Graphs without repeated cycle lengths (postprint) We study graphs without repeated cycle lengths. Let $f(n)$ be the maximum number of edges in an n-vertex graph such that any two cycles in the graph have different lengths. Erdős and Hajnal posed this problem in 1981 and asked whether \$f(n)=n+O(···

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Abstract

In 1975, P. Erd"{o}s proposed the problem of determining themaximum number $f(n)$ of edges in a graph of n vertices inwhich any two cycles are of different lengths. In this paper, it is proved that

$$f(n) \geq n + 36t$$

for $t = 1260r + 169$ $(r \geq 1)$ and $n \geq 540t^2 + \frac{175811}{2}t + \frac{7989}{2}$. Consequently,$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{2}{5}}$, which is better than the previous bounds $\sqrt{2}$ (see [2]), $\sqrt{2+\frac{2562}{6911}}$ (see [7]).

Combining this with Boros, Caro, F"uredi and Yuster' s upper bound, we get

$$1.98 \geq \limsup_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2.4}.$$

Full Text

Preamble

GRAPHS WITHOUT REPEATED CYCLE LENGTHS

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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number f(n) of edges in a graph of n vertices in which any two cycles are of different lengths. In this paper, it is proved that f(n) ≥ n + 36t for t = 1260r + 169 (r ≥ 1) and n ≥ 540t2 + 175811 t + 7989 2 + 2 5, which is better than the previous bounds. Consequently, 2 (see lim inf n→∞ (cid:113) [2]), 2 + 2562 (cid:113) f (n)−n√ 6911 (see [7]). Combining this with Boros, Caro, Füredi and Yuster’ s upper bound, we get 1.98 ≥ lim sup f(n) − n ≥ lim inf f(n) − n. We make the following conjecture:

Conjecture. f(n) − n

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Introduction

Let f(n) be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining f(n) (see [1], p.247, Problem 11). Shi[2] proved that f(n) ≥ n + [(8n − 23 + 1)/2] for n ≥ 3. Lai[3,4,5,6,7] proved that for n ≥ 6911 16 t2 + 514441 t− 3309665, t = 27720r +169, and for n ≥ e2m(2m + 3)/4, f(n) ≥ n + 32t − 1, f(n) < n − 2 + (cid:113) nln(4n/(2m + 3)) + 2n + log2(n + 6). Boros, Caro, Füredi and Yuster[8] proved that f(n) ≤ n + 1.98 n(1 + o(1)). In this paper, we construct a graph G having no two cycles with the same length which leads to the following result.

Theorem. Let t = 1260r + 169 (r ≥ 1), then f(n) ≥ n + 36t for n ≥ 540t2 + 175811 t + 7989.

2 Proof of the theorem

Proof. Let t = 1260r + 169 with r ≥ 1, let nt = 540t2 + 175811 2, and let n ≥ nt. We shall show that there exists a graph G on n vertices with n + 36t edges such that all cycles in G have distinct lengths. t + 7989

Now we construct the graph G which consists of a number of subgraphs: Bi, for indices i in the following ranges: 0 ≤ i ≤ 21t − 1, 27t ≤ i ≤ 28t + 64, 29t − 734 ≤ i ≤ 29t + 267, 30t − 531 ≤ i ≤ 30t + 57, 31t − 741 ≤ i ≤ 31t + 58, 32t − 740 ≤ i ≤ 32t + 57, 33t − 741 ≤ i ≤ 33t + 57, 34t − 741 ≤ i ≤ 34t + 52, 35t − 746 ≤ i ≤ 35t + 60, 36t − 738 ≤ i ≤ 36t + 60, 37t − 738 ≤ i ≤ 37t + 799, i = 21t + 2j + 1 (0 ≤ j ≤ t − 1), i = 21t + 2j (0 ≤ j ≤ t−1 2), and i = 26t.

Now we define these Bi' s. These subgraphs all have a common vertex x; otherwise, their vertex sets are pairwise disjoint. For 0 ≤ i ≤ t − 1, let the subgraph B21t+2i+1 consist of a cycle and a path: u(23t+2i+1)/2 i ···u25t+2i−1 and u(19t+2i−1)/2 i,1···u(19t+2i−1)/2. Based on this construction, B21t+2i+1 contains exactly three cycles of lengths: 21t + 2i + 1, 23t + 2i, 25t + 2i.

For 0 ≤ i ≤ t−3 2, let the subgraph B21t+2i consist of a cycle and a path: v(22t+2i+1)/2 i ···v25t+2i and v(9t+i−1)/2 i,1v2 i,1···v9t+i−1 v12t+i. Based on this construction, B21t+2i contains exactly three cycles of lengths: 21t + 2i, 22t + 2i + 1, 25t + 2i + 1.

For 0 ≤ i ≤ t−3 2, let the subgraph B23t+2i+1 consist of a cycle and a path: w(25t+2i+1)/2 i ···w26t+2i+1 and w(21t+2i−1)/2 i,1···w(21t+2i−1)/2. Based on this construction, B23t+2i+1 contains exactly three cycles of lengths: 23t + 2i + 1, 24t + 2i + 2, 26t + 2i + 2.

For 58 ≤ i ≤ t − 742, let the subgraph B27t+i−57 consist of a cycle C27t+i−57 = xy1 i ···y132t+11i+893 and ten paths sharing a common vertex x, with the other end vertices on the cycle C27t+i−57: y(17t−1)/2+i y(57t−103)/2+2i y(77t+315)/2+3i y(97t+313)/2+4i y(117t+313)/2+5i y(137t+311)/2+6i y(157t+309)/2+7i i,8y2 i,9y2 i,10y2 i,8···y(25t−1)/2 i,9···y(25t−1)/2 i,10 ···y(27t−1)/2 (cid:17) (cid:16)d+2 y(177t+297)/2+8i y(197t+301)/2+9i y(217t+305)/2+10i. As a cycle with d chords contains distinct cycles, B27t+i−57 contains exactly 66 cycles of lengths: 28t + i + 7, 27t + i − 57, 32t + i, 31t + i + 1, 36t + i + 3, 35t + i + 3, 40t + 2i + 209, 38t + 2i + 216, 44t + 2i − 6, 42t + 2i − 1, 48t + 3i + 158, 46t + 2i + 744, 52t + 3i − 1, 51t + 3i − 1, 56t + 3i + 746, 55t + 3i − 1, 61t + 4i − 2, 61t + 4i + 208, 65t + 4i + 740, 65t + 4i − 2, 72t + 5i − 8, 71t + 5i + 207, 80t + 6i + 156, 75t + 5i + 739, 84t + 6i − 3, 82t + 6i − 6, 92t + 7i + 203, 91t + 7i + 207, 101t + 8i + 209, 101t + 8i + 149, 111t + 9i + 151, 112t + 9i + 211, 122t + 10i + 952, 132t + 11i + 894, 30t + i, 29t + i + 210, 34t + i − 5, 33t + i, 38t + 2i − 51, 37t + i + 742, 42t + 2i, 40t + 2i, 46t + 2i + 5, 44t + 2i − 3, 50t + 3i + 209, 49t + 3i + 215, 54t + 3i − 4, 53t + 3i − 7, 59t + 4i + 215, 59t + 4i + 157, 63t + 4i − 5, 63t + 4i − 7, 70t + 5i + 214, 69t + 5i + 157, 74t + 5i − 3, 73t + 5i − 5, 82t + 6i + 201, 80t + 6i + 213, 90t + 7i + 155, 84t + 6i + 738, 93t + 7i − 4, 94t + 7i + 738, 103t + 8i + 205, 103t + 8i + 737, 113t + 9i + 946, 122t + 10i + 153.

B0 is a path with an end vertex x and length n − nt. Other Bi are simply cycles of length i. Then f(n) ≥ n + 36t for n ≥ nt. This completes the proof.

From the above theorem, we have lim inf f(n) − n (cid:115) which is better than the previous bounds 2 (see [2]), (cid:113) 2 + 2562 6911 (see [7]). Combining this with Boros, Caro, Füredi and Yuster' s upper bound, we get 1.98 ≥ lim sup f(n) − n ≥ lim inf f(n) − n. We make the following conjecture:

Conjecture. f(n) − n

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