly (I-4) (I-5) and (I-5) (I-6). This means (I-4) (I-5) (I-6). Thus, we can conclude that a certain subset D of $\mathbb{R}^3$ is included in G if $\{(l, m, n) : [l, m, n] \in D\}$ is well-defined.

(II) Clearly $S_1 \subseteq$ Trag as $A_1 \subseteq G$. So $(f, g, h) \in S_1 \quad S_2 \subseteq S_1 \subseteq$ Trag.

In fact $(f, g, h) \in$ Trag does not need to be mentioned since it follows from the fact that (f, g, h) is well-defined (see (I-1) (I-3) given above), which is implicit in the preceding expression "$(f, g, h) \in S_1 \quad S_2$".

(III) (d) The six statements (a), (a-1), (a-2), (b-1), (b-2), and Proposition 4.2(ii) are equivalent.

Clearly (a-1) (a-2). So (a) (a-1) (a-2). (Obviously the converse is true.) Proposition 4.2(ii) (a-1) and (a) (b-1) (b-2) have been shown in the above contents. To show (d) we only need to show that (b-2) Proposition 4.2(ii), a proof of which is given below.

Assume (b-2) holds. Let (a, b, c) and (a_1, b_1, c_1) be in Trag. Clearly if $[a, b, c] = [a_1, b_1, c_1]$ then $(a, b, c) = (a_1, b_1, c_1)$. Suppose $(a, b, c) = (a_1, b_1, c_1)$. Set $S_1 := \{(a_1, b_1, c_1)\}$, $S_2 := \{(a, b, c)\}$, $A_1 := \{[a_1, b_1, c_1]\}$ and $A_2 := \{[a, b, c]\}$. Clearly $S_1 = \{(e, f, g) : [e, f, g] \in A_1\}$ and $S_2 = \{(e, f, g) : [e, f, g] \in A_2\}$. We can see that $\{(e, f, g) : [e, f, g] \in A_1 \quad A_2\} =$ (by (b-2)) $S_1 \quad S_2 = \emptyset$ (see (V-1) below). So $A_1 \quad A_2 = \emptyset$; that is, $[a_1, b_1, c_1] = [a, b, c]$ (see (V-2) below). Thus Proposition 4.2(ii) holds. Hence (b-2) Proposition 4.2(ii). So (d) is proved.

(V-1) (a, b, c) and (a_1, b_1, c_1) are in Trag means that $[a, b, c]$ and $[a_1, b_1, c_1]$ are in G, which means that A_1 and A_2 are subsets of G. So (b-2) can be used here. We think that the fact that A_1 and A_2 are subsets of G does not need to be mentioned as it is implicit in the fact that (a, b, c) and (a_1, b_1, c_1) are in Trag (the contents in the first sentence of this paragraph indicate that these two facts are equivalent.).

(V-2) (α) Let B be a subset of $\mathbb{R}^3$. Put $S_B := \{(e, f, g) : [e, f, g] \in B\}$. Then $S_B = \emptyset$ if and only if $B = \emptyset$. Clearly if $B = \emptyset$ then $S_B = \emptyset$. Suppose $S_B = \emptyset$. Then S_B is well-defined. This means $B \subseteq G$ (see (I-4) (I-6)). So if $B \neq \emptyset$, then $S_B \neq \emptyset$, which is a contradiction. Thus $B = \emptyset$. Hence (α) holds. $(a_1, b_1, c_1) \in$ Trag means $[a_1, b_1, c_1] \in G$. So $A_1 \subseteq G$. Thus $A_1 \quad A_2 \subseteq G \subseteq \mathbb{R}^3$. Thus by ($\alpha$), $\{(e, f, g) : [e, f, g] \in A_1 \quad A_2\} = \emptyset$ if and only if $A_1 \quad A_2 = \emptyset$.

It is easy to see that the fact that $A_1 \quad A_2 \subseteq \mathbb{R}^3$ is implicit in the fact that $(a_1, b_1, c_1) \in$ Trag. Also (α) can be used directly without citing as it is easy to see. So we think we can directly write $\{(e, f, g) : [e, f, g] \in A_1 \quad A_2\} = \emptyset$ if and only if $A_1 \quad A_2 = \emptyset$.

Remark 4.9. (α) Suppose $(a) \Rightarrow (b)$. Clearly if $(a') \Rightarrow (a)$ and $(b) \Rightarrow (b')$, then $(a') \Rightarrow (b')$. Of course, $(c) = (c')$ is a special case of $(c) \Leftrightarrow (c')$, and $(c) \Leftrightarrow (c')$ is a special case of $(c) \Rightarrow (c')$.

In this paper, we give some conclusions in the form of "$(a) \Rightarrow (b)$". These include "Theorem 4.1(ii) Theorem 4.1(iv)" (see the proof of Theorem 4.1), "Proposition 4.2(ii) Proposition 4.2(iv)" (see the proof of Proposition 4.2) and some conclusions in Remark 4.3. We also give several equivalent forms of Theorem 4.1(ii), Theorem 4.1(iv), Proposition 4.2(ii), Proposition 4.2(iv), respectively. By (α), it is easy to obtain various conclusions in the form of "$(a) \Rightarrow (b)$" from certain conclusions in this paper. We will not list them one by one as they are easy to see. Below are a few examples.

We know that Remark 4.4($\bar{a}$) (Remark 4.4(a)) Theorem 4.1(ii), Remark 4.4($\bar{b}$) (Remark 4.4(b)) Proposition 4.2(ii), and Theorem 4.1(ii) Proposition 4.2(ii). So Remark 4.4($\bar{a}$) Remark 4.4($\bar{b}$).

We know that Remark 4.5($\bar{a}$) (Remark 4.5(a)) Theorem 4.1(iv), Remark 4.5($\bar{b}$) (Remark 4.5(b)) Proposition 4.2(iv), and Theorem 4.1(iv) Proposition 4.2(iv). So Remark 4.5($\bar{a}$) Remark 4.5($\bar{b}$).

5. Relationships Between Tag and Tap, and Between Trag and Trap

Let A be a set. A mapping $f : A \to A$ is said to be the identity mapping on A if $f(x) = x$ for each $x \in A$. A mapping g is said to be an identity mapping if there exists a set S and g is the identity mapping on S.

Define $\text{Trap}_1 := \{(a, b, c, d) : (a, b, c, d) \in \text{Trap and } b = c\}$. Clearly $\text{Trap}_1 \subsetneq \text{Trap}$ and $\text{Trap}_1 = \{(a, b, b, c) : (a, b, b, c) \in \text{Trap}\}$.

Proposition 5.1. (i) $\text{Trag} = \text{Trap}_1$. (ii) Define a mapping $K : \text{Trag} \to \text{Trap}_1$ as follows: for each $u \in \text{Trag}$, find $[a, b, c] \in G$ satisfying $u = (a, b, c)$, and then define $K(u)$ to be (a, b, b, c). Then K is the identity mapping on Trag.

Proof. Remark 3.4(ii) implies that $\text{Trag} \subseteq \text{Trap}_1$. Remark 3.4(v) implies that $\text{Trap}_1 \subseteq \text{Trag}$. (For each $a, b, c \in \mathbb{R}$, $(a, b, b, c) \in \text{Trap}$ if and only if $(a, b, b, c) \in \text{Trap}_1$.) So (i) is true.

We claim the following (a) and (b). (a) K is well-defined; that is, by virtue of K, for each $u \in \text{Trag}$, (a-1) $K(u)$ is one element, and (a-2) $K(u) \in \text{Trap}_1$. (b) For each $u \in \text{Trag}$, $K(u) = u$.

Let $u \in \text{Trag}$. Then there exists $[a, b, c] \in G$ satisfying $u = (a, b, c)$. Thus (a, b, b, c) is a value of $K(u)$ (formally, $K(u)$ may have multiple values). By Remark 3.4(ii), $(a, b, c) = u \in \text{Trag}$ implies that $(a, b, b, c) \in \text{Trap}$, which means $(a, b, b, c) \in \text{Trap}_1$ (see also (I) below), and that $(a, b, c) = (a, b, b, c)$. So to show (a) and (b), we only need to show (c) $K(u)$ is one element. (Suppose $K(u)$ is one element. Then $K(u) = (a, b, b, c)$. Hence $K(u) \in \text{Trap}_1$ and $K(u) = (a, b, b, c) = (a, b, c) = u$. So (a) and (b) hold.) Let $[a_1, b_1, c_1]$ be an element of G which satisfies $u = (a_1, b_1, c_1)$. Then (a_1, b_1, b_1, c_1) is a value of $K(u)$. To show (c), we only need to show that $(a, b, b, c) = (a_1, b_1, b_1, c_1)$. (If this is true, then $K(u)$ can only be the element (a, b, b, c), and so (c) holds.) Notice that $(a, b, c) = u = (a_1, b_1, c_1) \in \text{Trag}$. Thus $(a, b, b, c) = (a_1, b_1, b_1, c_1)$, as by Remark 3.4(ii), $(a, b, c) = (a, b, b, c)$ and $(a_1, b_1, c_1) = (a_1, b_1, b_1, c_1)$. Hence (c) is proved. So (a) and (b) hold.

Combining (i), (a) and (b) yields that K is the identity mapping on Trag. So (ii) is true. (Clearly that K is the identity mapping on Trag also implies (i), (a) and (b).) The proof is completed. (I) By Remark 3.4(i), $(a, b, c) = u \in \text{Trag}$ (obviously, $a, b, c \in \mathbb{R}$ in this case) implies that $(a, b, b, c) \in \text{Trap}$, which means $(a, b, b, c) \in \text{Trap}_1$.

Remark 5.2. In this remark, the symbols are consistent with those in the above proof of Proposition 5.1. (i) From the above proof of Proposition 5.1, we can see (i-1) Remark 3.4(ii)(v) implies Proposition 5.1; (i-2) (a) Remark 3.4(ii) implies (a) and (b). Obviously, combining (a) and (b) yields Remark 3.4(ii); Proposition 5.1(ii) implies Remark 3.4(ii)(v). (ii) The above proof of Proposition 5.1 will become a new proof of Proposition 5.1 if the contents from "Let $[a_1, b_1, c_1]$ be an element of G" to "Hence (c) is proved." in it are replaced by the contents in clause (ii-1). (ii-1) By Remark 4.4($\bar{b}$), $[a, b, c]$ is the unique element of G that satisfies $u = (a, b, c)$. Then $K(u)$ can only be the element (a, b, b, c). So (c) holds. (c) can be stated as "Let $u \in$ Trag. Then $K(u)$ is one element." (c) holds means that (a-1) holds. (iii) Clearly $\text{Trap}_1 = \text{Trag} = \{(a, b, c) : [a, b, c] \in G\} = \{(a, b, b, c) : [a, b, c] \in G\} = \{(a, b, b, c) : [a, b, b, c] \in T\}$, where the third equality follows from Remark 3.4(ii) or Proposition 5.1(ii), and the fourth equality follows from the fact that $[a, b, c] \in G$ if and only if $[a, b, b, c] \in T$.

Define $\text{Tap}_1 := \{(a, b, c, d) : (a, b, c, d) \in \text{Tap} \text{ and } b = c\}$. Clearly $\text{Tap}_1 \subsetneq \text{Tap}$ and $\text{Tap}_1 = \{(a, b, b, c) : (a, b, b, c) \in \text{Tap}\}$.

Proposition 5.3. (i) $\text{Tag} = \text{Tap}_1$. (ii) Define a mapping $L : \text{Tag} \to \text{Tap}_1$ as follows: for each $u \in$ Tag, find $[a, b, c] \in G_0$ satisfying $u = (a, b, c)$, and then define $L(u)$ to be (a, b, b, c). Then L is the identity mapping on Tag.

Proof. Remark 2.4(ii) implies that $\text{Tag} \subseteq \text{Tap}_1$. Remark 2.4(v) implies that $\text{Tap}_1 \subseteq \text{Tag}$. (For each a, b and c in $\mathbb{R}$, $(a, b, b, c) \in$ Tap if and only if $(a, b, b, c) \in \text{Tap}_1$.) So (i) is true.

We claim the following (a) and (b). (a) L is well-defined; that is, by virtue of L, for each element $u \in$ Tag, (a-1) $L(u)$ is one element, and (a-2) $L(u) \in \text{Tap}_1$. (b) For each $u \in$ Tag, $L(u) = u$.

Let $u \in$ Tag. We can find $[a, b, c] \in G_0$ satisfying $u = (a, b, c)$. Then (a, b, b, c) is a value of $L(u)$ (formally, $L(u)$ may have multiple values). By Remark 2.4(ii), $(a, b, c) = u \in$ Tag implies that $(a, b, b, c) \in$ Tap, which means $(a, b, b, c) \in \text{Tap}_1$ (see also (I) below), and that $(a, b, c) = (a, b, b, c)$. So to show (a) and (b), we only need to show (c) $L(u)$ is one element. (Suppose $L(u)$ is one element. Then $L(u) = (a, b, b, c)$. Hence $L(u) \in \text{Tap}_1$ and $L(u) = (a, b, b, c) = (a, b, c) = u$. So (a) and (b) hold.) Let $[a_1, b_1, c_1]$ be an element of G_0 which satisfies $u = (a_1, b_1, c_1)$. Then (a_1, b_1, b_1, c_1) is a value of $L(u)$. To show (c), we only need to show that $(a, b, b, c) = (a_1, b_1, b_1, c_1)$. (If this is true, then $L(u)$ can only be the element (a, b, b, c) in Tap_1, and so (c) holds.) Notice that $(a, b, c) = u = (a_1, b_1, c_1) \in$ Tag. Thus $(a, b, b, c) = (a_1, b_1, b_1, c_1)$, as by Remark 2.4(ii), $(a, b, c) = (a, b, b, c)$ and $(a_1, b_1, c_1) = (a_1, b_1, b_1, c_1)$. Hence (c) is proved. So (a) and (b) hold. (i), (a) and (b) hold if and only if (ii) holds. So (ii) is proved as (i), (a) and (b) are proved. (I) By Remark 2.4(i), $(a, b, c) = u \in$ Tag (obviously, $a, b, c \in \mathbb{R}$ in this case) implies that $(a, b, b, c) \in$ Tap, which means $(a, b, b, c) \in \text{Tap}_1$.

Remark 5.4. In this remark, the symbols are consistent with those in the above proof of Proposition 5.3. (i) From the above proof of Proposition 5.3, we can see (i-1) Remark 2.4(ii)(v) implies Proposition 5.3; (i-2) Remark 2.4(ii) implies (a) and (b). Obviously, combining (a) and (b) yields Remark 2.4(ii); Proposition 5.3(ii) implies Remark 2.4(ii)(v). (ii) The above proof of Proposition 5.3 remains true if the contents from "Let $[a_1, b_1, c_1]$ be an element of G_0" to "Hence (c) is proved." in it are replaced by the contents in clause (ii-1) or by the contents in clause (ii-2). (ii-1) By Remark 4.6($\bar{b}$), $[a, b, c]$ is the unique element of G_0 that satisfies $u = (a, b, c)$. Then, by the definition of L, $L(u)$ can only be the element

(a, b, b, c). Hence (c) holds. (ii-2) Note that $u \in \text{Tag} \subset \text{Trag}$ and $G_0 \subset G$. Thus, by the definitions of L and K, each value of $L(u)$ is a value of $K(u)$, where K is defined in Proposition 5.1. Hence $L(u)$ is one element, as $K(u)$ is one element and (a, b, b, c) is a value of $L(u)$. (c) can be stated as "Let $u \in \text{Tag}$. Then $L(u)$ is one element." (c) holds means that (a-1) holds. (iii) By Remark 4.5($\bar{b}$), we have the fact that for each $u \in \text{Tag}$, whether you perform the operation "find an $[a, b, c] \in G_0$ satisfying $u = (a, b, c)$" or the operation "find an $[a, b, c] \in G$ satisfying $u = (a, b, c)$", the same one element $[a, b, c]$ will be found. (Conversely, this fact also implies Remark 4.5($\bar{b}$).) So L is invariant if we replace "find an $[a, b, c] \in G_0$" by "find an $[a, b, c] \in G$" in the definition of L. In other words, we can also define L as follows:

Define a mapping $L : \text{Tag} \to \text{Tap}_1$ as follows: for each $u \in \text{Tag}$, find an $[a, b, c] \in G$ satisfying $u = (a, b, c)$, and then define $L(u)$ to be (a, b, b, c).

And if we define L in this way, then for each $u \in \text{Tag}$, $L(u) = K(u)$, and hence $L(u)$ is one element as $K(u)$ is one element. (iv) Clearly $\text{Tap}_1 = \text{Tag} = \{(a, b, c) : [a, b, c] \in G_0\} = \{(a, b, b, c) : [a, b, c] \in G_0\} = \{(a, b, b, c) : [a, b, b, c] \in T_0\}$, where the third equality follows from Remark 2.4(ii) or Proposition 5.3(ii), and the fourth equality follows from the fact that $[a, b, c] \in G_0$ if and only if $[a, b, b, c] \in T_0$.

6. Conclusions

In this paper, we establish the representation uniqueness of generalized triangular fuzzy numbers and generalized trapezoidal fuzzy numbers, which are (i) and (ii) listed below, respectively.

(i) (Remark 4.4($\bar{b}$)) For each $u \in \text{Trag}$, there exists $[a, b, c]$ which is the unique element of G satisfying $u = (a, b, c)$.

(ii) (Remark 4.4($\bar{a}$)) For each $u \in \text{Trap}$, there exists $[a, b, c, d]$ which is the unique element of T satisfying $u = (a, b, c, d)$.

We show the representation uniqueness of triangular fuzzy numbers and trapezoidal fuzzy numbers, which are (iii) and (iv) listed below, respectively.

(iii) (Remark 4.5($\bar{b}$)) For each $u \in \text{Tag}$, there exists $[a, b, c] \in G_0$ which is the unique element of G satisfying $u = (a, b, c)$.

(iv) (Remark 4.5($\bar{a}$)) For each $u \in \text{Tap}$, there exists $[a, b, c, d] \in T_0$ which is the unique element of T satisfying $u = (a, b, c, d)$.

We point out that (ii) (i) (iii) and (ii) (iv) (iii) (see Section 4). Furthermore, we obtain the following relationships among Tag, Trag, Tap and Trap: $\text{Tap} \subsetneq \text{Trap}$, $\text{Trag} \subsetneq \text{Trap}$, $\text{Tag} \subsetneq \text{Tap}$, and $\text{Tag} \subsetneq \text{Trag}$.

The results of this paper have potential effects on the analysis and applications of generalized triangular fuzzy numbers and generalized trapezoidal fuzzy numbers.

7. Supplementary Content

The following representation theorem should be a known conclusion. A proof for this theorem is given in [**?**]. See Theorem 3.1 in [**?**] and its proof.

In this paper we assume that $\sup\emptyset = 0$.

Theorem 7.1. Let Y be a nonempty set. If $u \in \mathcal{F}(Y)$, then for all $\alpha \in (0,1]$, $[u]_\alpha = \bigcap_{\beta<\alpha}[u]_\beta$.

Conversely, suppose that $\{v_\alpha : \alpha \in (0,1]\}$ is a family of sets in Y with $v_\alpha = \bigcap_{\beta<\alpha} v_\beta$ for all $\alpha \in (0,1]$. Define $u \in \mathcal{F}(Y)$ by $u(x) := \sup\{\alpha \in (0,1] : x \in v_\alpha\}$ for each $x \in Y$. Then u is the unique element of $\mathcal{F}(Y)$ which satisfies $[u]_\alpha = v_\alpha$ for all $\alpha \in (0,1]$; that is, u is the unique element of the set $\{w \in \mathcal{F}(Y) : [w]_\alpha = v_\alpha \text{ for all } \alpha \in (0,1]\}$.

Remark 7.2. Let Y be a nonempty set and let $\{v_\alpha : \alpha \in (0,1]\}$ be a family of sets in Y. Denote $S := \{w \in \mathcal{F}(Y) : [w]_\alpha = v_\alpha \text{ for all } \alpha \in (0,1]\}$. (i) Suppose that the statement "$v_\alpha = \bigcap_{\beta<\alpha} v_\beta$ for all $\alpha \in (0,1]$" does not hold. Then $S = \emptyset$. (ii) Suppose that $v_\alpha = \bigcap_{\beta<\alpha} v_\beta$ for all $\alpha \in (0,1]$. Then S is a singleton set. (iii) S is an empty set or a singleton set. (iv) Let $v \in \mathcal{F}(Y)$. Then the set $S(v) := \{w \in \mathcal{F}(Y) : [w]_\alpha = [v]_\alpha \text{ for all } \alpha \in (0,1]\}$ is a singleton set. (v) Let $v \in \mathcal{F}(Y)$. Then $S(v) = \{v\}$.

We show (i). By Theorem 7.1, for each $w \in \mathcal{F}(Y)$, it holds that $[w]_\alpha = \bigcap_{\beta<\alpha}[w]_\beta$ for all $\alpha \in (0,1]$. Thus $S = \emptyset$. So (i) is proved.

By Theorem 7.1, (ii) holds. (iii) follows immediately from (i) and (ii).

We show (iv). Since $v \in \mathcal{F}(Y)$, by Theorem 7.1, for all $\alpha \in (0,1]$, $[v]_\alpha = \bigcap_{\beta<\alpha}[v]_\beta$. Then by (ii), $S(v)$ is a singleton set. So (iv) is proved.

It is easy to see (a) for each $v \in \mathcal{F}(Y)$, $v \in S(v)$; and (b) for each $v \in \mathcal{F}(Y)$, $S(v) \neq \emptyset$. ((a) (b).) Clearly (a) and (iv) hold if and only if (v) holds. So (v) holds.

Combining (b) and (iii) yields (iv).

The following Proposition 7.3 should be known. We cannot find the original reference which gave this conclusion, so we give a proof here for the self-containing of this paper.

Let Y be a nonempty set and $u \in \mathcal{F}(Y)$. Then $[u]_0$ is well-defined if and only if Y is a topological space.

Proposition 7.3. Let Y be a nonempty set, $x \in Y$ and $u, v \in \mathcal{F}(Y)$. (i) (i-1) $u(x) = \sup\{\alpha \in (0,1] : x \in [u]_\alpha\}$. (i-2) If $[u]_0$ is well-defined, then $u(x) = \sup\{\alpha \in [0,1] : x \in [u]_\alpha\}$. (ii) (ii-1) If for each $\alpha \in (0,1]$, $[u]_\alpha = [v]_\alpha$, then $u = v$. (ii-2) Assume that $u = v$. Then (ii-2a) for each $\alpha \in (0,1]$, $[u]_\alpha = [v]_\alpha$; (ii-2b) Y is a topological space if and only if $[u]_0 = [v]_0$. (ii-3)

$u = v$ if and only if for each $\alpha \in (0, 1]$, $[u]_\alpha = [v]_\alpha$. (ii-4) Assume that Y is a topological space. Then $u = v$ if and only if for each $\alpha \in [0, 1]$, $[u]_\alpha = [v]_\alpha$.

Proof. First we show (i). Put $u(x) = \xi$. If $\xi > 0$, then $\{\alpha \in (0, 1] : x \in [u]_\alpha\} = (0, \xi]$, and so $\sup\{\alpha \in (0, 1] : x \in [u]_\alpha\} = \sup(0, \xi] = \xi = u(x)$. If $\xi = 0$, then $\{\alpha \in (0, 1] : x \in [u]_\alpha\} = \emptyset$, and so $\sup\{\alpha \in (0, 1] : x \in [u]_\alpha\} = 0 = u(x)$. Thus (i-1) holds.

Below we show ($\bar{a}$) If $[u]_0$ is well-defined, then $\sup S_1 = \sup S_2$, where $S_1 := \{\alpha \in (0, 1] : x \in [u]_\alpha\}$ and $S_2 := \{\alpha \in [0, 1] : x \in [u]_\alpha\}$.

Case (I). Assume $S_1 = \emptyset$. Then $\sup S_1 = 0$, and it holds that $S_2 = \emptyset$ or $S_2 = \{0\}$. Clearly $\sup S_2 = 0$ regardless of $S_2 = \emptyset$ or $S_2 = \{0\}$. So $\sup S_1 = \sup S_2$.

Case (II). Assume $S_1 \neq \emptyset$. Then there exists $\alpha \in (0, 1]$ with $x \in [u]_\alpha$. So $x \in [u]_0$ as $[u]_\alpha \subseteq [u]_0$. Thus we obtain (a) $S_2 = S_1 \cup \{0\}$. Clearly we have (b) $\sup S_1 > 0$. Thus $\sup S_2 =$ (by (a)) $\sup(S_1 \cup \{0\}) = \sup S_1 \vee \sup\{0\} = \sup S_1 \vee 0$ (i.e., $(\sup S_1) \vee 0$) $=$ (by (b)) $\sup S_1$. The proof of ($\bar{a}$) is completed.

Combining (i-1) and ($\bar{a}$) yields (i-2). Hence (i) is proved. (Obviously, combining (i-1) and (i-2) yields ($\bar{a}$).)

Now we show (ii-1). Notice that for each $y \in Y$, $v(y) =$ (by (i-1)) $\sup\{\alpha \in (0, 1] : y \in [v]_\alpha\} = \sup\{\alpha \in (0, 1] : y \in [u]_\alpha\} =$ (by (i-1)) $u(y)$. So $u = v$ as $u, v \in \mathcal{F}(Y)$. This proof of (ii-1) is essentially given in the proof of Theorem 3.1 in [**?**]. (see also (I) below)

Now we show (ii-2). (ii-2a) holds obviously. Assume Y is a topological space. This means that $[u]_0$ and $[v]_0$ are well-defined. Then $u = v$ implies that $[u]_0 = [v]_0$. If $[u]_0 = [v]_0$ then $[u]_0$ and $[v]_0$ are well-defined, which means that Y is a topological space. So (ii-2b) holds. (ii-3) follows immediately from (ii-1) and (ii-2a). (ii-4) follows immediately from (ii-1) and (ii-2).

(I) We can see that (ii-1) is equivalent to (ii-1)' Given $v \in \mathcal{F}(Y)$, if $u \in S(v)$ then $u = v$. It holds that (a) for each $v \in \mathcal{F}(Y)$, $v \in S(v)$. Clearly (a) and (ii-1)' hold if and only if Remark 7.2(v) holds (we use (b) to denote this statement.). Below (I-1) and (I-2) are two proofs of (ii-1). (I-1) Since Remark 7.2(v) is proved, by (b), (ii-1)' holds; that is, (ii-1) holds. (I-2) Suppose that u and v are in $\mathcal{F}(Y)$ satisfying for all $\alpha \in (0, 1]$, $[u]_\alpha = [v]_\alpha$. Then it holds that (c) u and v are in $S(v)$. By Remark 7.2(iv), we have (d) $S(v)$ is a singleton set. By (c) and (d), we have that $u = v$. So (ii-1) is proved. ((d) also follows from (c) and Remark 7.2(iii).) In some sense, all the proofs of (ii-1) given in this paper are essentially the same.

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First, the corresponding author of this paper independently gave all contents of ChinaXiv:202507.00428, which include the contents from the beginning of this section to the "(□)" at the end of the proof of Proposition 7.3 (see ChinaXiv:202507.00428 at https://chinaxiv.org/abs/202507.00428). Then we gave the rest of this paper.

The corresponding author of this paper also independently gave at least the following contents of this paper: all sentences that contain the expression "the unique element of", Remarks 4.4, 4.5 and 4.6, and clauses (i), (ii), (iii) and (iv) of Section 6.

Proposition 7.4. Let $u \in \mathcal{F}(\mathbb{R})$ and $(a,b,c,d) \in$ Trap. Then $u = (a,b,c,d)$ if and only if for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a, c+(1-\xi)(d-c)]$.

Proof. We prove the "only if" part. Suppose $u = (a,b,c,d)$. Then, by Definition 3.2 and easy calculations, (1) holds (see also (I) below).

We prove the "if" part. Suppose (1) holds. By Proposition 7.3(i-2), $u(x) = \sup\{\alpha \in [0,1] : x \in [u]_\alpha\}$ for each $x \in \mathbb{R}$. From this, by easy calculations, we can obtain that $u(x) = (a,b,c,d)(x)$ for each $x \in \mathbb{R}$ (see also (II) below). This means $u = (a,b,c,d)$ as both u and (a,b,c,d) are in $\mathcal{F}(\mathbb{R})$.

Another proof of the "if" part is as follows. Suppose (1) holds. Note that the "only if" part says that for each $\xi \in [0,1]$, $[(a,b,c,d)]_\xi = [\xi(b-a)+a, c+(1-\xi)(d-c)]$. Thus (1) means that for each $\xi \in [0,1]$, $[u]_\xi = [(a,b,c,d)]_\xi$. As both u and (a,b,c,d) are in $\mathcal{F}(\mathbb{R})$, by Proposition 7.3(ii-1), $u = (a,b,c,d)$. This proof is based on the result of the "only if" part.

(I) One way to perform these calculations is to do it based on watching the graphs of the membership functions of (a,b,c,d). In this way, it is easy to calculate that for each $\xi \in [0,1]$, $[(a,b,c,d)]_\xi = [\xi(b-a)+a, c+(1-\xi)(d-c)]$ in all the four cases $a < b \le c < d$, $a = b \le c < d$, $a < b \le c = d$ and $a = b \le c = d$. So (1) holds as we suppose that $u = (a,b,c,d)$.

(II) One way to perform these calculations is to do it based on watching the graphs of the cut sets $[u]_\alpha$, $\alpha \in [0,1]$. In this way, it is easy to calculate that $u(x) = (a,b,c,d)(x)$ for each $x \in \mathbb{R}$ in all the four cases $a < b \le c < d$, $a = b \le c < d$, $a < b \le c = d$ and $a = b \le c = d$.

For $u \in \mathcal{F}(\mathbb{R})$, we call u a 1-dimensional compact fuzzy number if u has the following properties: (i) $[u]_1 \neq \emptyset$; and (ii) for each $\alpha \in [0,1]$, $[u]_\alpha$ is a compact interval of $\mathbb{R}$. The set of all 1-dimensional compact fuzzy numbers is denoted by E.

Let $u \in$ Trap. Denote $u = (a,b,c,d)$. By Proposition 7.4, $[u]_1 = [b,c] \neq \emptyset$ and for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a, c+(1-\xi)(d-c)]$ is a compact interval of $\mathbb{R}$. Also $u \in \mathcal{F}(\mathbb{R})$. Thus $u \in E$. So Trap $\subseteq E$. Below Example 7.5 shows that E Trap $\neq \emptyset$. Hence Trap $\subsetneq E$. So Tap $\subsetneq$ Trap $\subsetneq E$ and Tag $\subsetneq$ Trag $\subsetneq$ Trap $\subsetneq E$, where the first $\subsetneq$, the third $\subsetneq$ and the fourth $\subsetneq$ have already been given earlier in this paper.

Example 7.5. Define $u \in \mathcal{F}(\mathbb{R})$ by

$$u(x) = \begin{cases} e^{-x}, & \text{if } x \in [0,1], \\ 0, & \text{if } x \in \mathbb{R} \quad [0,1]. \end{cases}$$

Then

$$[u]_\alpha = \begin{cases} [0, -\ln\alpha], & \text{if } \alpha \in [e^{-1}, 1], \\ [0,1], & \text{if } \alpha \in [0, e^{-1}]. \end{cases}$$

Thus $[u]_1 = \{0\} \neq \emptyset$, and for each $\alpha \in [0,1]$, $[u]_\alpha$ is a compact interval of $\mathbb{R}$. So $u \in E$. We claim $u \notin \text{Trap}$. Suppose $u \in \text{Trap}$. Denote $u = (a,b,c,d)$. Then $\{0\} = [u]_1 = [b,c]$ and $[0,1] = [u]_0 = [a,d]$, where the second and fourth equalities follow from Proposition 7.4 or Theorem 4.1(i). So $a = b = c = 0$ and $d = 1$. Hence, by Proposition 7.4, $[u]_{e^{-1}} = [e^{-1}(0-0)+0, 0+(1-e^{-1})(1-0)] = [0, 1-e^{-1}]$. However, by (2), $[u]_{e^{-1}} = [0,1]$. This is a contradiction. Thus $u \notin \text{Trap}$.

Corollary 7.6. Let $u \in \mathscr{F}(\mathbb{R})$ and $(a,b,c,d) \in \text{Tap}$. Then $u = (a,b,c,d)$ if and only if for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a, c+(1-\xi)(d-c)]$.

Proof. Note that $(a,b,c,d) \in \text{Tap}$ implies $(a,b,c,d) \in \text{Trap}$ (see also (I) below). So the desired result follows immediately from Proposition 7.4. (I) Tap $\subsetneq$ Trap. This means that if $w \in \text{Tap}$ then $w \in \text{Trap}$ but the converse is false.

From the above proof of Corollary 7.6, we can see that Corollary 7.6 is a corollary of Proposition 7.4.

Proposition 7.7. Let $u \in \mathscr{F}(\mathbb{R})$ and $(a,b,c) \in \text{Trag}$. Then $u = (a,b,c)$ if and only if for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a, b+(1-\xi)(c-b)]$.

Proof. By Remark 3.4(ii), (a,b,c) is the (a,b,b,c) in Trap. And, by Proposition 7.4, $u = (a,b,b,c)$ if and only if for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a, b+(1-\xi)(c-b)]$. So we obtain the desired result.

From the above proof of Proposition 7.7, we can see that Proposition 7.7 is a corollary of Proposition 7.4.

Corollary 7.8. Let $u \in \mathscr{F}(\mathbb{R})$ and $(a,b,c) \in \text{Tag}$. Then $u = (a,b,c)$ if and only if for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a, b+(1-\xi)(c-b)]$.

Proof. $(a,b,c) \in \text{Tag}$ implies $(a,b,c) \in \text{Trag}$ (see also (I) below). So the desired result follows immediately from Proposition 7.7. (I) Tag $\subsetneq$ Trag. This means that if $w \in \text{Tag}$ then $w \in \text{Trag}$ but the converse is false.

From the above proof of Corollary 7.8, we can see that Corollary 7.8 is a corollary of Proposition 7.7. So Corollary 7.8 is a corollary of Proposition 7.4.

Theorem 7.9. Let u and v be in Trap. Then $u = v$ if and only if there exist two distinct elements λ and τ in $[0,1]$ with $[u]_\lambda = [v]_\lambda$ and $[u]_\tau = [v]_\tau$.

Proof. Clearly $u = v$ implies that $[u]_\xi = [v]_\xi$ for all $\xi \in [0,1]$. So the "only if" part is true.

Conversely, assume there exist two distinct elements λ and τ in $[0,1]$ with $[u]_\lambda = [v]_\lambda$ and $[u]_\tau = [v]_\tau$. Denote $u := (a,b,c,d)$ and $v := (a_1,b_1,c_1,d_1)$. Then

$$[\lambda(b-a)+a, c+(1-\lambda)(d-c)] = [u]_\lambda = [v]_\lambda = [\lambda(b_1-a_1)+a_1, c_1+(1-\lambda)(d_1-c_1)],$$

$$[\tau(b-a)+a, c+(1-\tau)(d-c)] = [u]_\tau = [v]_\tau = [\tau(b_1-a_1)+a_1, c_1+(1-\tau)(d_1-c_1)],$$

where the first and third equalities in (3) and the first and third equalities in (4) follow from Proposition 7.4. (3) implies (5) and (6) below. (4) implies (7) and (8) below.

$$\lambda(b-a)+a = \lambda(b_1-a_1)+a_1,$$
$$c+(1-\lambda)(d-c) = c_1+(1-\lambda)(d_1-c_1),$$
$$\tau(b-a)+a = \tau(b_1-a_1)+a_1,$$
$$c+(1-\tau)(d-c) = c_1+(1-\tau)(d_1-c_1).$$

We claim (a) (5) and (7) hold if and only if $a = a_1$ and $b = b_1$; and (b) (6) and (8) hold if and only if $c = c_1$ and $d = d_1$.

We show (a). Obviously $a = a_1$ and $b = b_1$ implies (5) and (7). Conversely, suppose (5) and (7) hold. Computing (5)−(7), we obtain (c) $(\lambda-\tau)(b-a) = (\lambda-\tau)(b_1-a_1)$. (c) is equivalent to (d) $(b-a) = (b_1-a_1)$, as $\lambda \neq \tau$. Computing (5)−$\lambda\cdot$(d), we obtain (e) $a = a_1$. Computing (d) + (e), we obtain $b = b_1$. Thus (a) is proved.

We show (b). Obviously $c = c_1$ and $d = d_1$ implies (6) and (8). Conversely, suppose (6) and (8) hold. Computing (6)−(8), we obtain $(\bar{c})$ $(\tau-\lambda)(d-c) = (\tau-\lambda)(d_1-c_1)$. $(\bar{c})$ is equivalent to $(\bar{d})$ $(d-c) = (d_1-c_1)$, as $\lambda \neq \tau$. Computing (6)$-(1-\lambda)\cdot(\bar{d})$, we obtain $(\bar{e})$ $c = c_1$. Computing $(\bar{d})+(\bar{e})$, we obtain $d = d_1$. Thus (b) is proved.

The above proofs of (a) and (b) are similar. See also (I) below.

By (5), (6), (7), (8), (a) and (b), we have $a = a_1$, $b = b_1$, $c = c_1$ and $d = d_1$. Then obviously $u = v$ (see also (II) below). So the "if" part is true.

The proof is completed. (I) We can also show (a) and (b) as follows. Below two proofs of (a) and (b) are similar.

The following is a proof of (a). We see (5) and (7) as a system of linear equations in 2 unknowns a and b. We use (A) to denote this system of linear equations. Clearly (a) means that $a = a_1$ and $b = b_1$ is the unique solution of (A). Obviously, by (5) and (7), $a = a_1$ and $b = b_1$ is a solution of (A). So to show (a), we only need to show that (A) has a unique solution.

We can write (A) as

$$\begin{cases} \lambda b+(1-\lambda)a = \lambda(b_1-a_1)+a_1, \\ \tau b+(1-\tau)a = \tau(b_1-a_1)+a_1. \end{cases}$$

We can see that (A) is square. Computing the determinant of the coefficient matrix of (A), we obtain

$$\begin{vmatrix} \lambda & 1-\lambda \\ \tau & 1-\tau \end{vmatrix} = \lambda(1-\tau)-(1-\lambda)\tau = \lambda-\tau \neq 0.$$

Thus (A) has a unique solution. So (a) is proved.

The following is a proof of (b). We see (6) and (8) as a system of linear equations in 2 unknowns c and d. We use (B) to denote this system of linear equations. Clearly (b) means that $c = c_1$ and $d = d_1$ is the unique solution of (B). Obviously, by (6) and (8), $c = c_1$ and $d = d_1$ is a solution of (B). So to show (b), we only need to show that (B) has a unique solution.

We can write (B) as

$$\begin{cases} \lambda c + (1-\lambda)d = c_1 + (1-\lambda)(d_1 - c_1), \\ \tau c + (1-\tau)d = c_1 + (1-\tau)(d_1 - c_1). \end{cases}$$

We can see that (B) is square. Computing the determinant of the coefficient matrix of (B), we obtain

$$\begin{vmatrix} \lambda & 1-\lambda \\ \tau & 1-\tau \end{vmatrix} = \lambda(1-\tau) - (1-\lambda)\tau = \lambda - \tau \neq 0.$$

Thus (B) has a unique solution. So (b) is proved.

(II) In fact, by Theorem 4.1(ii), $u = v$ is equivalent to $[a, b, c, d] = [a_1, b_1, c_1, d_1]$, which means $a = a_1$, $b = b_1$, $c = c_1$ and $d = d_1$. In other words, $u = v$ if and only if $a = a_1$, $b = b_1$, $c = c_1$ and $d = d_1$.

Corollary 7.10. (i) Let u and v be in Tap. Then $u = v$ if and only if there exist two distinct elements λ and τ in $[0, 1]$ with $[u]_\lambda = [v]_\lambda$ and $[u]_\tau = [v]_\tau$. (ii) Let u and v be in Trag. Then $u = v$ if and only if there exist two distinct elements λ and τ in $[0, 1]$ with $[u]_\lambda = [v]_\lambda$ and $[u]_\tau = [v]_\tau$. (iii) Let u and v be in Tag. Then $u = v$ if and only if there exist two distinct elements λ and τ in $[0, 1]$ with $[u]_\lambda = [v]_\lambda$ and $[u]_\tau = [v]_\tau$.

Proof. We show (i). u and v are in Tap implies u and v are in Trap. So (i) follows immediately from Theorem 7.9. (This proof indicates that (i) is a corollary of Theorem 7.9.)

We show (ii). u and v are in Trag implies u and v are in Trap (see also (I) below). So (ii) follows immediately from Theorem 7.9. (This proof indicates that (ii) is a corollary of Theorem 7.9.)

We show (iii). u and v are in Tag implies u and v are in Trag. So (iii) follows immediately from (ii). (This proof indicates that (iii) is a corollary of (ii). So (iii) is a corollary of Theorem 7.9.) (I) Trag $\subsetneq$ Trap. This means that if $w \in$ Trag then $w \in$ Trap but the converse is false.

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Note: Figure translations are in progress. See original paper for figures.