Stability and Hopf bifurcation analysis of fractional double-delay prey-predator model under PD control strategies

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Abstract

A fractional-order proportional-derivative controller is designed to address bifurcation issues in a dual-time-delay fractional-order predator-prey model. By selecting different delays as bifurcation parameters, the stability and Hopf bifurcation conditions of the controlled system are derived. The results show that the fractional order, delays and control parameters play an important role on the stability and Hopf bifurcation of the system. By selecting reasonable system parameters (fractional order, delays, and control parameters), suitable system control strategies can be devised. Finally, the key findings of this study are verified through numerical examples.

Full Text

Preamble

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Abstract

A fractional-order proportional-derivative controller is designed to address bifurcation issues in a dual-time-delay fractional-order predator-prey model. By

selecting different delays as bifurcation parameters, we derive the stability and Hopf bifurcation conditions of the controlled system. The results demonstrate that the fractional order, delays, and control parameters play a crucial role in determining the stability and Hopf bifurcation of the system. By selecting appropriate system parameters (fractional order, delays, and control parameters), suitable control strategies can be devised. Finally, the key findings of this study are verified through numerical examples.

Keywords: Prey-predator model, Caputo derivative, Hopf bifurcation, Fractional-order PD controller

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1. Introduction

Predator-prey relationships have attracted significant attention from biophysicists in recent decades. The functional response of predators constitutes an important component in studying these relationships, as it reflects the effect of predation on changes in prey density at any given time. Functional responses are generally classified into two types: prey-dependent and predator-dependent. The classical Holling types I-IV [1–4] are prey-dependent, while the Hassell-Varley type [5], Beddington-DeAngelis type [6, 7], and Crowley-Martin type [8] represent predator-dependent functional responses.

Time delay plays a crucial role in species modeling, as factors such as food digestion, species migration, pregnancy, and maturation all involve time delays. Jun et al. [9] demonstrated that time delay can affect system stability, showing that when the delay exceeds a critical value, the system oscillates periodically and becomes unstable. Kumar and Dubey [10] studied a prey-predator system with prey shelter and gestation delay, revealing that time delay triggered by the fear effect of prey causes the system to produce Hopf bifurcation. Cui et al. [11] investigated prey-predator models with fear delay and pregnancy delay, demonstrating that both prey and predator oscillate periodically and reach equilibrium in finite time when the delay is smaller than the critical bifurcation value.

Fractional calculus exhibits long memory and hereditary properties, enabling it to describe and analyze the changes and behaviors of complex systems more accurately than integer-order calculus [12]. It has significant applications in numerous fields, including viscoelastic materials [13], heat conduction [14], biological systems [15], neuronal conduction [16], control systems [17], and signal processing [18]. Mondal et al. Li [19] established a fractional delayed zooplankton-phytoplankton system and showed that delay plays an important role in the stability and timing of Hopf bifurcation. Additional interesting studies on fractional order can be found in [20-23].

The proportional-derivative (PD) controller has been widely used in various applications, including mechanical engineering [24], automotive control systems [25], and robot control [26]. Dupont [27] investigated how the PD controller can be used to achieve stable movement at very low speeds. Ding et al. [28] studied the bifurcation control problem for a class of complex networks with small-world network models and time delays using the PD controller, which can delay or advance the occurrence of Hopf bifurcation by setting appropriate control parameters. Lu et al. [29] applied the PD controller to control the bifurcation problem of a time-delayed fractional-order prey-predator system, successfully delaying the generation of Hopf bifurcation and illustrating the relationship between Hopf bifurcation points and fractional orders.

The structure of this paper is as follows: Section 2 introduces the fundamental theorems and definitions of stability for fractional-order delay systems. Section 3 presents the research model and provides some basic instructions. Section 4 establishes sufficient conditions for the Hopf bifurcation of the corresponding equilibrium points of the system considering multiple time delays. Section 5 employs numerical simulations to validate the analytical findings and demonstrate the efficacy of the control strategy. Finally, Section 6 summarizes the obtained results as the conclusion of this paper.

2. Preliminary and Model Description

Definition 1. The Caputo fractional derivative is defined as:

$$ {}_{0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{n-\alpha-1}} d\tau, $$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}^+$, and Γ is the Gamma function in the form $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$.

Theorem 1. [30] Consider the following fractional-order system:

$$ {}_{0}^{C}D_t^{\alpha} x(t) = f(t), \quad x(0) = x_0, $$

where $0 < \alpha \leq 1$. The equilibrium points of system (2) are locally asymptotically stable if all eigenvalues λ of the Jacobian matrix $J = \frac{\partial f}{\partial x}$ satisfy the condition $|\arg(\lambda)| > \frac{\alpha\pi}{2}$. Furthermore, if $|\arg(\lambda)| = \frac{\alpha\pi}{2}$, then the system may undergo a Hopf bifurcation.

Definition 2. Deng et al. [31] considered the following fractional-order system:

$$
\begin{aligned}
{}_0^C D_t^{\alpha_1} x_1(t) &= a_{11} x_1(t-\tau_{11}) + \cdots + a_{1n} x_n(t-\tau_{1n}),\\
{}_0^C D_t^{\alpha_2} x_2(t) &= a_{21} x_1(t-\tau_{21}) + \cdots + a_{2n} x_n(t-\tau_{2n}),\\
&\vdots\\
{}_0^C D_t^{\alpha_n} x_n(t) &= a_{n1} x_1(t-\tau_{n1}) + \cdots + a_{nn} x_n(t-\tau_{nn}),
\end{aligned}
$$

where $0 < \alpha_i \leq 1$ for $i = 1, 2, \cdots, n$. We select the initial value $x_i(t) = \vartheta_i(t)$ in the domain $-\tau_{\max} \leq t \leq 0$, $i = 1, 2, \cdots, n$, where $\tau_{\max} = \max_{1\leq i,j\leq n} \tau_{i,j}$. The characteristic equation for system (3) can be expressed as:

$$
\begin{vmatrix}
s^{\alpha_1} - a_{11}e^{-s\tau_{11}} & \cdots & -a_{n1}e^{-s\tau_{n1}}\\
\vdots & \ddots & \vdots\\
-a_{1n}e^{-s\tau_{1n}} & \cdots & s^{\alpha_n} - a_{nn}e^{-s\tau_{nn}}
\end{vmatrix} = 0.
$$

We can obtain some stability results in [31].

[Figure 1: see original paper]

3. Model Descriptions

In [11], Cui et al. studied a prey-predator model with double delays and a Beddington-DeAngelis functional response:

$$
\begin{aligned}
\frac{dx(t)}{dt} &= kx(t)\left(1 + fy(t-\tau_1)\right) - \alpha x^2(t) - \frac{px(t)y(t)}{ax(t) + by(t) + c},\\
\frac{dy(t)}{dt} &= \frac{\mu p x(t-\tau_2) y(t-\tau_2)}{ax(t-\tau_2) + by(t-\tau_2) + c} - dy(t) - hy^2(t),
\end{aligned}
$$

where $x(t)$ and $y(t)$ represent the prey and predator populations, respectively, with initial conditions $x(0) > 0$ and $y(0) > 0$. The parameters f, p, a, b, c, μ, d, h, k, and α are all positive constants. Additionally, τ_1 denotes the fear delay and τ_2 represents the gestation delay. System (4) exhibits three positive equilibrium points: $E_0(0, 0)$, $E_1(\frac{k}{\alpha}, 0)$, and $E^*(x^*, y^*)$. From [11], we can derive:

$$
x^* = \frac{(by^* + c)(d + hy^*)}{\mu p - a(d + hy^*)},
$$

The interior equilibrium point exists if and only if $\mu p - a(d + hy^*) > 0$, and the equilibrium value y^* must satisfy the equation:

$$
Ay^4 + By^3 + Cy^2 + Dy + E = 0,
$$

where

$A = a^2h^2fp + b^2\alpha fh\mu p,$
$B = \mu pb\alpha(2cfh + bdf + bh) + a^2h^2p + 2ahfp(ad - \mu p),$
$C = \mu pb\alpha(bd + 2cdf + 2ch) + h\mu p(abk + c^2\alpha f) + fp(ad - \mu p)^2 + 2ahp(ad - \mu p),$
$D = c\mu p(2bd\alpha + ahk + c\alpha h + c\alpha h + cd\alpha f) + \mu pbk(ad - \mu p) + p(ad - \mu p)^2,$
$E = \mu pc^2d\alpha + \mu pck(ad - \mu p).$

In this paper, we develop a Caputo derivative fractional-order prey-predator model described as follows:

$$
\begin{aligned}
{}_0^C D_t^\alpha x(t) &= kx(t)\left(1 + fy(t-\tau_1)\right) - \alpha x^2(t) - \frac{px(t)y(t)}{ax(t) + by(t) + c},\\
{}_0^C D_t^\alpha y(t) &= \frac{\mu px(t-\tau_2)y(t-\tau_2)}{ax(t-\tau_2) + by(t-\tau_2) + c} - dy(t) - hy^2(t).
\end{aligned}
$$

To regulate the Hopf bifurcation in system (5), we propose the following controller:

$$
u(t) = k_p(y - y^*) + k_d\, {}_0^C D_t^\alpha (y - y^*),
$$

where $\alpha \in (0, 1]$, and k_p and k_d are the control gains.

Applying the controller $u(t)$ to the second equation of system (5), we obtain:

$$
\begin{aligned}
{}_0^C D_t^\alpha x(t) &= kx(t)\left(1 + fy(t-\tau_1)\right) - \alpha x^2(t) - \frac{px(t)y(t)}{ax(t) + by(t) + c},\\
{}_0^C D_t^\alpha y(t) &= \frac{\mu px(t-\tau_2)y(t-\tau_2)}{ax(t-\tau_2) + by(t-\tau_2) + c} - dy(t) - hy^2(t) + k_p(y - y^*) + k_d\, {}_0^C D_t^\alpha (y - y^*).
\end{aligned}
$$

Remark 1: When $k_d = 0$ and $k_p \neq 0$, the controller (6) transforms into a linear feedback controller. When $k_p = 0$ and $k_d \neq 0$, the controller (6) becomes a neutral delay feedback controller. When $k_p \neq 0$ and $k_d \neq 0$, the controller (6) is a fractional-order PD controller.

Remark 2: Currently, several common bifurcation control strategies have been designed for integer-order and fractional-order systems, including dynamic feedback control [32, 33], delay feedback control [34, 35], and hybrid control [36, 37]. Notably, in recent studies [29] and [38], the fractional PD controller was used to realize bifurcation control of fractional-order delay systems.

4. Stability and Hopf Bifurcation of Controlled System (7)

In this section, we investigate the Hopf bifurcation of the controlled system (7). The characteristic equation of system (7) is:

$$\begin{vmatrix} s^{\alpha} - a_{11} & -b_{21}e^{-s\tau_2} \\ -a_{12} - b_{12}e^{-s\tau_1} & s^{\alpha} - a_{22} - b_{22}e^{-s\tau_2} \end{vmatrix} = 0,$$

where

$$\begin{aligned} a_{11} &= \frac{apx^*y^*}{(ax^* + by^* + c)^2}, \\ a_{12} &= 1 + fy^* - 2\alpha x^* - \frac{px^*}{ax^* + by^* + c} + \frac{apx^*y^*}{(ax^* + by^* + c)^2}, \\ b_{12} &= -\frac{kfx^*}{(1 + fy^*)^2}, \\ a_{22} &= \frac{(-d - 2hy + k_p)}{1 - k_d}, \\ b_{21} &= \frac{\mu bpx^*y^*}{(ax^* + by^* + c)^2(1 - k_d)} i, \\ b_{22} &= \frac{\mu bpx^*y^*}{(ax^* + by^* + c)^2(1 - k_d)} i. \end{aligned}$$

We can obtain:

$$s^{2\alpha} + A_1 s^{\alpha} + A_2 + e^{-s\tau_2}(A_3 s^{\alpha} + A_4) + A_5 e^{-s(\tau_1 + \tau_2)} = 0,$$

where

$$\begin{aligned} A_1 &= -(a_{11} + a_{22}), \\ A_2 &= a_{11}a_{22}, \\ A_3 &= -b_{22}, \\ A_4 &= a_{11}b_{22} - a_{12}b_{21}, \\ A_5 &= -b_{12}b_{21}. \end{aligned}$$

Case 1: $\tau_1 = 0$, $\tau_2 = 0$.

The characteristic equation (8) becomes:

$$s^{2\alpha} + (A_1 + A_3)s^{\alpha} + A_2 + A_4 + A_5 = 0.$$

Theorem 2. When $\tau_1 = 0$ and $\tau_2 = 0$, the equilibrium point $E^*(x^*, y^*)$ of system (7) is locally asymptotically stable if $A_1 + A_3 > 0$ and $A_2 + A_4 + A_5 > 0$.

Proof. If $A_1 + A_3 > 0$ and $A_2 + A_4 + A_5 > 0$, then according to the fractional Routh-Hurwitz criteria, the roots of (9) have no positive real parts. Therefore, $E^*(x^*, y^*)$ of system (7) is locally asymptotically stable.

Case 2: $\tau_1 > 0$, $\tau_2 = 0$.

The characteristic equation (8) can be described as:

$$s^{2\alpha} + (A_1 + A_3)s^\alpha + A_2 + A_4 + A_5 e^{-s\tau_1} = 0.$$

Let $s = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$. Substituting into equation (10) and separating the real and imaginary parts, we obtain:

$$\omega^{2\alpha}\cos(\alpha\pi) + (A_1 + A_3)\omega^\alpha\cos\frac{\alpha\pi}{2} + A_2 + A_4 = -A_5\cos(\omega\tau_1),$$
$$\omega^{2\alpha}\sin(\alpha\pi) + (A_1 + A_3)\omega^\alpha\sin\frac{\alpha\pi}{2} = A_5\sin(\omega\tau_1).$$

Squaring equations (11) and (12) and adding them together yields:

$$\omega^{4\alpha}+2(A_1+A_3)\omega^{3\alpha}\cos\frac{\alpha\pi}{2}+2(A_2+A_4)\cos(\alpha\pi)\omega^{2\alpha}+2(A_1+A_3)(A_2+A_4)\omega^\alpha\cos\frac{\alpha\pi}{2}+(A_1+A_3)^2+(A_2+A_4)^2-A_5^2$$

We define the following function:

$$H_1(\omega) = \omega^{4\alpha}+2(A_1+A_3)\omega^{3\alpha}\cos\frac{\alpha\pi}{2}+(A_1+A_3)^2\omega^{2\alpha}+2(A_2+A_4)\cos(\alpha\pi)\omega^{2\alpha}+2(A_1+A_3)(A_2+A_4)\omega^\alpha\cos\frac{\alpha\pi}{2}+(A$$

Let $\phi = \omega^\alpha$. Equation (14) becomes:

$$H_1(\phi) = \phi^4+2(A_1+A_3)\phi^3\cos\frac{\alpha\pi}{2}+(A_1+A_3)^2\phi^2+2(A_2+A_4)\cos(\alpha\pi)\phi^2+2(A_1+A_3)(A_2+A_4)\phi\cos\frac{\alpha\pi}{2}+(A_2+A_4)$$

If (15) has a positive root ϕ_0, then (14) has a corresponding positive root $\omega_0 = \phi_0^{1/\alpha}$. Note that $H_1(0) = (A_2+A_4)^2 - A_5^2$ and $\lim_{\phi\to\infty} H_1(\phi) = +\infty$. We assume that $H_1(0) < 0$. According to Descartes' rule of signs, equation (15) has at least one positive root ϕ_0. From equation (11), we can deduce:

$$\tau_1^{(j)} = \frac{1}{\omega_0}\arccos\left[\frac{-\omega_0^{2\alpha}\cos(\alpha\pi) - (A_1 + A_3)\omega_0^\alpha\cos\frac{\alpha\pi}{2} - A_2 - A_4}{A_5}\right] + \frac{2j\pi}{\omega_0},$$

where $j = 0, 1, 2, ...,$ and ω_0 is the greatest positive root of $H_1(\omega) = 0$.

We define:

$$\tau_{10} = \min\{\tau_1^{(j)}\}.$$

We now prove the transversality condition $\mathrm{Re}\left[\frac{ds(\tau_1)}{d\tau_1}\right]_{s=i\omega_0} \neq 0$.

By differentiating equation (10) with respect to τ_1, we obtain:

$$\frac{ds(\tau_1)}{d\tau_1} = \frac{A_5 s^2 e^{-s\tau_1}}{2\alpha s^{2\alpha} + \alpha(A_1 + A_3)s^{\alpha} + A_5 s^2 e^{-s\tau_1}}.$$

Substituting $s = i\omega_0$ into the above equation yields:

$$\left.\frac{ds(\tau_1)}{d\tau_1}\right|_{s=i\omega_0} = \frac{A_5 (i\omega_0)^2 e^{-i\omega_0\tau_1}}{2\alpha(i\omega_0)^{2\alpha} + \alpha(A_1 + A_3)(i\omega_0)^{\alpha} + A_5(i\omega_0)^2 e^{-i\omega_0\tau_1}}.$$

It follows that:

$$\left.\frac{ds(\tau_1)}{d\tau_1}\right|_{s=i\omega_0} = \frac{-A_5\omega_0^2\cos(\omega_0\tau_1) + iA_5\omega_0^2\sin(\omega_0\tau_1)}{2\alpha\omega_0^{2\alpha}\cos(\omega_0\tau_1 + \alpha\pi) + \alpha(A_1 + A_3)\omega_0^{\alpha}\cos(\omega_0\tau_1 + \frac{\alpha\pi}{2}) - A_5\omega_0^2\cos(\omega_0\tau_1) + i[2\alpha\omega_0^{2\alpha}\sin(\omega_0\tau_1 + \alpha}$$

If $\mathrm{Re}\left[\frac{ds(\tau_1)}{d\tau_1}\right]_{s=i\omega_0} \neq 0$, it means that the transversality condition is satisfied and the Hopf bifurcation occurs at $E^*(x^*, y^*)$ for $\tau_1 = \tau_{10}$.

[Figure 4: see original paper]

[Figure 5: see original paper]

Theorem 3. When $\tau_1 > 0$ and $\tau_2 = 0$, if $(A_2 + A_4)^2 < A_5^2$ and $\mathrm{Re}\left[\frac{ds(\tau_1)}{d\tau_1}\right]_{s=i\omega_0} \neq 0$ hold, the system (7) is locally asymptotically stable at the equilibrium point $E^*(x^*, y^*)$ for all $\tau_1 \in (0, \tau_{10})$. When $\tau_1 > \tau_{10}$, the system (7) becomes unstable. Furthermore, when $\tau_1 = \tau_{10}$, a Hopf bifurcation occurs at the equilibrium point $E^*(x^*, y^*)$.

Case 3: $\tau_1 = 0$, $\tau_2 > 0$.

The characteristic equation (8) becomes:

$$s^{2\alpha} + A_1 s^{\alpha} + A_2 + e^{-s\tau_2}(A_3 s^{\alpha} + A_4 + A_5) = 0.$$

Assuming $s = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ is a solution to equation (17), substituting $s = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ into equation (17) and separating the real and imaginary parts yields:

$$\eta_{21}\cos(\omega\tau_2) + \eta_{22}\sin(\omega\tau_2) = \gamma_{21},$$
$$\eta_{22}\cos(\omega\tau_2) - \eta_{21}\sin(\omega\tau_2) = \gamma_{22},$$

where

$$\begin{aligned}
\eta_{21} &= -(A_3\omega^\alpha\cos\frac{\alpha\pi}{2} + A_4 + A_5),\\
\eta_{22} &= -A_3\omega^\alpha\sin\frac{\alpha\pi}{2},\\
\gamma_{21} &= \omega^{2\alpha}\cos(\alpha\pi) + A_1\omega^\alpha\cos\frac{\alpha\pi}{2} + A_2,\\
\gamma_{22} &= \omega^{2\alpha}\sin(\alpha\pi) + A_1\omega^\alpha\sin\frac{\alpha\pi}{2}.
\end{aligned}$$

We define the following function:

$$H_2(\omega) = \gamma_{21}^2 + \gamma_{22}^2 - \eta_{21}^2 - \eta_{22}^2.$$

Then, we have:

$$H_2(\omega) = \omega^{4\alpha} + A_{11}\omega^{3\alpha} + A_{12}\omega^{2\alpha} + A_{13}\omega^\alpha + A_{14},$$

where

$$\begin{aligned}
A_{11} &= 2A_1\cos\frac{\alpha\pi}{2},\\
A_{12} &= 2A_2\cos(\alpha\pi) + A_1^2 - A_3^2,\\
A_{13} &= 2A_1A_2\cos\frac{\alpha\pi}{2} - 2A_3(A_4 + A_5)\cos\frac{\alpha\pi}{2},\\
A_{14} &= A_2^2 - (A_4 + A_5)^2.
\end{aligned}$$

When $A_{14} < 0$, equation (21) has at least one positive root ω_0. Suppose equation (21) has four positive roots $\omega = \omega_k$ $(k = 1, 2, 3, 4)$, and define the largest positive root as ω_0.

From equations (18) and (19), we obtain:

$$\cos(\omega\tau_2) = \frac{\gamma_{21}\eta_{21} + \gamma_{22}\eta_{22}}{\eta_{21}^2 + \eta_{22}^2}.$$

The bifurcation point is:

$$\tau_2^{(j)} = \frac{1}{\omega_0}\arccos\left(\frac{\gamma_{21}\eta_{21} + \gamma_{22}\eta_{22}}{\eta_{21}^2 + \eta_{22}^2}\right) + \frac{2j\pi}{\omega_0},$$

where $j = 0, 1, 2, ...$ and $k = 1, 2, 3, 4$.

We define:

$$\tau_{20} = \min\{\tau_2^{(j)}\}.$$

By differentiating equation (17) with respect to τ_2, we obtain:

$$\left.\frac{ds(\tau_2)}{d\tau_2}\right|_{s=i\omega_0} = \frac{O_1O_3 + O_2O_4}{1 + O_2},$$

where

$$\begin{aligned}
O_1 &= -\omega_0^2 A_3 \omega_0^\alpha \cos\left(\frac{\alpha\pi}{2} - \omega_0\tau_2\right) - (A_4 + A_5)\omega_0^2 \cos(\omega_0\tau_2), \\
O_2 &= -\omega_0^2 A_3 \omega_0^\alpha \sin\left(\frac{\alpha\pi}{2} - \omega_0\tau_2\right) - (A_4 + A_5)\omega_0^2 \sin(\omega_0\tau_2), \\
O_3 &= 2\alpha\omega_0^{2\alpha} \cos(\alpha\pi) + A_3\alpha\omega_0^\alpha \sin\left(\omega_0\tau_2 - \frac{\alpha\pi}{2}\right) + A_1\alpha\omega_0^\alpha \cos\left(\omega_0\tau_2 - \frac{\alpha\pi}{2}\right), \\
O_4 &= 2\alpha\omega_0^{2\alpha} \sin(\alpha\pi) - A_3\alpha\omega_0^\alpha \cos\left(\omega_0\tau_2 - \frac{\alpha\pi}{2}\right) + A_1\alpha\omega_0^\alpha \sin\left(\omega_0\tau_2 - \frac{\alpha\pi}{2}\right).
\end{aligned}$$

If $\frac{O_1O_3+O_2O_4}{1+O_2} \neq 0$, there exists a Hopf bifurcation at $\tau_2 = \tau_{20}$ in the system.

[Figure 6: see original paper]

[Figure 7: see original paper]

Theorem 4. Supposing $\frac{O_1O_3+O_2O_4}{1+O_2} \neq 0$. For system (7), the following results can be obtained:

- (i) If $A_2^2 < (A_4 + A_5)^2$, the equilibrium point $E^*(x^*, y^*)$ is locally asymptotically stable for any $0 < \tau_2 < \tau_{20}$.
- (ii) If $A_2^2 < (A_4 + A_5)^2$, the equilibrium point $E^*(x^*, y^*)$ is unstable for $\tau_2 > \tau_{20}$.

In addition, the system (7) will undergo a Hopf bifurcation at $\tau_2 = \tau_{20}$.

Case 4: $\tau_1 > 0$, $0 < \tau_2^* < \tau_{20}$.

The derivation process is the same as Case 3, and the detailed derivation is shown in the Appendix. Our conclusions are as follows.

[Figure 8: see original paper]

[Figure 9: see original paper]

Theorem 5. Assuming $(A_4\cos(\omega_1\tau_2^*) + A_5)^2 + A_4^2\sin^2(\omega_1\tau_2^*) < A_5^2$, and $Q_1Q_3 + Q_2Q_4 \neq 0$, the following results can be obtained for system (7):

(i) The equilibrium point $E^*(x^*, y^*)$ is locally asymptotically stable for any $\tau_1 \in (0, \tau_1^0)$.

(ii) The equilibrium point $E^*(x^*, y^*)$ is unstable for $\tau_1 > \tau_1^0$. In addition, the system (7) will undergo a Hopf bifurcation at $\tau_1 = \tau_1^0$.

Case 5: $\tau_2 > 0$, $0 < \tau_1^* < \tau_{10}$.

Theorem 6. Supposing $A_2^2 < (A_5 \cos(\omega_2\tau_1^*) - A_4)^2 + A_5^2 \sin^2(\omega_2\tau_1^*)$, and $M_1M_3 + M_2M_4 \neq 0$. For system (7), the following results can be obtained:

(i) The equilibrium point $E^*(x^*, y^*)$ is locally asymptotically stable for any $\tau_2 \in (0, \tau_2^0)$.

(ii) The equilibrium point $E^*(x^*, y^*)$ is unstable for $\tau_2 > \tau_2^0$. In addition, the system (7) will undergo a Hopf bifurcation at $\tau_2 = \tau_2^0$.

5. Numerical Simulation

In this section, we conduct numerical simulations to demonstrate the impact of fractional order and time delay on the stability and Hopf bifurcation of system (7). We utilized MATLAB 2021a software to simulate the fractional-order delay system with the predictor-corrector scheme [39].

According to [11], we choose the parameter values as follows: $f = 1.5$, $p = 4$, $a = 4.8$, $b = 5$, $c = 2.1$, $\mu = 1.2$, $d = 0.1$, $h = 0.01$, $k = 0.9$, $\alpha = 0.6$.

The initial values are all chosen as $(x(0), y(0)) = (0.5, 0.5)$, and the equilibrium point E^* is calculated as $(x^*, y^*) = (0.1219, 0.5702)$.

[Figure 1: see original paper] demonstrates the impact of the fractional order as a parameter on the dynamical behavior of system (5). As the value of fractional order decreases, the convergence rate of the system can be increased.

Case 1: When $\tau_1 = \tau_2 = 0$, choosing $\alpha = 0.95$ and $k_p = k_d = 0.1$, the system (7) is locally asymptotically stable at the equilibrium point $E^*(0.1219, 0.5702)$, as shown in Figure 2. When $\alpha = 0.98$, the system (7) experiences a Hopf bifurcation at E^*, which is illustrated in Figure 3. Other parameters are the same as in (25). This indicates that a Hopf bifurcation occurs in the system under the action of the controller. However, because of the effect of fractional order on the stability of the system, we adjust the value of fractional order to realize the stability control of the controlled system again.

Case 2: By setting $k_p = k_d = 0.1$, the conditions of Theorem 3 can be satisfied. It is obtained from (16) that $\tau_{10} = 2.5815$. When $\tau_1 = 2.2 < \tau_{10}$, the system (7) is locally asymptotically stable at the equilibrium point $E^*(0.1219, 0.5702)$, as illustrated in Figure 4. The system (7) generates a Hopf bifurcation at E^* when $\tau_1 = 3 > \tau_{10}$, as displayed in Figure 5.

Case 3: We set $k_p = 0.1$ and $k_d = 0.1$ in the fractional-order PD controller (6), satisfying the conditions of Theorem 4. From (24), we obtain $\tau_{20} = 1.4589$.

The system (7) demonstrates an asymptotically stable equilibrium when $\tau_2 = 1.3$ such that $\tau_2 < \tau_{20}$ (refer to Figure 6). Conversely, when $\tau_1 = 2 > \tau_{20}$, the equilibrium point E^* becomes unstable, and a Hopf bifurcation occurs, as illustrated in Figure 7.

Case 4: We assume $\tau_1 > 0$ and $\tau_2 = 0.5 < \tau_{20}$, taking the gain parameters $k_p = k_d = 0.1$, satisfying the conditions of Theorem 5. According to the derivation of Case 3, we can obtain $\tau_1^0 = 2.7354$. The system (7) exhibits an asymptotically stable equilibrium at $\tau_1 = 2.5 < \tau_1^0$ (see Figure 8). It is exhibited in Figure 9 that a Hopf bifurcation happens when $\tau_1 = 3 > \tau_1^0$.

Case 5: For $\alpha = 0.9$, $k_p = k_d = 0.1$, and $\tau_1 = 1.5$, we calculate $\tau_2^0 = 0.5964$. According to the conclusion of Theorem 6, the equilibrium point of system (7) is locally asymptotically stable when $\tau_2 = 0.3 < \tau_2^0$. When $\tau_2 = 0.7 > \tau_2^0$, system (7) loses its stability, and a Hopf bifurcation occurs with a stable periodic solution, as shown in Figure 10. Figure 11 depicts the stable periodic solution.

Figures 12 and 13 demonstrate the impact of multiple sets of gain parameters and fractional order α on the critical values τ_1 and τ_2. In an uncontrolled system ($k_p = k_d = 0$), the critical values τ_1 and τ_2 both decrease with an increase in the fractional order α. Keeping the fractional order α constant, we observe that the critical values τ_1 and τ_2 decrease with increasing control parameters.

[Figure 12: see original paper] shows that the critical value τ_1 decreases slowly and then increases rapidly with the increase of the gain parameter k_d, and the critical value increases with the decrease of the fractional order α. When $k_d = 2$, the critical value first decreases rapidly and then increases slowly with the increase of k_p, and increases with the decrease of fractional order α, which is shown in [Figure 15: see original paper].

As shown in [Figure 16: see original paper], as the gain parameter k_d increases, the critical value τ_2 shows a trend of decreasing first and then increasing, and with the decrease of the fractional order α, the critical value also increases. When $k_d = 2$, with the increase of the proportional gain k_p, the change curve of the critical value τ_2 is w-shaped, which first decreases and then increases and then decreases and increases again. At the same time, the critical value τ_2 increases as the fractional order α decreases, as shown in [Figure 17: see original paper].

6. Conclusion

In this paper, we extend the integer-order predator-prey model with double delays to fractional order and apply a fractional-order PD controller to regulate the fractional-order double-delay model. First, we provide the stability conditions of the controlled system at the equilibrium point. Meanwhile, we discuss the sufficient conditions for the Hopf bifurcation of the controlled system at the equilibrium point under different delays. The research shows that when the delay exceeds a critical value, the system undergoes a Hopf bifurcation. The stability of the system is affected by the fractional order, delay, and control parameters. A suitable control strategy for the system can be designed by selecting appropriate system parameters (fractional order, delay, and control parameters). The control parameters of the fractional-order PD controller can be adjusted over a wide range. The conclusions of this paper have certain reference value for studying the dynamic behavior of prey-predator systems.

The application of fractional-order PD controllers is extensive and can be further advanced to cater to more complex systems. Our future research goal is to focus on bifurcation control for high-dimensional fractional systems.

Credit Authors Statement

Zhuang Cui wrote the manuscript, analyzed the data, and wrote code for the model. Yan Zhou initiated the concept of the study, developed the model, and acted as corresponding author. Wei Zhang proofread and critically reviewed the manuscript. Ruimei Li critically reviewed the manuscript.

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Appendix: Fractional Order System with $\tau_1 > 0$ and $\tau_2 > 0$

When $\tau_1 > 0$ and $\tau_2 = \tau_2^* < \tau_{20}$, the characteristic equation (8) of system (7) becomes:

$$s^{2\alpha} + A_1 s^\alpha + A_2 + e^{-s\tau_2^*}(A_3 s^\alpha + A_4) + A_5 e^{-s(\tau_1+\tau_2^*)} = 0.$$

We substitute $s = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ into (26) and separate the real and imaginary parts, obtaining:

$$\eta_{31}\cos(\omega\tau) + \eta_{32}\sin(\omega\tau) = \gamma_{31},$$
$$\eta_{32}\cos(\omega\tau) - \eta_{31}\sin(\omega\tau) = \gamma_{32},$$

where

$$\eta_{31} = -A_5 \cos(\omega\tau_2),$$
$$\eta_{32} = A_5 \sin(\omega\tau_2),$$
$$\gamma_{31} = \omega^{2\alpha} \cos\alpha\pi + A_1\omega^{\alpha} \cos\frac{\alpha\pi}{2} + A_2 + A_3\omega^{\alpha} \cos\left(\frac{\alpha\pi}{2} - \omega\tau_2\right) + A_4 \cos(\omega\tau_2),$$
$$\gamma_{32} = \omega^{2\alpha} \sin\alpha\pi + A_1\omega^{\alpha} \sin\frac{\alpha\pi}{2} + A_3\omega^{\alpha} \sin\left(\frac{\alpha\pi}{2} - \omega\tau_2\right) - A_4 \sin(\omega\tau_2).$$

By rearranging (27) and (28), we get:

$$\cos(\omega\tau) = \frac{\gamma_{31}\eta_{31} + \gamma_{32}\eta_{32}}{\eta_{31}^2 + \eta_{32}^2},$$
$$\sin(\omega\tau) = \frac{\gamma_{31}\eta_{32} - \gamma_{32}\eta_{31}}{\eta_{31}^2 + \eta_{32}^2}.$$

As a consequence of $\sin^2(\omega\tau) + \cos^2(\omega\tau) = 1$, we have:

$$\omega^{4\alpha} + V_{21}\omega^{3\alpha} + V_{22}\omega^{2\alpha} + V_{23}\omega^{\alpha} + V_{24} = 0,$$

where

$$V_{21} = 2A_1 \cos\frac{\alpha\pi}{2},$$
$$V_{22} = A_1^2 + A_3^2 + 2A_1A_3 \cos(\omega\tau_2^* + \alpha\pi) + 2A_2 \cos(\alpha\pi) + 2A_4 \cos(\omega\tau_2^*),$$
$$V_{23} = 2A_1A_4 \cos(\omega\tau_2^* - \frac{\alpha\pi}{2}) + 2A_2A_3 \cos(\omega\tau_2^* + \frac{\alpha\pi}{2}) + 2A_2A_1 \cos\frac{\alpha\pi}{2},$$
$$V_{24} = A_2^2 + A_4^2 - A_5^2 + 2A_2A_4 \cos(\omega\tau_2^*).$$

Let us assume that $H_1(\omega) = \omega^{4\alpha} + V_{21}\omega^{3\alpha} + V_{22}\omega^{2\alpha} + V_{23}\omega^{\alpha} + V_{24}$. Given that $H_1(0) = V_{24} < 0$ and $\lim_{\omega\to\infty} H_1(\omega) = +\infty$, there must be at least one ω_1 for which $H_1(\omega_1) = 0$.

According to (29), we can deduce that the bifurcation point is:

$$\tau^{(j)} = \frac{1}{\omega_1} \arccos\left(\frac{\gamma_{31}\eta_{31} + \gamma_{32}\eta_{32}}{\eta_{31}^2 + \eta_{32}^2}\right) + \frac{2j\pi}{\omega_1},$$

where $j = 0, 1, 2, ...$

We define:

$$\tau_1^0 = \min\{\tau^{(j)}\}.$$

To validate the transversality condition, we differentiate both sides of equation (26) with respect to τ_1. This yields:

$$\left.\frac{ds(\tau_1)}{d\tau_1}\right|_{\tau_1=\tau_1^0} = \frac{Q_1Q_3 + Q_2Q_4}{1 + Q_2},$$

where

$$Q_1 = -\omega_1^2 A_5 \cos\omega_1(\tau_1 + \tau_2^*) - \omega_1^{2+\alpha} A_3 \cos\left(\frac{\alpha\pi}{2} - \omega_1\tau_2^*\right) - A_4\omega_1^2 \cos(\omega_1\tau_2^*),$$
$$Q_2 = -\omega_1^2 A_5 \sin\omega_1(\tau_1 + \tau_2^*) + \omega_1^{2+\alpha} A_3 \sin\left(\frac{\alpha\pi}{2} - \omega_1\tau_2^*\right) + A_4\omega_1^2 \sin(\omega_1\tau_2^*),$$
$$Q_3 = 2\alpha\omega_1^{2\alpha}\cos(\alpha\pi) + \alpha A_1\omega_1^{\alpha}\cos\frac{\alpha\pi}{2} - A_3\tau_2^*\omega_1^{\alpha+1}\cos\left(\frac{\alpha\pi}{2} - \omega_1\tau_2^*\right) - \tau_2^* A_4\omega_1\cos(\omega_1\tau_2^* - \frac{\pi}{2}) + A_3\alpha\omega_1^{\alpha}\cos\frac{\alpha\pi}{2}\, c$$
$$Q_4 = 2\alpha\omega_1^{2\alpha}\sin(\alpha\pi) + \alpha A_1\omega_1^{\alpha}\sin\frac{\alpha\pi}{2} + A_3\tau_2^*\omega_1^{\alpha+1}\sin\left(\frac{\alpha\pi}{2} - \omega_1\tau_2^*\right) + A_4\tau_2^*\sin(\omega_1\tau_2^* - \frac{\pi}{2}) + A_3\alpha\omega_1^{\alpha}\sin\frac{\alpha\pi}{2}\cos(\omega$$

When $\tau_2 > 0$ and $\tau_1 = \tau_1^* < \tau_{10}$, the proof of Theorem 6 is the same as in Case 4.

When $\tau_2 > 0$ and $\tau_1 = \tau_1^* < \tau_{10}$, the characteristic equation (8) of system (7) becomes:

$$s^{2\alpha} + A_1 s^{\alpha} + A_2 + e^{-s\tau_1^*}(A_3 s^{\alpha} + A_4) + A_5 e^{-s(\tau_1^* + \tau_2)} = 0.$$

Let

$$\eta_{41} = -A_5\cos(\omega\tau_1^*) - A_3\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2} - \omega\tau_1^*\right) - A_4,$$
$$\eta_{42} = A_5\sin(\omega\tau_1^*) - A_3\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2} - \omega\tau_1^*\right),$$
$$\gamma_{41} = \omega^{2\alpha}\cos(\alpha\pi) + A_1\omega^{\alpha}\cos\frac{\alpha\pi}{2} + A_2,$$
$$\gamma_{42} = \omega^{2\alpha}\sin(\alpha\pi) + A_1\omega^{\alpha}\sin\frac{\alpha\pi}{2}.$$

Then we have:

$$H_2(\omega) = \omega^{4\alpha} + V_{11}\omega^{3\alpha} + V_{12}\omega^{2\alpha} + V_{13}\omega^{\alpha} + V_{14},$$

where

$$
\begin{aligned}
V_{31} &= 2A_1 \cos\frac{\alpha\pi}{2},\\
V_{32} &= A_1^2 - A_3^2 + 2A_2\cos(\alpha\pi),\\
V_{33} &= 2A_1A_2\cos\frac{\alpha\pi}{2} - 2A_3A_4\cos\left(\frac{\alpha\pi}{2} - \omega\tau_1^*\right) - 2A_3A_5\cos\left(\omega\tau_1^* - \frac{\alpha\pi}{2}\right),\\
V_{34} &= A_2^2 - A_4^2 - A_5^2 - 2A_4A_5\cos(\omega\tau_1^*).
\end{aligned}
$$

The bifurcation point is:

$$
\tau^{(j)} = \frac{1}{\omega_2}\arccos\left(\frac{\gamma_{41}\eta_{41} + \gamma_{42}\eta_{42}}{\eta_{41}^2 + \eta_{42}^2}\right) + \frac{2j\pi}{\omega_2},
$$

where $j = 0, 1, 2, ...,$ and ω_2 is the largest positive root in (36).

The Hopf bifurcation point of system (7) is defined as:

$$
\tau_2^0 = \min\{\tau^{(j)}\}, \quad j = 0, 1, 2, \cdots.
$$

$$
\begin{aligned}
M_1 &= -\omega_2^2 A_5\cos\omega_2(\tau_1^* + \tau_2) - \omega_2^{2+\alpha}A_3\cos\left(\frac{\alpha\pi}{2} - \omega_2\tau_2\right) - A_4\omega_2^2\cos(\omega_2\tau_2),\\
M_2 &= \omega_2^2 A_5\sin\omega_2(\tau_1^* + \tau_2) + \omega_2^{2+\alpha}A_3\sin\left(\frac{\alpha\pi}{2} - \omega_2\tau_2\right) + A_4\omega_2^2\sin(\omega_2\tau_2),\\
M_3 &= 2\alpha\omega_2^{2\alpha}\cos(\alpha\pi) + \alpha A_1\omega_2^\alpha\cos\frac{\alpha\pi}{2} - A_3\tau_2\omega_2^{\alpha+1}\cos\left(\frac{\alpha\pi}{2} - \omega_2\tau_2\right) - \tau_2A_4\omega_2\cos(\omega_2\tau_2 - \frac{\pi}{2}) + A_3\alpha\omega_2^\alpha\cos\frac{\alpha\pi}{2}\, \mathrm{c}\\
M_4 &= 2\alpha\omega_2^{2\alpha}\sin(\alpha\pi) + \alpha A_1\omega_2^\alpha\sin\frac{\alpha\pi}{2} + A_3\tau_2\omega_2^{\alpha+1}\sin\left(\frac{\alpha\pi}{2} - \omega_2\tau_2\right) + A_4\tau_2\sin(\omega_2\tau_2 - \frac{\pi}{2}) + A_3\alpha\omega_2^\alpha\sin\frac{\alpha\pi}{2}\cos(
\end{aligned}
$$

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