# A Minimum Energy Principle with Variable Loads and Its Application to Structural Optimization

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## Abstract

In mechanical structures, conventional energy principles are typically formulated for fixed loads. However, in engineering structures such as bridges, loads often need to be adjusted to achieve reasonable internal forces and material savings. Nevertheless, theoretical investigations in this domain remain scarce. To resolve this problem, this paper establishes a constrained functional extremum problem and proposes a minimum energy principle incorporating movable loads in mechanics: when the virtual work of the variable loads vanishes, that is, when the variable loads are orthogonal to their corresponding displacements, the system energy attains its minimum, thereby demonstrating that the principle of minimum potential energy is a special case of this principle. It is further noted that this orthogonality condition can serve as a supplementary condition to the fundamental equations of mechanics. This paper also proposes specific computational methods for implementing this theory in structural optimization: the forced displacement method, the large stiffness method, and the generalized unit load method. This theory enables the transformation of variable load control into displacement control, facilitating convenient solution via tools such as the finite element method. Several structural examples are presented to illustrate the broad practical applicability of the proposed theory.

## Full Text

## A Minimum Energy Principle with Variable Loads and Its Application in Structural Optimization

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In structural mechanics, conventional energy principles are formulated for fixed loads. However, in engineering structures such as bridges, load adjustment is often necessary to achieve reasonable internal force distribution and material savings. Nevertheless, theoretical research in this area remains limited. To address this gap, this paper establishes a constrained functional extremum problem and proposes a minimum energy principle with variable loads in mechanics: the system energy is minimized when the virtual work of variable loads equals zero, i.e., when the variable loads are orthogonal to their corresponding displacements. The principle of minimum potential energy is shown to be a special case of this principle. Furthermore, the orthogonality condition can serve as a supplementary equation to the fundamental equations of mechanics. This paper also presents specific computational methods for applying this theory to structural optimization: the forced displacement method, the large stiffness method, and the generalized unit load method. This theory transforms the control of variable loads into control of displacements, enabling convenient solution using finite element tools. Multiple structural examples are provided to demonstrate the broad practical value of the proposed theory.

**Keywords:** minimum energy principle, orthogonality, variable loads, bridges, structural optimization

In engineering structures, active load control is often required to ensure reasonable force distribution and economic efficiency. This need is particularly urgent in structural engineering, such as: configuring prestressing tendons to achieve reasonable internal force distribution in prestressed concrete beams; adjusting cable forces in cable-stayed bridges for optimal structural performance; adjusting hanger forces in arch bridges and tie forces in tied-arch bridges; and adjusting tension forces in spatial cable-net structures. The influence matrix method is commonly used and remains dominant, while intelligent algorithms and data-driven technologies have grown rapidly in recent years. Fleming (1979) [?] proposed the influence matrix method, establishing its foundation. Wang P.H., et al. (1993) [?] developed a linear programming model for cable force optimization in cable-stayed bridges based on the influence matrix. Xiao Rucheng (1997) [?] introduced a bending energy minimization criterion using the influence matrix to optimize cable forces and minimize the bending potential energy of cable-stayed bridge girders. Tibert (2002) [?] proposed the force density method for form-finding of spatial cable-net structures. Li Yongle (2010) [?] presented a genetic algorithm-based optimization of hanger forces in tied-arch bridges with a multi-objective fitness function. Zhang et al. (2021) [?] proposed a hybrid PSO-GA algorithm that combines global search capability of PSO with local optimization ability of GA, significantly improving computational efficiency. Wang Hao (2023) [?] developed a digital twin-driven intelligent control method for prestressed steel structures. Li Hui (2022) [?] developed an LSTM prediction model for cable force variation in cable-stayed bridges. All these optimization problems can be reduced to optimal control problems of variable loads.

Energy principles form the cornerstone of structural analysis. The principle of virtual displacements was proposed by John Bernoulli in 1717, and Maxwell established the displacement reciprocity theorem in 1864. In 1872, Betti generalized the displacement reciprocity theorem to the work reciprocity theorem. Castigliano proposed Castigliano' s first and second theorems in 1879, and Engesser introduced the complementary energy method in 1889. Reissner proposed a generalized variational principle for two types of variables in elasticity in 1950 [?], and Hu Haichang proposed a generalized variational principle for three types of variables in 1954 [?]. Long Yuqiu introduced a partitioned mixed generalized variational principle in 1983 [?]. All these structural energy principles were derived for fixed loads. The applicability of these energy principles to structural optimization problems with variable loads requires further investigation.

Structural energy principles can all be derived using variational methods, which essentially involve solving extremum problems of functionals. The orthogonality condition holds a central position in functional extremum problems, contact mechanics, and variational methods. Courant & Hilbert (1953) [?] first rigorously proved the orthogonality of constrained variational problems, proposing dual space orthogonality and laying the mathematical foundation for energy orthogonality principles in elasticity (such as modal analysis of beams). Lions & Stampacchia (1967) [?] established a general framework for variational inequalities, proving that the solution $u$ and multiplier $\lambda$ satisfy complementary orthogonality conditions, and noting that orthogonality is essentially an expression of Hilbert' s projection theorem. Duvaut & Lions (1976) [?] formulated the Signorini contact problem as a variational inequality, rigorously proving the orthogonality between contact pressure $\lambda$ and displacement gap $u - g$, and introducing the concept of dual pairing to explain the physical meaning of orthogonality as "contact forces do work only in the effective contact zone." Oden & Reddy (1983) [?] systematically discussed orthogonality modifications under non-homogeneous boundary conditions, noting that non-homogeneous boundaries require additional correction terms to orthogonality conditions. Kikuchi & Oden (1988) [?] proposed orthogonality conditions for contact problems. Brezzi & Fortin (1991) [?] introduced the inf-sup condition to ensure numerical stability of orthogonality in mixed finite element methods, solving non-standard orthogonality problems (such as Stokes equations) and providing theoretical tools for non-matching mesh discretization in contact problems. Reddy (2002) [?] systematically discussed orthogonality under non-homogeneous boundaries, specifically proving for beam differential equations that the weighted orthogonality between constraint force $q(x)$ and displacement $w(x)$ is a necessary condition for energy extremum. Wriggers (2002) [?] extended orthogonality to frictional contact: tangential friction force $\lambda_t$ and slip amount $s$ satisfy $\int \lambda_t \cdot s \, ds = 0$. Arnold et al. (2005) [?] used discontinuous Galerkin methods to preserve discrete orthogonality on non-conforming meshes, resolving numerical oscillations in Mortar methods and proving that orthogonality error relates to mesh size $h$ and polynomial degree $p$ as $O(h^{p+1/2})$.

This paper employs variational methods and utilizes orthogonality conditions for functional extremum problems to extend the minimum energy principle in mechanics to cases with variable loads and applies it to structural optimization design.

### 2.1 Problem Description

Consider the following functional extremum problem:

$$J[w,f]=\int\left[\frac{EI(w'')^2}{2}+\lambda_1(x)(q(x)-C_1)\right]dx+\int\left[\frac{EA(u')^2}{2}+q_0w+qw+f_0u+fu\right]dx$$

The Lagrangian functional is:

$$L=\int\left[\frac{EI(w'')^2}{2}+\frac{EA(u')^2}{2}+q_0w+q(w+\lambda_1)-\lambda_1C_1+f_0u+f(u+\lambda_2)-\lambda_2C_2\right]dx$$

Where: - $w(x)$ is the vertical displacement field (free variable) - $u(x)$ is the axial displacement field (free variable) - $q(x)$ is the constraint force satisfying $q(x)<C_1$ (for all $x\in[0,1]$) - $f(x)$ is the constraint force satisfying $f(x)<C_2$ (for all $x\in[0,1]$) - $EI$ is flexural stiffness, $EA$ is axial stiffness - $q_0$ is distributed vertical load (constant), $f_0$ is distributed axial load (constant)

KKT conditions include: - **Primal feasibility:** $q(x)\le C_1$, $f(x)\le C_2$ - **Dual feasibility:** $\lambda_1(x)\ge 0$, $\lambda_2(x)\ge 0$ - **Complementary slackness:** $\lambda_1(x)(q(x)-C_1)=0$, $\lambda_2(x)(f(x)-C_2)=0$ - **Extremum condition:** Variation of functional $L$ with respect to $w,u,q,f$ equals zero

### 2.2 Variational Derivation

Taking variation with respect to $w$, the first variation of the functional is:

$$\delta J=\int\left[EIw''\delta w''+q_0\delta w+q\delta w\right]dx=\int EIw''\delta w''dx+\int(q_0+q)\delta w dx$$

Applying integration by parts twice:

$$\int EIw''\delta w''dx=\left[EIw''\delta w'\right]_0^L-\left[EIw'''\delta w\right]_0^L+\int EIw''''\delta w dx$$

Therefore, the extremum condition yields:

**Euler-Lagrange equation:**

$$EIw''''+q_0+q=0$$

**Boundary conditions:**

$$w''(0)=w''(1)=0,\quad w'''(0)=w'''(1)=0$$

Similarly for variation with respect to $u(x)$:

$$-EAu''+f_0+f=0$$

### 2.3 Orthogonality Proof for Variable Loads and Displacements

For a general function $q(x)$, expand using basis functions:

$$q(x) = \sum a_k \psi_k(x)$$

where $\{\psi_k(x)\}$ are basis functions (e.g., polynomials or trigonometric functions).

**Optimization condition:**

$$\int (w + \lambda_1)\psi_k dx = 0 \quad \forall k$$

When $q = \sum Q_i \delta(x - L_i)$ represents concentrated loads, based on the properties of Dirac functions:

$$\int q(w + \lambda_1)dx = \sum Q_i(w_i + \lambda_{1i}) = 0$$

In the above equation, $Q_i$ satisfies the constraints of characteristic functions. When unconstrained:

$$w_i + \lambda_{1i} = 0$$

For a general function $f(x)$, expand using basis functions:

$$f(x) = \sum a_k \phi_k(x)$$

where $\{\phi_k(x)\}$ are basis functions.

**Optimization condition:**

$$\int (u + \lambda_2)\phi_k dx = 0 \quad \forall k$$

When $f = \sum F_i \delta(x - L_i)$ represents concentrated loads:

$$\int f(u + \lambda_2)dx = \sum F_i(u_i + \lambda_{2i}) = 0$$

**Special case:** When $q, f$ are not subject to inequality constraints, the orthogonality conditions simplify to:

$$\int qwdx = 0, \quad \int fudx = 0$$

If $q, f$ are constants, they can be taken outside the integral:

$$\int wdx = 0, \quad \int udx = 0$$

Based on these orthogonality conditions, when a structure is subjected to variable loads, minimizing structural energy requires that the virtual work (external work) done by variable loads equals zero. Specifically, when variable loads are constant, the integral of displacement at the load locations must be zero. This orthogonality condition transforms the problem of solving for optimal variable loads into a displacement problem, facilitating computational implementation, especially when combined with finite element displacement conditions.

## 3 Finite Element Discretization and Proof

Previous chapters presented theoretical analysis of the minimum virtual work principle. For engineering applications, finite element discretization can be employed, which is analyzed in detail below. Finite element discretization is conventional, and specific discretization processes are not analyzed in detail here. For convenience, matrix notation is adopted. After discretization, the finite element equation becomes:

$$Ku = F + Eq$$

where $u, F$ are column vectors with $n$ elements, $K$ is an $n \times n$ matrix, $q$ is a control load vector with $m$ elements ($m \leq n$), and $E$ is a position expansion matrix with elements of 0 and 1 to enable incorporation of control loads $q$ into the finite element equilibrium equations.

Assume control load $q$ has $m_1$ equality constraints:

$$h(q) = 0$$

where $h$ represents a vector function with $m_1$ elements and 0 denotes $[0, 0, ..., 0]^\top$. Additionally, $p$ inequality constraints exist:

$$g(q) \leq C$$

where $C = [c_1, c_2, ..., c_p]^\top$ is a constant vector.

Since $q$ is an $m$-dimensional vector with $m_1$ constraints, its dimension can be reduced to $m_2 = m - m_1$, meaning there are only $m_2$ independent variables:

$$q = [q_1, q_2, ..., q_{m_1}, q_{m_1+1}, ..., q_m]^\top$$

where $[q_{m_1+1}, q_{m_1+2}, ..., q_m]^\top$ represents independent variables and $[q_1, q_2, ..., q_{m_1}]^\top$ represents dependent variables.

The $m_1$ equality constraint equations can be expressed in terms of $q_{m_1+1}, ..., q_m$:

$$\begin{cases} q_1 = f_1(q_{m_1+1}, q_{m_1+2}, ..., q_m) \\ q_2 = f_2(q_{m_1+1}, q_{m_1+2}, ..., q_m) \\ \vdots \\ q_{m_1} = f_{m_1}(q_{m_1+1}, q_{m_1+2}, ..., q_m) \end{cases}$$

Based on structural virtual work and introducing Lagrange multipliers, the virtual work expression becomes:

$$J = (F + Eq)^\top w - \frac{1}{2} w^\top K w + \mu^\top h + \lambda^\top g$$

where $E$ is the load position expansion matrix. Taking variation yields:

$$\delta J = (\delta w)^\top [F + Eq - Kw] + (\delta q)^\top [E^\top w + \left(\frac{\partial h}{\partial q}\right)^\top \mu + \left(\frac{\partial g}{\partial q}\right)^\top \lambda] = 0$$

**Analysis of $\delta w$ term:** Since $\delta w$ is arbitrary, the equilibrium equation is obtained:

$$Kw = F + Eq$$

**Analysis of $\delta q$ term:** Since $\delta q$ is an $m$-dimensional vector with $m_1$ constraints, it can be reduced to $m_2 = m - m_1$ independent variables. Therefore, the control equation cannot be obtained directly by setting $\delta q = 0$. Let:

$$q = [q_1, q_2, ..., q_{m_1}, q_{m_1+1}, ..., q_m]^\top$$

For convenience, denote independent variables as $[q_{m_1+1}, q_{m_1+2}, ..., q_m]^\top$ and dependent variables as $[q_1, q_2, ..., q_{m_1}]^\top$. The $m_1$ equality constraints can be expressed as:

$$\begin{cases} f_1 = f_1(q_{m_1+1}, q_{m_1+2}, ..., q_m) \\ f_2 = f_2(q_{m_1+1}, q_{m_1+2}, ..., q_m) \\ \vdots \\ f_{m_1} = f_{m_1}(q_{m_1+1}, q_{m_1+2}, ..., q_m) \end{cases}$$

Taking variation yields the Jacobian matrix of $q$ with respect to independent variables. Substituting into $\delta J$ gives:

$$(\delta q)^\top [E^\top w + \left(\frac{\partial h}{\partial q}\right)^\top \mu + \left(\frac{\partial g}{\partial q}\right)^\top \lambda] = (\delta \tilde{q})^\top [E^\top w + \left(\frac{\partial h}{\partial q}\right)^\top \mu + \left(\frac{\partial g}{\partial q}\right)^\top \lambda] = 0$$

Since $\tilde{q}$ is arbitrary:

$$[E^\top w + \left(\frac{\partial h}{\partial q}\right)^\top \mu + \left(\frac{\partial g}{\partial q}\right)^\top \lambda] = 0$$

This is the control equation for minimizing system energy.

Additionally, according to KKT conditions, Lagrange multipliers $\lambda_i$ for inequality constraints must satisfy dual feasibility and complementary slackness conditions.

### 3.2 Orthogonality Proof for Control Loads and Displacements

Since $q$ is an $m$-dimensional vector subject to $m_1$ constraints, its dimension reduces to $m_2 = m - m_1$. Thus, $q$ has $m_2$ independent basis vectors $\phi_i, i = 1, 2, ..., m_2$:

$$q = \sum_{i=1}^{m_2} a_i \phi_i$$

If constraints $g(q), h(q)$ are linear functions, let:

$$g(q) = Aq - b = \sum_{i=m_2+1} q_i \phi_i - b, \quad \lambda^\top g(q) = \lambda^\top \left( \sum q_i \phi_i - b \right)$$

$$h(q) = Bq - c = \sum q_i \phi_i - c, \quad \mu^\top h(q) = \mu^\top \left( \sum q_i \phi_i - c \right)$$

Then energy $J$ can be viewed as a function of $q_i$. Substituting into the expression yields:

$$J(q_i) = (F + E \sum q_i \phi_i)^\top w - \frac{1}{2} w^\top K w + \mu^\top (\sum q_i \phi_i - c) + \lambda^\top (\sum q_i \phi_i - b)$$

Taking derivative gives the optimality condition:

$$\frac{\partial J}{\partial q_i} = (E\phi_i)^\top w + \mu^\top \phi_i + \lambda^\top \phi_i = 0$$

Expanding:

$$(Eq)^\top w + \mu^\top h(q) + \lambda^\top g(q) = \sum q_i (E\phi_i)^\top w + \mu^\top (\sum q_i \phi_i - c) + \lambda^\top (\sum q_i \phi_i - b)$$

$$= \sum q_i [(E\phi_i)^\top w + \mu^\top \phi_i + \lambda^\top \phi_i] - \mu^\top c - \lambda^\top b = -\mu^\top c - \lambda^\top b$$

If $b = 0, c = 0$, we obtain:

$$(Eq)^\top w + \mu^\top h(q) + \lambda^\top g(q) = 0$$

This is the orthogonality condition. If $b \neq 0, c \neq 0$, it is equivalent to applying constants $b, c$ to the structure first, then applying the orthogonality condition.

### 3.3 Special Cases and Case Analysis

Several special cases are presented to illustrate the derived equations:

1) **When control load $q$ is unconstrained:** Equation (51) simplifies to $E^\top w = 0$

2) **When control load $q$ has only equality constraints:** Equation (51) simplifies to:

$$\left( \frac{\partial q}{\partial \tilde{q}} \right)^\top [E^\top w + \left( \frac{\partial h}{\partial q} \right)^\top \mu] = 0$$

**Example:** For two nodes with control loads $q = [q_1, q_2]$ subject to $h(q_1, q_2) = q_1 - q_2 = 0$, using $q_2$ as the independent variable gives:

$$\frac{\partial q}{\partial \tilde{q}} = [1, -1]^\top$$

Substituting into (58) yields:

$$[1, -1] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w_1 - w_2 = 0$$

This indicates that when loads at two nodes are equal, minimizing system energy requires equal displacements at these nodes, making the virtual work of control loads zero. This simple example can be applied to cable force adjustment in cable-stayed bridges and arch bridges.

If the equality constraint is $h(q_1, q_2) = q_1 + q_2 = 0$, the orthogonality condition gives $w_1 - w_2 = 0$ and $q_1 w_1 + q_2 w_2 = 0$, making the virtual work zero.

If $h(q_1, q_2) = q_1 - 2q_2 = 0$, we get $2w_1 + w_2 = 0$ and $q_1 w_1 + q_2 w_2 = 0$. These three simple cases demonstrate that system energy is minimized when virtual work of control loads equals zero, proving the minimum virtual work principle.

3) **When control load $q$ has only inequality constraints:** Equation (51) simplifies to:

$$E^\top w + \left(\frac{\partial g}{\partial q}\right)^\top \lambda = 0$$

**Example:** For a single inequality constraint $q < q_0$, i.e., $g(q) = q - q_0 \leq 0$.

## 4 Algorithms and Applications

### 4.1 Cantilever Beam with Constrained Concentrated Load at Tip

To verify the derived minimum energy principle with inequality constraints, consider a cantilever beam of length $L$ with fixed left end and free right end, subjected to uniformly distributed load $q$ (positive downward) and concentrated force $F$ at the free end (positive upward, with $F \leq C$). Find the variable load $F$ that minimizes structural elastic potential energy:

$$U(F) = \int_0^L \frac{EI(w'')^2}{2} dx - \int_0^L q w(x) dx + F w(L)$$

According to the minimum potential energy principle, introducing Lagrange multiplier $\lambda$, the generalized functional of potential energy is:

$$\Pi[w] = \int_0^L \frac{EI(w'')^2}{2} dx - \int_0^L q w dx + F w(L) + \lambda(F - C)$$

Variation yields the Euler-Lagrange equation and orthogonality condition:

$$EIw'''' = q$$

**Boundary conditions:**

$$w(0) = 0, \quad w'(0) = 0, \quad w''(L) = 0, \quad EIw'''(L) = F$$

**Orthogonality condition:**

$$w(L) + \lambda = 0$$

**KKT conditions:**

$$\lambda(F - C) = 0, \quad \lambda \geq 0, \quad F \leq C$$

Solving the differential equation $EIw'''' = q$ with boundary conditions gives the deflection curve:

$$w(x) = \frac{qx^4}{24EI} - \frac{qLx^3}{6EI} + \frac{qL^2x^2}{4EI} + \frac{Fx^3}{6EI} - \frac{FLx^2}{2EI}$$

The tip deflection is:

$$w(L) = \frac{3qL^4 - 8FL^3}{24EI}$$

**Case analysis based on complementary slackness:**

- **Case 1 (Constraint inactive):** $F < C, \lambda = 0$ gives $w(L) = 0$, yielding $F = \frac{3qL}{8}$ when $C \geq \frac{3qL}{8}$
- **Case 2 (Constraint active):** $F = C, \lambda \geq 0$ gives $\lambda = -w(L) = -\frac{3qL^4 - 8CL^3}{24EI} \geq 0$, valid when $C < \frac{3qL}{8}$

**Special cases: - Unconstrained ($C \to +\infty$):** Reduces to classical solution $F = \frac{3qL}{8}$ **- Fully constrained ($C = 0$):** $F = 0$, only uniform load acts

[Figure 1: see original paper]

**4.2 Cantilever Beam with Distributed Force Under Specific Function Constraints** This section verifies orthogonality for distributed forces under specific function constraints. The control load is a distributed force $f$. Find $f$ that minimizes strain energy as shown in Figure 1.

From equation (??), the Euler equation for the cantilever beam element is:

$$EIy'' - q = 0, \quad 0 \leq x < L_1$$

$$EIy'' - q - f = 0, \quad L_1 < x < L$$

**Boundary conditions:**

$$y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad EIy'''(L) = 0$$

**Continuity conditions at $x = L_1$:**

$$y_1(L_1) = y_2(L_1), \quad y_1'(L_1) = y_2'(L_1), \quad y_1''(L_1) = y_2''(L_1)$$

**Control condition from (??):**

$$\int_{L_1}^{L} y_2 dx = 0$$

This equation shows that internal energy is minimized when the displacement integral at control load $F$ equals zero, i.e., when the virtual work done by $F$ is zero. This validates the proposed minimum virtual work principle for cantilever beams under distributed loads.

The differential equation solutions are:

$$y_1(x) = \frac{qx^4}{24EI} - \frac{qLx^3}{6EI} + \frac{qL^{2x^2}}{4EI}$$

$$y_2(x) = \frac{(q+f)x^4}{24EI} - \frac{(q+f)Lx^3}{6EI} + \frac{(q+f)L^{2x^2}}{4EI} - \frac{fL_1^4}{24EI} + \frac{fL_1^{3x}}{6EI}$$

The total virtual work is:

$$W(F) = \frac{1}{120EI}\left(3L^5f^2 + 6L^5fq + 3L^5q^2 - 10L^2f^2L_1^3 - 10L^2fqL_1^3 + 5Lf^2L_1^4 + 5LfqL_1^4 + 2f^2L_1^5 - fqL_1^5\right)$$

From the control equation (??):

$$\frac{\partial W}{\partial f} = \frac{1}{120EI}\left(6L^5f + 6L^5q - 20L^2fL_1^3 - 10L^2qL_1^3 + 10LfL_1^4 + 5LqL_1^4 + 4fL_1^5 - qL_1^5\right) = 0$$

Solving gives:

$$f = -\frac{6L^5q - 10L^2qL_1^3 + 5LqL_1^4 - qL_1^5}{6L^5 - 20L^2L_1^3 + 10LL_1^4 + 4L_1^5}$$

Using extremum conditions to verify, $W'(F)$ yields the same result. For visualization, assuming $E = 1, I = 1, q = 1, L = 40, L_1 = 20$, the displacement, controlled load work, fixed load virtual work, and strain energy curves versus $F$ are shown in Figure 2.

As $f$ decreases, the negative work done by the controlled load increases and internal energy decreases. When $F = -1.18$, $\int y_2 dx = 0$, the controlled load work reaches its minimum, and strain energy is minimized. As $f$ continues to decrease, $y_L$ changes from positive to negative (downward to upward deformation), the controlled load begins positive work, total work increases, and internal energy rises. When $f = -2.36$, the controlled load' s effect exceeds the fixed load, causing adverse effects.

These results confirm the correctness of the proposed minimum virtual work principle, which effectively calculates minimum potential energy (minimum sum of squared moments) to obtain optimal control loads and maximum material savings.

[Figure 2: see original paper]

**4.3 Cable Force and Counterweight Adjustment for Cable-Stayed Bridges** This section demonstrates that discrete cable forces and counterweights satisfy constrained orthogonality and presents a generalized unit load method. When variable loads are free, displacements can be directly set to zero. When two forces between nodes are equal and opposite, orthogonality requires equal node displacements, allowing the large stiffness method (forcing zero displacement between points). For more complex constraints, the generalized unit load method is used, illustrated with a cable-stayed bridge counterweight example.

For simplified analysis, consider a cable-stayed bridge with two towers, main span $L_1 + L_2 + L_1$, tower heights $H_{11}$ above and $H - 2$ below deck, girder moment of inertia $EI_b$, tower moment of inertia $EI_t$, and 3 pairs of cables per tower. The elevation layout is shown below.

**Optimal cable force analysis:** Since cables are not parallel to displacement directions, the orthogonality condition is:

$$T_{si}u_{ti} + T_{xi}w_{bi} = T_{si}u_{ti}\cos\theta_i + T_{xi}w_{bi}\sin\theta_i = 0$$

Neglecting cable self-weight ($T_{si} = T_{xi}$) simplifies to:

$$u_{ti}\cos\theta_i + w_{bi}\sin\theta_i = 0$$

**Side span counterweight analysis:** Cable-stayed bridges typically require side span counterweights to address negative support reactions and main span deflection under loads. Assume counterweight loads $F_1, F_2, F_3$ satisfy:

$$F_1 = 2F_2$$

Let $F_2 = F$, then $F_1 = 2F$. The counterweight loads become $[2F, F, F_3]^\top$. Taking generalized unit loads ($F = 1, F_3 = 1$) gives counterweights $[2, 1, 1]^\top$. Since $F, F_3$ are independent, they can be considered separately. For $F_3$, directly set $w_3 = 0$. For $F$, the orthogonality condition is:

$$2Fw_1 + Fw_2 = 0 \implies 2w_1 + w_2 = 0$$

To find $F$, apply the counterweight to displacements from original loads to get virtual work $V_0 = 2Fw_{10} + Fw_{20}$, then apply only the counterweight to get $V_1 = 2Fw_{11} + Fw_{21}$. Orthogonality requires:

$$V_0 + FV_1 = 0 \implies F = -\frac{V_0}{V_1}$$

This process is the generalized unit load method.

## Conclusions

This paper proposes a minimum energy principle with variable loads and verifies it through multiple examples. Main conclusions are:

1. **Innovative principle:** The minimum energy principle with variable loads is creatively proposed—system energy is minimized when virtual work of variable loads equals zero, i.e., when variable loads are orthogonal to their displacements.
2. **Fundamental equations:** Basic equations for mechanical systems with variable loads are derived, including geometric, equilibrium, constitutive, boundary conditions, and orthogonality conditions. The orthogonality condition serves as an additional control equation. Without it, the equations reduce to those for fixed-boundary systems.
3. **Transformation:** The principle transforms variable load control into displacement control, enabling seamless integration with finite element analysis without conventional optimization methods.
4. **Computational methods:** Specific methods are proposed: forced displacement method for freely variable loads, large stiffness method for equal/opposite loads between two nodes, and generalized unit load method for complex constraints.
5. **Verification:** Multiple simplified examples verify the theory and provide optimization design methods, demonstrating broad applicability and practical value for structural optimization.

This paper systematically proposes a minimum energy principle with variable loads, derives orthogonality conditions, transforms variable load optimization into displacement boundary conditions, and extends energy principles and fundamental mechanical equations. It offers high theoretical and practical value for bridge and structural engineering optimization, advancing structural optimization development.

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