Some notes on representation theorem of fuzzy sets

Authors: Huan Huang, Huan Huang

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Abstract

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Full Text

Some Notes on the Representation Theorem of Fuzzy Sets

Huan Huang
Department of Mathematics, Jimei University, Xiamen 361021, China

Abstract

This paper presents some notes on the representation theorem of fuzzy sets.

Keywords: fuzzy sets, representation theorem

1. Introduction

This work provides additional observations and remarks on the representation theorem of fuzzy sets, supplementing existing results in the literature.

2. Fuzzy Sets

Let N denote the set of all positive integers and R^m the m-dimensional Euclidean space; we write R^1 simply as R. For any nonempty set Y, P(Y) denotes its power set (the set of all subsets of Y), while F(Y) denotes the set of all fuzzy sets in Y, defined as functions from Y to [0, 1]. Given u   F(Y) and α (0, 1], the α-cut [u]α *of u is defined by [u]*α := {x   Y : u(x) $\geq$ α}.

When Y is a topological space, C(Y) denotes the set of all nonempty closed subsets of Y, and K(Y) denotes the set of all nonempty compact subsets of Y. For u   F(Y), the 0-cut [u]_0 is defined by [u]_0 := {x   Y : u(x) > 0}, where

S denotes the topological closure of S in Y. The set [u]_0 is called the support of u and is also denoted by supp u. For comprehensive coverage of fuzzy theory and applications, we refer readers to [**?**, **?**, **?**, **?**, **?**, **?**].

3. Main Results

The following representation theorem should be a known conclusion; a proof is provided in [**?**] (see Theorem 3.1 in [**?**] and its proof). In this paper we assume that sup = 0.

Theorem 3.1. Let Y be a nonempty set. If u F(Y), then for all α (0, 1], [u]α = {β<α}[u]β. *Conversely, suppose that {vα : α* (0, 1]} is a family of sets in Y with v_α = *{β<α}vβ* for all α (0, 1]. Define u F(Y) by u(x) := sup{α (0, 1] : x v_α} for each x Y. Then u is the unique element of F(Y) which satisfies [u]α = *vα* for all α (0, 1]; that is, u is the unique element of the set {w F(Y) : [w]α = *vα* for all α (0, 1]}.

Remark 3.2. Let Y be a nonempty set and let {v_α : α (0, 1]} be a family of sets in Y. Denote S := {w F(Y) : [w]α = *vα* for all α (0, 1]}. The following statements hold:

- (i) If the condition "v_α = *{β<α}vβ* for all α (0, 1]" fails, then S = . To see this, note that by Theorem 3.1, any w F(Y) must satisfy [w]α = {β<α}[w]_β for all α (0, 1]. Consequently, no such w can exist, and S must be empty.
- (ii) If v_α = *{β<α}vβ* for all α (0, 1], then S is a singleton set. This follows directly from Theorem 3.1.
- (iii) S is either empty or a singleton set, which is an immediate consequence of (i) and (ii).
- (iv) For any v F(Y), the set S(v) := {w F(Y) : [w]α = *[v]α* for all α (0, 1]} is a singleton set. Since v F(Y), Theorem 3.1 ensures that [v]α = {β<α}[v]_β for all α (0, 1], and by (ii), S(v) must be a singleton.
- (v) For any v F(Y), we have S(v) = {v}. It is easy to see that (a) v S(v) for each v F(Y), and (b) S(v) $\neq$. Clearly, (a) and (iv) together imply (v), establishing the result.

Proposition 3.3. The following results should be known. We were unable to locate the original reference for this conclusion, so we provide a proof here for the self-containedness of this paper. Let Y be a nonempty set and u F(Y). Then [u]_0 is well-defined if and only if Y is a topological space.

Let Y be a nonempty set, x Y, and u, v F(Y). The following hold:

- (i) (i-1) u(x) = sup{α (0, 1] : x [u]_α}. (i-2) If [u]*0 is well-defined, then u(x) = sup{α [0, 1] : x [u]α*}.

(ii) (ii-1) If [u]α = *[v]*α for each α (0, 1], then u = v. (ii-2) Assume that u = v. Then (ii-2a) [u]α = *[v]*α for each α (0, 1]; (ii-2b) Y is a topological space if and only if [u]*0 = [v]*0. (ii-3) u = v if and only if [u]α = *[v]*α for each α (0, 1]. (ii-4) If Y is a topological space, then u = v if and only if [u]α = *[v]*α for each α [0, 1].

Proof. We first prove (i). Let u(x) = . If   > 0, then {α (0, 1] : x   [u]α*} = (0,], and thus sup{*α *(0, 1] : x   [u]*α} = sup(0,] =   = u(x). If   = 0, then {α (0, 1] : x   [u]α*} = , and sup{*α *(0, 1] : x   [u]*α} = 0 = u(x). Therefore (i-1) holds.

To establish (i-2), we first show that if [u]*0 is well-defined, then sup S_1 = sup S_2, where S_1 := {*α *(0, 1] : x   [u]*α} and S_2 := {α [0, 1] : x   [u]α*}. Consider two cases. If S_1 = , then sup S_1 = 0, and S_2 is either   or {0}. In either case, sup S_2 = 0, so sup S_1 = sup S_2. If S_1 ≠ , then there exists* α *(0, 1] with x   [u]*α. Since [u]_α [u]_0, we have x   [u]_0, which implies S_2 = S_1   {0}. Moreover, sup S_1 > 0, so sup S_2 = sup(S_1   {0}) = sup S_1   sup{0} = sup S_1   0 = sup S_1. This completes the proof of the auxiliary statement. Combining this with (i-1) yields (i-2), and consequently (i) is proved.

Next, we prove (ii-1). For any y   Y, we have v(y) = sup{α (0, 1] : y   [v]α*} = sup{*α *(0, 1] : y   [u]*α} = u(y), where the equalities follow from (i-1). Since u, v   F(Y), we conclude u = v. This proof of (ii-1) is essentially given in the proof of Theorem 3.1 in [**?**].

For (ii-2), part (ii-2a) holds trivially. For (ii-2b), if Y is a topological space, then [u]_0 and [v]_0 are well-defined, and u = v implies [u]_0 = [v]_0. Conversely, if [u]_0 = [v]_0, then both [u]_0 and [v]_0 are well-defined, which means Y must be a topological space. Thus (ii-2b) holds.

Statement (ii-3) follows immediately from (ii-1) and (ii-2a), while (ii-4) follows from (ii-1) and (ii-2).

We observe that (ii-1) is equivalent to (ii-1) : Given v   F(Y), if u   S(v) then u = v. It holds that v   S(v) for each v   F(Y). Clearly, this property together with (ii-1) is equivalent to Remark 3.2(v). We provide two proofs of (ii-1). First, since Remark 3.2(v) has been established, (ii-1) follows, and thus (ii-1) holds. Alternatively, suppose u, v   F(Y) satisfy [u]α = *[v]*α for all α (0, 1]. Then both u and v belong to S(v). By Remark 3.2(iv), S(v) is a singleton set, which forces u = v, proving (ii-1). In some sense, all proofs of (ii-1) given in this paper are essentially equivalent.

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Note: Figure translations are in progress. See original paper for figures.