Space -charge driven resonant beam halo in high-intensity hadron synchrotrons

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Abstract

With the significant development of high-intensity hadron (proton and heavy ion) accelerator facilities, the space charge effect has become a major limiting factor for increasing beam intensity because it can drive particle resonance, forming beam halos and causing beam quality degradation or even beam loss. In studies on space charge, the particle-core model (PCM) has been widely adopted to describe halo particle formation. In this paper, we generalize the conventional PCM to include dispersion to investigate the physical mechanism of the beam halo in high-intensity synchrotrons. In particular, a "1:1 parametric resonance" driven by the combined effects of space charge and dispersion is identified. A large dispersion is proven to have a damping effect on the 2:1 parametric resonance. The analysis based on the generalized PCM agrees with particle-in-cell simulations. A beam halo with large mismatch oscillations is also discussed.

Full Text

Preamble

Space-Charge Driven Resonant Beam Halo in High-Intensity Hadron Synchrotrons

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With the significant development of high-intensity hadron (proton and heavy ion) accelerator facilities worldwide, the space charge effect has emerged as a major limiting factor for increasing beam intensity. Space charge can drive particle resonances, leading to beam halo formation and causing beam quality degradation or even beam loss. In studies of space charge dynamics, the particle-core model (PCM) has been widely adopted to describe halo particle formation. In this paper, we generalize the conventional PCM to include dispersion effects, enabling investigation of the physical mechanisms underlying beam halo formation in high-intensity synchrotrons. In particular, we identify a novel "1:1 parametric resonance" driven by the combined effects of space charge and dispersion. We also demonstrate that large dispersion has a damping effect on the 2:1 parametric resonance. Our analytical results based on the generalized PCM show excellent agreement with particle-in-cell simulations. Additionally, we discuss beam halo formation in the presence of large mismatch oscillations.

Keywords: Particle-core model, Space charge, Beam halo

Introduction

In recent years, an increasing number of high-intensity proton and heavy-ion accelerators have been proposed, are under construction, or have begun operation worldwide for various scientific and industrial applications. For such high-intensity machines, understanding the mechanisms of intense beam dynamics and controlling beam losses are crucial for both machine design and operation. One of the most important contributors to beam quality degradation is the formation of beam halos driven by space charge effects [?]. The primary driving mechanism for beam halo formation is parametric resonance, which can cause severe beam losses [?]. Uncontrolled beam losses can lead to serious consequences, including residual activation of beam pipes, quenching of superconducting magnets, vacuum degradation, and radiation damage to insulation materials [?].

Among the various approaches for studying beam halos, the particle-core model (PCM) has been one of the most widely employed analytical tools [?]. Compared with particle-in-cell (PIC) simulations, the PCM provides an analytical framework for investigating beam dynamics, particularly the mechanisms of halo formation, without requiring extensive particle tracking that can be computationally expensive. In this model, the dynamic behavior of an intense beam core is described by the evolution of beam envelopes [?, ?], while the motion of individual test particles is influenced by the space charge fields generated by beam-core mismatch oscillations [?]. The situation becomes more complex in high-intensity hadron synchrotrons, where the combined effects of space charge and dispersion influence the motion of circulating beams [?, ?, ?, ?]. To properly analyze beam halo formation in circular machines, the conventional PCM must be generalized to include dispersion effects.

In this paper, we investigate the dynamics of halo particles under both moderate and strong space charge conditions, where 2:1 and higher-order resonances, or even chaotic behavior, can exist. Building upon the generalization of the conventional PCM to include dispersion effects for high-intensity synchrotrons, we identify a novel "1:1 parametric resonance" driven by the dispersion mode. We also explain the damping effect observed by Ikegami et al. [?] from the perspective of oscillation modes. Furthermore, for beams with large mismatch oscillations, we discuss both high-order and low-order resonances driven by corresponding beam-core oscillation modes.

The remainder of this paper is organized as follows. Following this introduction, we briefly review the fundamentals of the PCM method in Sec. II. Section III investigates single-particle dynamics for round and elliptical beams. In Sec. IV, we generalize the PCM to include dispersion and discuss in detail the 1:1 parametric resonance driven by the dispersion mode. Section V presents an analysis of high-order modes in large beam mismatch oscillations. Finally, Sec. VI provides a summary of our findings.

II. Fundamentals

A. Beam-Core Oscillations

In the PCM, beams are assumed to have uniform spatial density in the transverse plane (KV distribution) because the dynamics of individual particles are insensitive to the detailed distribution of the beam core. An envelope approach is employed to describe mismatch oscillations of the beam core, with beam halo formation driven by space-charge interactions between collective envelope oscillation modes and single particles.

We begin with a coasting beam propagating through a uniformly focusing structure, which can describe the average dynamic behavior of beams in an alternating-gradient focusing channel [?] (the smooth approximation method). For simplicity, we neglect impedance effects from the beam pipe and all chromatic terms. We adopt x and y to represent the transverse degrees of freedom in the horizontal and vertical directions, respectively, and s as the longitudinal coordinate. The "pseudo" Hamiltonian for beam envelope oscillations in such a transport system is:

$$H_{\text{env}} = p_x^2 + p_y^2 + \kappa_{x,0}^2\sigma_x^2 + \kappa_{y,0}^2\sigma_y^2 + \frac{K_{\text{sc}}}{\sigma_x+\sigma_y} + \frac{\varepsilon_x^2}{\sigma_x^2} + \frac{\varepsilon_y^2}{\sigma_y^2}$$

where $\sigma_{x,y}$ represents the RMS transverse beam size (for KV beams, the total transverse beam size is $2\sigma_{x,y}$). The derivatives $\sigma_{p_{x,y}}$ are conjugate variables defined by $d\sigma_{x,y}/ds = \sigma_{p_{x,y}}$. The parameters $\kappa_{x,0}$ and $\kappa_{y,0}$ are the external transverse focusing gradients in x and y, respectively, while ε_x and ε_y are the transverse RMS emittances. The space charge perveance K_{sc} is defined as $K_{\text{sc}} = 2N_L r_c/(\beta^2\gamma^3)$, where N_L is the number of particles per unit length, r_c is the classical proton radius, and β and γ are relativistic factors.

The RMS envelope equations derived from this Hamiltonian are:

$$\frac{d^2\sigma_x}{ds^2} + \kappa_{x,0}^2\sigma_x - \frac{K_{\rm sc}}{\sigma_x+\sigma_y} - \frac{\varepsilon_x^2}{\sigma_x^3} = 0$$

$$\frac{d^2\sigma_y}{ds^2} + \kappa_{y,0}^2\sigma_y - \frac{K_{\rm sc}}{\sigma_x+\sigma_y} - \frac{\varepsilon_y^2}{\sigma_y^3} = 0$$

Under constant focusing, the matched RMS beam sizes $\sigma_{x,m}$ and $\sigma_{y,m}$ can be obtained straightforwardly from the corresponding algebraic equations:

$$\kappa_{x,0}^2\sigma_{x,m} - \frac{K_{\rm sc}}{\sigma_{x,m}+\sigma_{y,m}} - \frac{\varepsilon_x^2}{\sigma_{x,m}^3} = 0$$

$$\kappa_{y,0}^2\sigma_{y,m} - \frac{K_{\rm sc}}{\sigma_{x,m}+\sigma_{y,m}} - \frac{\varepsilon_y^2}{\sigma_{y,m}^3} = 0$$

Here, the subscript "m" denotes the matched case.

These envelope equations can be converted into dimensionless form using the following variables and parameters:

$$\hat\sigma_x = \sigma_x/\sigma_{x,m}, \quad \hat\sigma_y = \sigma_y/\sigma_{x,m}, \quad \hat\sigma_{p_x} = \sigma_{p_x}/(\kappa_{x,0}\sigma_{x,m}), \quad \hat\sigma_{p_y} = \sigma_{p_y}/(\kappa_{x,0}\sigma_{x,m}), \quad \tau = \kappa_{x,0}s$$

$$\mu_x = \frac{\Delta\kappa_x^2}{\kappa_{x,0}^2}, \quad r = \frac{\sigma_{y,m}}{\sigma_{x,m}}, \quad \eta = \frac{\kappa_{y,0}}{\kappa_{x,0}}, \quad \varepsilon_r = \frac{\varepsilon_y}{\varepsilon_x}$$

where $\Delta\kappa_x^2 = K_{\rm sc}/[2\sigma_{x,m}(\sigma_{x,m}+\sigma_{y,m})]$ represents the space-charge tune depression in the matched case. For constant focusing, the dimensionless matched beam sizes are $\hat\sigma_{x,m} = 1$ and $\hat\sigma_{y,m} = r$.

The dimensionless Hamiltonian and envelope equations become:

$$\hat H_{\rm env} = \hat p_x^2 + \hat p_y^2 + \hat\sigma_x^2 + \eta^2\hat\sigma_y^2 - \mu_x(1+r)\ln(\hat\sigma_x+\hat\sigma_y) + \frac{1-\mu_x}{2\hat\sigma_x^2} + \frac{r(1-\mu_x)}{2\hat\sigma_y^2}$$

$$\frac{d^2\hat\sigma_x}{d\tau^2} + \hat\sigma_x - \frac{\mu_x(1+r)}{\hat\sigma_x+\hat\sigma_y} - \frac{1-\mu_x}{\hat\sigma_x^3} = 0$$

$$\frac{d^2\hat\sigma_y}{d\tau^2} + \eta^2\hat\sigma_y - \frac{\mu_x(1+r)}{\hat\sigma_x+\hat\sigma_y} - \frac{r(1-\mu_x)}{\hat\sigma_y^3} = 0$$

In practice, beams are never perfectly matched due to magnetic errors or misalignment of lattice elements. Consequently, $\{\hat{\sigma}_x, \hat{\sigma}_y\}$ differs slightly from the matched solution $\{\hat{\sigma}_{x,m}, \hat{\sigma}_{y,m}\}$, a condition referred to as beam-coherent mismatch oscillation. In the PCM, such mismatch oscillations provide the mechanism for energy transfer from the beam core to single particles via space charge, forming halo particles when resonance occurs.

A mismatched beam in a constant-focusing channel can be expressed as small perturbations $(\xi, \zeta, \xi_p, \zeta_p)$ on the matched solutions:

$$\hat{\sigma}_x = \hat{\sigma}_{x,m} + \xi = 1 + \xi, \quad \hat{\sigma}_y = \hat{\sigma}_{y,m} + \zeta = r + \zeta, \quad \hat{\sigma}_{p_x} = \hat{\sigma}_{p_x,m} + \xi_p, \quad \hat{\sigma}_{p_y} = \hat{\sigma}_{p_y,m} + \zeta_p$$

Substituting these into the Hamiltonian yields the Hamiltonian for envelope perturbations:

$$\hat{H}_{\mathrm{per}} = (\hat{\sigma}_{p_x,m} + \xi_p)^2 + (\hat{\sigma}_{p_y,m} + \zeta_p)^2 + (1+\xi)^2 + \eta^2 (r+\zeta)^2 - \mu_x (1+r) \ln(1+r+\xi+\zeta) + \frac{1-\mu_x}{2(1+\xi)^2} + \frac{r(1-\mu_x)}{2(r+\zeta)^2}$$

By performing a Taylor expansion and keeping linear terms, we obtain the equations of motion for envelope perturbations in matrix form (discussed further in Appendix A):

$$\frac{d^2}{d\tau^2}\begin{pmatrix}\xi \\ \zeta\end{pmatrix} + \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}\begin{pmatrix}\xi \\ \zeta\end{pmatrix} = 0$$

with coefficients:

$$a_0 = 4(1-\mu_x) + \frac{r+2}{r+1}, \quad a_1 = -\frac{1}{r+1}, \quad a_2 = 4\left(\eta^2 - \frac{2r+1}{r+1}\right)$$

B. Single-Particle Motion with Space Charge

The motion of a single test particle under the influence of the beam core' s space charge is governed by:

$$\frac{d^2 x}{ds^2} + \kappa_{x,0}^2 x = \frac{2K_{\mathrm{sc}} x}{\sigma_x(\sigma_x + \sigma_y)} \quad \text{for } |x| < 2\sigma_x$$

$$\frac{d^2 x}{ds^2} + \kappa_{x,0}^2 x = \frac{2K_{\mathrm{sc}} x}{\sqrt{x^2 + 4(\sigma_y^2 - \sigma_x^2)}\left(\sqrt{x^2 + 4(\sigma_y^2 - \sigma_x^2)} + 2\sigma_x\right)} \quad \text{for } |x| > 2\sigma_x$$

where x is the horizontal displacement from the beam core center. Using the dimensionless parameters defined previously, these equations become:

$$\frac{d^2\hat{x}}{d\tau^2} + \hat{x} = \frac{\mu_x(1+r)\hat{x}}{\hat{\sigma}_x(\hat{\sigma}_x + \hat{\sigma}_y)} \quad \text{for } |\hat{x}| < \hat{\sigma}_x$$

$$\frac{d^2\hat{x}}{d\tau^2} + \hat{x} = \frac{\mu_x(1+r)\hat{x}}{\sqrt{\hat{x}^2 + (\hat{\sigma}_y^2 - \hat{\sigma}_x^2)}(\sqrt{\hat{x}^2 + (\hat{\sigma}_y^2 - \hat{\sigma}_x^2)} + \hat{\sigma}_x)} \quad \text{for } |\hat{x}| > \hat{\sigma}_x$$

where the dimensionless horizontal displacement is defined as $\hat{x} = x/(2\sigma_{x,m})$. These equations show that a test particle experiences different wavenumbers when traveling inside versus outside the beam core due to varying space charge strength.

For particles inside the beam core, the wavenumber reaches its minimum:

$$k_{p,\min} = \sqrt{1-\mu_x}$$

Substituting the perturbation expressions into the particle motion equation yields:

$$\frac{d^2\hat{x}}{d\tau^2} + (1-\mu_x)\hat{x} = f(\tau)$$

where $f(\tau)$ describes the oscillation of the beam core:

$$f(\tau) = -\frac{\mu_x}{1+r}\left\{[(2+r)C_{11} + C_{21}]\cos(k_b\tau) + [(2+r)C_{12} + C_{22}]\cos(k_q\tau)\right\}$$

The coefficient matrix in the envelope perturbation equations is symmetric and can be decomposed as $A = U \cdot \mathrm{diag}\{k_b^2, k_q^2\} \cdot U^T$, where U is the eigenvector matrix. Here, k_b and k_q represent the wavenumbers of the "breathing mode" and "quadrupole mode," respectively. The general solutions are:

$$\xi(\tau) = C_{11}\cos(k_b\tau) + C_{12}\cos(k_q\tau)$$

$$\zeta(\tau) = C_{21}\cos(k_b\tau) + C_{22}\cos(k_q\tau)$$

with coefficients $C_{ij} = \sum_{k=1}^{2} U_{ij}U_{jk}^T\alpha_k$, where $\alpha_1 = \xi(0)$ and $\alpha_2 = \zeta(0)$. We use τ as the independent variable and consider mismatch without initial momentum perturbations ($\xi_p(0) = \zeta_p(0) = 0$), so the solutions contain only cosine terms. The envelope oscillation patterns depend on initial conditions: a pure breathing mode occurs when $\zeta(0) = -(U_{21}/U_{22})\xi(0)$, while a pure quadrupole mode occurs when $\zeta(0) = -(U_{11}/U_{12})\xi(0)$. Generally, beam-core oscillations are characterized by a superposition of these two modes.

When the wavenumbers of a test particle and the beam core satisfy the resonance condition $k_p = k_b/2$ or $k_p = k_q/2$, a 2:1 parametric resonance occurs, leading to beam halo formation [**?**]. To illustrate this, we plot wavenumbers as functions of beam current (in units of normalized space-charge tune depression μ_x) for two representative cases: round and elliptical beams [Figure 1: see original paper].

For round beams in a symmetric focusing channel with initial equal mismatch perturbations ($\xi(0) = \zeta(0) = 0.05$), only the breathing mode is excited. We select four test particles with different initial dimensionless horizontal displacements $\hat{x}_1 < 1.0 < \hat{x}_2 < \hat{x}_3 < \hat{x}_4$ (with zero momentum) and numerically solve the equations of motion. The resulting Poincaré maps for moderate ($\mu_x = 0.1$) and strong ($\mu_x = 0.8$) space charge are shown in [Figure 2: see original paper].

For $\mu_x = 0.1$, particles with initial positions $\hat{x}_1$ and $\hat{x}_2$ exhibit regular elliptical motion, while the third particle with initial position $\hat{x}_3$ shows large excursions, indicating resonance. The "lock" of the particle' s wavenumber at $k = k_b/2$ is characteristic of parametric resonance. PIC simulations using PyORBIT with 16 equal cells confirm these numerical results .

For $\mu_x = 0.8$, the dynamics become more complex. In addition to the 2:1 resonance island, three smaller islands appear, corresponding to a 3:1 parametric resonance where the wavenumber locks at $k = (1/3)k_b$. The 3:1 resonance is much weaker than the 2:1 resonance, as evidenced by the smaller island areas. The condition $k_{p,\min} < k_b/3 < 3k_b/8 < k_{p,\max}$ supports the existence of both 3:1 and 8:3 resonance islands, while $k_b/4 < k_{p,\min}$ shows that 4:1 or higher resonances cannot be excited [Figure 3: see original paper].

For elliptical beams ($r \neq 1$), the wavenumber analysis reveals that for $\mu_x > 0.39$, both breathing and quadrupole modes can drive 2:1 resonance. For $\mu_x < 0.39$, only the quadrupole mode can drive resonance since $k_b/2 > k_{p,\max}$ [FIGURE:1(b)]. When both modes are present simultaneously (mixed modes), chaotic behavior appears in addition to the resonance islands [Figure 4: see original paper].

III. Resonance and Chaos in the Beam Halo

In the presence of space charge, particle motion around the beam core is periodic. When resonance occurs, particles can absorb energy from the beam core and develop much larger amplitudes, forming halo particles. Furthermore, under strong space charge, particles exhibit chaotic behavior due to the superposition of different modes. This section investigates 2:1 and higher-order parametric resonances and chaos in beam halo formation in detail.

For round beams with mixed modes ($\xi(0) \neq \zeta(0)$), both breathing and quadrupole modes can drive 2:1 parametric resonance when the test particle wavenumber satisfies $k = k_q/2$ or $k = k_b/2$ [Figure 4: see original paper]. Chaotic phenomena appear in the mixed-mode case, characterized by random particle wavenumbers, in contrast to the "lock" characteristic of parametric resonance. PIC simulations for moderate space charge ($\mu_x = 0.1$) show good agreement with numerical calculations, while some discrepancy appears for strong space charge ($\mu_x = 0.8$) due to disturbances of the uniform particle distribution during self-consistent tracking.

For elliptical beams, moderate space charge ($\mu_x = 0.1$) allows only the quadrupole mode to induce 2:1 resonance. As space charge increases ($\mu_x \geq 0.5$), both quadrupole and breathing modes can drive resonance, with the two resonance islands approaching each other. For strong space charge ($\mu_x = 0.8$), the proximity of the two resonance islands causes chaotic phenomena around them [Figure 5: see original paper]. PIC simulations using parameters from confirm the numerical calculations.

IV. Beam Halo Formation in High-Intensity Synchrotrons

This section investigates beam halo formation driven by resonant interactions between single particles and the beam core in high-intensity synchrotrons, where the combined effects of space charge and dispersion must be considered [?, ?, ?, ?]. We generalize the conventional PCM to include dispersion, noting that the mechanism discussed here differs from space-charge structural resonances driven by high-order terms in the space-charge potential [?].

A. Generalized PCM with Dispersion

For beams traveling in a constant-focusing bending channel, the transverse beam dynamics can be described by envelope equations including the dispersion function [?, ?]:

$$\frac{d^2\sigma_x}{ds^2} + \kappa_{x,0}^2\sigma_x - \frac{K_{\mathrm{sc}}}{2X(X+Y)} - \frac{\varepsilon_x^2}{\sigma_x^3} = 0$$

$$\frac{d^2\sigma_y}{ds^2} + \kappa_{y,0}^2\sigma_y - \frac{K_{\mathrm{sc}}}{2Y(X+Y)} - \frac{\varepsilon_y^2}{\sigma_y^3} = 0$$

$$\frac{d^2 D_\delta}{ds^2} + \kappa_{x,0}^2 D_\delta - \frac{K_{\mathrm{sc}}}{2X(X+Y)} = 0$$

where $X = \sqrt{\sigma_x^2 + D_\delta^2}$ is the total RMS horizontal beam size, σ_x is the betatron beam size, and $D_\delta \equiv D_x\sigma_p$ is the "dispersion beam size." The subscript "d" denotes the case with dispersion.

These equations can be derived from the dispersion-modified envelope Hamiltonian:

$$H_{\text{env},d} = p_{\sigma_x}^2 + p_{\sigma_y}^2 + p_{D_\delta}^2 + V_{\text{env},d}(\sigma_x, \sigma_y, D_\delta)$$

with

$$V_{\text{env},d} = \kappa_{x,0}^2\sigma_x^2 + \kappa_{y,0}^2\sigma_y^2 + \kappa_{x,0}^2 D_\delta^2 - K_{\text{sc}}\ln(X+Y) + \frac{\varepsilon_x^2}{\sigma_x^2} + \frac{\varepsilon_y^2}{\sigma_y^2}$$

The matched equations with dispersion are:

$$\kappa_{x,0}^2\sigma_{x,m} - \frac{K_{\text{sc}}}{2X_m(X_m+Y_m)} - \frac{\varepsilon_x^2}{\sigma_{x,m}^3} = 0$$

$$\kappa_{y,0}^2\sigma_{y,m} - \frac{K_{\text{sc}}}{2Y_m(X_m+Y_m)} - \frac{\varepsilon_y^2}{\sigma_{y,m}^3} = 0$$

$$\kappa_{x,0}^2 D_{\delta,m} - \frac{K_{\text{sc}}}{2X_m(X_m+Y_m)} = 0$$

where X_m, Y_m, and $D_{\delta,m}$ are the matched solutions.

Using the dimensionless parameters from Eq. (4), the Hamiltonian can be rewritten as:

$$\hat{H}_{d,\text{env}} = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_{D_\delta}^2 + \hat{\sigma}_x^2 + \eta^2\hat{\sigma}_y^2 + \hat{D}_\delta^2 - \mu_{x,d}(1+R)\ln(\hat{X}+\hat{Y}) + \frac{1-\mu_{x,d}}{2\hat{\sigma}_x^2} + \frac{r_d(1-\mu_{x,d})}{2\hat{\sigma}_y^2} + \frac{(1-\mu_{x,d})\cos^2\theta}{2\sin^2\theta}$$

The dimensionless variables related to dispersion are defined as:

$$\hat{X} = X/\sigma_{x,m}, \quad \hat{Y} = Y/\sigma_{x,m}, \quad \hat{D}_\delta = D_\delta/\sigma_{x,m}$$

$$\sin\theta = \sigma_{x,m}/X_m, \quad \cos\theta = D_{\delta,m}/X_m, \quad R = Y_m/X_m$$

$$\varepsilon_{r,d} = \varepsilon_{dy}/\varepsilon_{dx}, \quad \mu_{x,d} = \Delta\kappa_{x,d}^2/\kappa_{x,0}^2$$

where $\sin\theta$ is the betatron ratio, $\cos\theta$ is the dispersion ratio, and $\Delta\kappa_{x,d}^2 = K_{\text{sc}}/[2X_m(X_m+\sigma_{y,m})]$ represents the space-charge tune depression with dispersion.

We introduce the "dispersion strength" $\Lambda \equiv D_{\delta,m}^{(0)}/\sigma_{x,m}^{(0)}$ to characterize the ratio of dispersion motion to betatron motion in the zero-current case. For typical synchrotrons such as the CSNS RCS [?, ?], where energy spread is typically less than 1%, we have $0 < \Lambda < 1.0$.

The dimensionless dispersion-modified envelope equations are:

$$\frac{d^2\hat{\sigma}_x}{d\tau^2} + \hat{\sigma}_x - \frac{\mu_{x,d}(1+R)\hat{\sigma}_x}{\hat{X}(\hat{X}+\hat{Y})\sin^2\theta} - \frac{1-\mu_{x,d}}{\hat{\sigma}_x^3} = 0$$

$$\frac{d^2\hat{\sigma}_y}{d\tau^2} + \eta^2\hat{\sigma}_y - \frac{\mu_{x,d}(1+R)\hat{\sigma}_y}{\hat{Y}(\hat{X}+\hat{Y})} - \frac{r_d(1-\mu_{x,d})}{\hat{\sigma}_y^3} = 0$$

$$\frac{d^2\hat{D}_\delta}{d\tau^2} + \hat{D}_\delta - \frac{\mu_{x,d}(1+R)\hat{D}_\delta}{\hat{X}(\hat{X}+\hat{Y})\sin^2\theta} - \frac{(1-\mu_{x,d})\cos\theta}{\sin\theta} = 0$$

The matched beam sizes are $\hat{\sigma}_{x,m} = 1$, $\hat{\sigma}_{y,m} = r$, $\hat{D}_{\delta,m} = \cot\theta$, and $\hat{X}_m = \csc\theta$. Substituting mismatch perturbations $\hat{\sigma}_x = 1+\xi$, $\hat{\sigma}_y = r+\zeta$, and $\hat{D}_\delta = \cot\theta + d$ into these equations yields the perturbation equations (detailed in Appendix B):

$$\frac{d^2}{d\tau^2}\begin{pmatrix}\xi\\ \zeta\\ d\end{pmatrix} + \begin{pmatrix} b_0 & b_1 & b_2\\ b_1 & b_3 & b_4\\ b_2 & b_4 & b_5\end{pmatrix}\begin{pmatrix}\xi\\ \zeta\\ d\end{pmatrix} = 0$$

The symmetric coefficient matrix $B = \{b_{ij}\}$ can be decomposed as $B = U_d \cdot \mathrm{diag}\{k_b^2, k_q^2, k_d^2\} \cdot U_d^T$, where the dispersion mode k_d can be identified. The general solution is:

$$\xi = D_{11}\cos(k_b\tau) + D_{12}\cos(k_q\tau) + D_{13}\cos(k_d\tau)$$

$$\zeta = D_{21}\cos(k_b\tau) + D_{22}\cos(k_q\tau) + D_{23}\cos(k_d\tau)$$

$$d = D_{31}\cos(k_b\tau) + D_{32}\cos(k_q\tau) + D_{33}\cos(k_d\tau)$$

where $D_{ij} = \sum_{k=1}^{3} U_{d,ij}U_{d,jk}^T\alpha_k$ with $\alpha_1 = \xi(0)$, $\alpha_2 = \zeta(0)$, and $\alpha_3 = d(0)$.

In the presence of space charge and dispersion, single-particle motion is governed by:

$$\frac{d^2\hat{x}_d}{d\tau^2} + \hat{x}_d = \frac{\mu_{d,x}(1+R)\sin^2\theta\hat{x}_d}{\hat{X}(\hat{X}+\hat{Y})} \quad \text{for } |\hat{x}_d| < \hat{X}\sin\theta$$

$$\frac{d^2\hat{x}_d}{d\tau^2}+\hat{x}_d=\frac{\mu_{d,x}(1+R)\hat{x}_d}{\sqrt{\hat{x}_d^2+(\hat{Y}^2-\hat{X}^2)\sin^2\theta}(\sqrt{\hat{x}_d^2+(\hat{Y}^2-\hat{X}^2)\sin^2\theta+\hat{X}})} \quad \text{for } |\hat{x}_d|>\hat{X}\sin\theta$$

where $\hat{x}_d = x/(2X_m)$ and $\hat{\delta} = \delta/\sigma_p$ is the momentum spread ratio. For $\sigma_p = 0$ ($\Lambda = 0$, $\sin\theta = 1$, $R = r$), these equations reduce to the dispersion-free case.

B. Dispersion-Induced 1:1 Parametric Resonance

An important question is whether the dispersion mode can excite 2:1 parametric resonance and induce beam halo, similar to the envelope modes. Analysis shows that the half-wavenumber of the dispersion mode is always below the lower limit of test particle wavenumbers, indicating that the dispersion mode cannot induce 2:1 resonance [Figure 6: see original paper]. However, since $k_{p,\min} < k_d < k_{p,\max}$, the dispersion mode can generate a "1:1" parametric resonance.

For pure modes (only one oscillation mode present), parametric resonances can be analyzed using the dispersion-modified PCM. Two resonance islands appear for breathing and quadrupole modes (2:1 resonance), while a "crescent moon" island appears for the dispersion mode, identified as 1:1 resonance [Figure 7: see original paper]. In mixed-mode cases, both 2:1 resonances from envelope modes and 1:1 resonance from the dispersion mode coexist, with chaos observed around the islands.

The 1:1 resonance is clearly demonstrated in [Figure 8: see original paper], where the particle wavenumber locks to the dispersion mode wavenumber within the crescent moon island. PIC simulations using parameters from confirm the numerical calculations.

C. Alleviation of Beam Halo Using Strong Dispersion

1. Motion of Single Particles with Zero Momentum Deviation For synchronous particles ($\hat{\delta} = 0$) in high-intensity synchrotrons, strong dispersion ($\Lambda = 1.0$) can suppress resonance. For $\mu_x < 0.66$, the quadrupole mode cannot excite 2:1 resonance because $k_q/2 < k_{p,\min}$, while the breathing mode still can. The areas of 2:1 resonance islands decrease with increasing dispersion strength, and for $\Lambda = 1.0$, the quadrupole-driven resonance disappears entirely [Figure 9: see original paper]. The dispersion mode itself cannot induce 2:1 resonance.

For mixed modes, chaos weakens as dispersion strength increases [Figure 10: see original paper]. PIC simulations for $\Lambda = 0.2$ and $\Lambda = 1.0$ show good agreement with PCM calculations.

2. Motion of Single Particles with Large Momentum Deviation For particles with large momentum deviation, the stable fixed point (SFP) can move outside the beam core due to dispersion effects. As shown in [Figure 11: see original paper], for sufficiently large $\hat{\delta}$, the SFPs are located outside the beam core. Scanning the maximum excursion of "edge particles"(initially at $\hat{x}_d(0) = 1$) reveals that for particles with outer SFPs ($\hat{x}_{\max} > 1$), we obtain $k_{p,\min} > 1-\mu_x$. For large enough $\hat{\delta}$, $k_{p,\min} > k_b/2$, meaning the 2:1 resonance cannot be driven by the breathing mode. Thus, large off-momentum deviations can dampen the 2:1 resonance.

For $\hat{\delta} = 3.0$ and $\Lambda = 1.0$, only the 1:1 resonance driven by the dispersion mode is observed, while breathing and quadrupole mode resonances are suppressed [Figure 13: see original paper]. PIC simulations confirm these numerical results.

V. High-Order Mode in Large Mismatch Oscillations

Previous analyses were based on perturbation theory with small-amplitude oscillations, neglecting high-order terms. This section investigates single-particle motion driven by large-amplitude beam-core mismatch oscillations, where high-order modes become significant. We distinguish these from high-order resonances of low-order modes (e.g., the 8:3 resonance in [Figure 2: see original paper] is an eighth-order resonance driven by the low-order breathing mode).

For a round beam in a symmetric focusing channel, the envelope equation can be written as:

$$\frac{d^2\hat{\sigma}}{d\tau^2} + \hat{\sigma} - \frac{1-\mu}{\hat{\sigma}^3} = 0$$

The solution can be expressed as a Fourier series:

$$\hat{\sigma} = 1 + a_1 \cos(2\Omega\tau) + a_2 \cos(4\Omega\tau) + \cdots + a_n \cos(2n\Omega\tau)$$

where 2Ω is the breathing mode wavenumber (replacing k_b for convenience) and $2n\Omega$ ($n > 1$) represents high-order modes.

The single-particle equations of motion can be approximated by a cubic term [**?**]:

$$\frac{d^2\hat{x}}{d\tau^2} + \hat{x} - \mu\left(\frac{\hat{x}}{\hat{\sigma}^2} - \frac{\hat{x}^3}{\hat{\sigma}^4}\right) = 0$$

Substituting the series solution yields the Hamiltonian:

$$\hat{H}_L = \frac{1}{2}\left(P^2 + \omega^2\hat{x}^2\right) + \sum_{i=1}^{n} h_i \cos(2i\Omega\tau)\hat{x}^2 + \alpha\hat{x}^4$$

with parameters $\omega^2 = 1-\mu$, $h_i = -2\mu a_i/(1-\mu)$, and $\alpha = \mu/4$.

Using a canonical transformation, the Hamiltonian becomes $\hat{H}_L = \hat{H}_{L,0} + \hat{H}_{L,1} + \cdots + \hat{H}_{L,n}$, where $\hat{H}_{L,0}$ and $\hat{H}_{L,n}$ $(n > 0)$ represent the lowest-order (breathing) and high-order modes, respectively. Averaging over one oscillation period $T = 2\pi/\Omega$ gives $\langle \hat{H}_{L,n} \rangle = 0$ for $n > 0$ (detailed in Appendix C), indicating that high-order modes do not contribute to the 2:1 resonance.

Numerical solution of the envelope equation with 40% mismatch ($\hat{\sigma}(0) = 0.6$) confirms the presence of high-order modes up to sixth order, with frequencies that are multiples of the fundamental mode and amplitudes that decrease with increasing order [Figure 14: see original paper]. The Poincaré section for single particles under 40% mismatch shows 2:1 resonance islands driven by the lowest-order mode ($k = k_b/2$), with no evidence of resonance driven by high-order modes [Figure 15: see original paper]. A PIC simulation with 200,000 macro particles tracked for 500 turns shows a "peanut" shape halo distribution that agrees with the numerically calculated Poincaré section contour.

Chaos formation can be analyzed by distinguishing two types of particle-core resonances: (1) low-order resonances driven by high-order beam oscillation modes (proven non-existent by Eq. (44)), and (2) high-order resonances driven by low-order beam oscillation modes. Using perturbation theory with 40% mismatch while neglecting high-order modes (since they don' t contribute to resonances), we identify several high-order resonance islands: 3:1, 4:1, skew 4:1, 7:2, and 8:3 [Figure 16: see original paper]. The chaos region can be divided into inner and outer regions: outer chaos arises from mixing of the 2:1 resonance with higher-order resonances (3:1 and 8:3), while inner chaos near the beam core is weaker and caused by mixing of high-order resonances (4:1 and skew 4:1).

VI. Summary

We have analyzed beam halo formation driven by parametric resonance between single particles and the beam core in high-intensity synchrotrons. In the absence of dispersion, we observe several high-order resonances in addition to the dominant 2:1 resonance, with chaos arising from the mixture of parametric resonances that can be weakened by elliptical beam asymmetry. In the presence of combined space charge and dispersion effects, we find that the dispersion mode can drive a 1:1 parametric resonance and have discussed its physical mechanism in detail. Additionally, we have demonstrated that strong dispersion can alleviate beam halo formation.

For large-mismatch oscillations, we proved that while higher-order modes exist, they are unable to drive 2:1 parametric resonance. We expect that the 1:1 parametric resonance will have important implications for the design and operation of high-intensity synchrotrons. Further investigation of the role of synchrotron motion in beam halo formation is warranted.

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Appendices

Appendix A: Equations of Motion for Envelope Perturbations

For the Hamiltonian of envelope perturbations in Sec. II:

$$\hat{H}_{\text{per}} = (\hat{\sigma}_{p_x,m}+\xi_p)^2+(\hat{\sigma}_{p_y,m}+\zeta_p)^2+(1+\xi)^2+\eta^2(r+\zeta)^2-\mu_x(1+r)\ln(1+r+\xi+\zeta)+\frac{1-\mu_x}{2(1+\xi)^2}+\frac{r(1-\mu_x)}{2(r+\zeta)^2}$$

The equations of motion are derived from Hamilton's equations. After neglecting higher-order terms, we obtain the linearized equations that form the basis for the matrix representation in Eq. (10).

Appendix B: Equations of Motion for Envelope Perturbations with Dispersion

The Hamiltonian with dispersion from Sec. IV is:

$$\hat{H}_{\text{env}} = \hat{p}_x^2+\hat{p}_y^2+\hat{p}_{D_\delta}^2+\hat{\sigma}_x^2+\eta^2\hat{\sigma}_y^2+\hat{D}_\delta^2-\mu_{x,d}(1+R)\ln(\hat{X}+\hat{Y})+\frac{1-\mu_{x,d}}{2\hat{\sigma}_x^2}+\frac{r_d(1-\mu_{x,d})}{2\hat{\sigma}_y^2}+\frac{(1-\mu_{x,d})\cos^2\theta}{2\sin^2\theta}$$

Substituting perturbation variables and performing Taylor expansion yields the linearized equations for envelope perturbations with dispersion, resulting in the matrix equation (33) with coefficients b_0 through b_5 as defined in the main text.

Appendix C: Calculation of Higher-Order Mode Contributions

The Hamiltonian for lowest- and higher-order modes in Eq. (43) is:

$$\hat{H}_{L,0} = \frac{1}{2}\left(P^2+\omega^2Q^2\right)+\alpha Q^4+h_1Q^2\cos(2\Omega\tau)$$

$$\hat{H}_{L,n} = h_{n+1}Q^2\cos[2(n+1)\Omega\tau] \quad (n>0)$$

Averaging $\hat{H}_{L,0}$ over one period $T = 2\pi/\Omega$ yields the effective Hamiltonian for the 2:1 resonance. For $\hat{H}_{L,n}$ with $n > 0$, the average over one period is zero because $\langle\cos^2(\Omega\tau)\cos[2(n+1)\Omega\tau]\rangle = 0$, $\langle\sin^2(\Omega\tau)\cos[2(n+1)\Omega\tau]\rangle = 0$, and $\langle\sin(\Omega\tau)\cos(\Omega\tau)\cos[2(n+1)\Omega\tau]\rangle = 0$. This proves that high-order modes cannot excite the 2:1 parametric resonance.

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