Preservation of Exponential Stability for Two Spatially Discretized Port-Hamiltonian Systems

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Full Text

Preamble

Exponential Stability Preserving of Two Spatially Discretized Port-Hamiltonian Systems

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Abstract

For the ideal transmission line governed by telegrapher's equations, a mixed finite element method—a generalization of popular spatially discretized schemes—has been proposed. This numerical approximation scheme preserves both the Dirac structure and passivity, ensuring that the spatially discretized system retains its port-Hamiltonian nature. In this paper, we apply this method to spatially discretize two infinite-dimensional port-Hamiltonian systems with variable coefficients and boundary controls. We then investigate the preservation of exponential stability in the resulting semi-discretized systems, demonstrating their uniform exponential stability with respect to discretization parameters. For both semi-discretized models, the uniform exponential stabilities are derived through frequency domain analysis. Finally, numerical simulations validate the effectiveness of this semi-discrete scheme.

Keywords: Port-Hamiltonian system, exponential stabilization, mixed finite element, semi-discretization, frequency domain.

1. Introduction

It is widely recognized that control systems described by partial differential equations (PDEs) are inherently infinite-dimensional. A pivotal challenge in simulating and controlling these systems lies in achieving finite-dimensional approximations. To address this, various numerical approximation schemes have been devised in the field of PDE numerical computation, aiming to preserve system structures. A natural first step is spatial discretization of the PDEs, while maintaining the continuity of the time variable. This process, known as semi-discretization, offers significant advantages. Firstly, the resulting semi-discretized models are finite-dimensional, allowing for their analysis and design by control engineers. Secondly, if these models retain numerous physical properties of the original PDEs, they can be regarded as physical modes rather than approximations of the infinite-dimensional system. Consequently, over the past two decades, extensive research has focused on preserving geometric structure [?], [?], [?], passivity [?, ?], [?], controllability [?], [?], observability [?], and exponential stability [?, ?, ?, ?, ?, ?, ?].

The quest to maintain exponential stability in numerical approximations dates back to the 1980s, when Gibson et al. [?, ?] examined linear quadratic optimal control. Concurrently, Banks et al. [?] investigated numerical approximations for linear quadratic regulator problems and solutions to operator Riccati equations for distributed parameter systems with unbounded input and boundary control, highlighting the critical role of exponential stability. In 1990, Banks et al. [?] pioneered the study of exponentially stable approximations for wave equations with boundary dampers, noting that popular discretization techniques like finite differences or finite elements often fail to maintain a uniform decay rate in semi-discretized systems. Later, Infante et al. [?] provided a mathematical explanation for this phenomenon, revealing that spurious high-frequency modes generated during spatial semi-discretization can compromise this preservation.

A significant first attempt involved researchers modifying the finite difference scheme to restore uniform exponential stability. For example, Tebou and Zuazua introduced an artificial numerical viscosity term into the finite difference semi-discretization of the 1-D wave equation with boundary control [?], thereby restoring uniform exponential stability in the new discrete systems. Similar techniques have been applied to achieve uniform controllability [?] and uniformly exponentially stable approximations for a class of abstract second-order evolution equations [?]. More recently, Liu and Guo designed a novel semi-discretized order reduction finite difference scheme to discuss uniform approximation of the 1-D wave equation [?]. They formulated the wave equation as a first-order port-Hamiltonian system and employed the central finite difference scheme. These semi-discretized systems exhibited uniform exponential stability without any additional measures. Using this semi-discretization scheme, they analyzed the uniform exponential stability of the wave equation with local viscosity and observer-based control [?], the Schrödinger equation [?], the Timoshenko beam [?], and the heat-wave coupled system [?].

However, while the finite difference scheme offers numerous conveniences and advantages for dealing with PDEs with constant coefficients, discussing the uniform exponential stability of its semi-discretization for PDEs with varying coefficients poses certain difficulties, often making it challenging to validate key conclusions [?]. Therefore, it is imperative to explore alternative semi-discretization schemes for PDEs with varying coefficients that are suitable for investigating uniform exponential stability.

To our knowledge, the mixed finite elements method may be the most suitable approximating scheme for the ideal transmission line described by the telegrapher's equations [?]. This assertion is based on three key reasons. Firstly, the spatially discretized system remains a port-Hamiltonian system, and the geometric structure of the spatially discretized system on a partition interval is identical to that of the entire spatial domain. This greatly facilitates the analysis of discrete dynamics. Secondly, the discretized scheme derived from this mixed finite element method exhibits similarities to the order reduction finite difference method presented in [?]. Specifically, the temporal derivative component of the spatially discretized system corresponds to a weighted averaging of two adjacent node functions (see Remark 2.1 below), in contrast to the temporal derivative component of the order reduction finite difference method. This similarity allows for leveraging the extensive research on uniform exponential stability approximations for the order reduction finite difference method in the context of mixed finite elements. Finally, although it is exceptionally challenging to find an appropriate energy multiplier, the frequency domain characterization of uniform exponential stability for a family of contractive semigroups [?][?][?] can be applied to this scheme.

In this paper, we conduct a comprehensive study on both the ideal transmission line and the Timoshenko beam. The paper is structured as follows. In Section 2, we revisit the spatial discretization method based on mixed finite elements for the ideal transmission line. In Section 3, we delve into the exponential stability preservation of the resulting semi-discretized system for the ideal transmission line, focusing on ensuring its uniform exponential stability with respect to the discretization parameters. In Section 4, we propose a novel semi-discretized scheme for the Timoshenko beam and analyze its exponential stability preservation using frequency domain analysis. Finally, in Section 5, we conduct several numerical simulations to validate the effectiveness of our theoretical findings.

2.1. Continuous PDE System

The telegrapher's equations for the transmission line with a boundary resistor are given by PDEs with variable coefficients:

$$\begin{cases} \dfrac{\partial q(t,z)}{\partial t} = -\dfrac{\partial I(t,z)}{\partial z}, \quad V(t,0)=0, \quad V(t,S)=RI(t,S), \\ \dfrac{\partial \phi(t,z)}{\partial t} = -\dfrac{\partial V(t,z)}{\partial z}, \quad q(0,z)=q_0(z), \quad \phi(0,z)=\phi_0(z), \end{cases} \tag{2.1}$$

where $q(\cdot,\cdot)$ denotes the charge density, $\phi(\cdot,\cdot)$ is the flux density, and the current $I(\cdot,\cdot)$ and voltage $V(\cdot,\cdot)$ are defined as:

$$I(t,z) = \frac{\phi(t,z)}{L(z)}, \quad V(t,z) = \frac{q(t,z)}{C(z)} \tag{2.2}$$

with $C(z)$ and $L(z)$ representing the distributed capacitance and distributed inductance of the line, respectively. The spatial variable z belongs to the domain $I=[0,S]$ with $S>0$, and the voltage at $z=0$ is set to zero, while a resistor $R>0$ is placed at the other end, resulting in $V(t,S)=RI(t,S)$.

The energy of the system (2.1) is expressed as

$$E(t) = \int_0^S \left(\frac{q(t,z)^2}{2C(z)} + \frac{\phi(t,z)^2}{2L(z)} \right) dz. \tag{2.3}$$

Under certain assumptions, the energy along the solution to (2.1) decays exponentially. This finding was primarily derived from exercise 9.1 in [**?**].

Theorem 2.1. Assuming $C(\cdot), L(\cdot) \in C^\infty(I)$ and $q_0, \phi_0 \in L^2(0,S)$, the system (2.1) admits a unique solution in the state space $X := L^2(0,S) \times L^2(0,S)$. Furthermore, there exist two positive constants M and ω such that the energy along the solution to (2.1) satisfies

$$E(t) \leq Me^{-\omega t}E(0). \tag{2.4}$$

Drawing inspiration from the concepts and notations in [**?**], the telegrapher's equations (2.1) can be reformulated in a geometric version as

$$\begin{cases} \dfrac{\partial q(t,z)}{\partial t} = -de_\phi(t,z), \\ \dfrac{\partial \phi(t,z)}{\partial t} = -de_q(t,z), \end{cases}$$

where $q(t,z), \phi(t,z)$ are interpreted as one-forms, and $e_\phi(t,z), e_q(t,z)$ are zero-forms given by

$$e_q = \delta_q H, \quad e_\phi = \delta_\phi H, \quad H = \int_I \left(\frac{*q(z)}{2C(z)} q(z) + \frac{*\phi(z)}{2L(z)} \phi(z) \right) \tag{2.5}$$

with d representing the exterior derivative, $*$ the Hodge star operator, and δ the variational derivative.

2.2. Semi-Discretization of a Part of the Transmission Line

This subsection is adapted from select parts of [**?**] to ensure the self-contained nature of the paper.

Initially, a semi-discretization process is applied to a segment of the transmission line spanning between two points a and b ($0 \leq a < b \leq S$), with $I_{ab} = [a,b]$.

Step 1: Approximations of one-forms on the domain I_{ab}. On the interval I_{ab}, the energy variables $q(t,z)$ and $\phi(t,z)$, as well as the infinitesimal charge rate $\partial q(t,z)/\partial t$ and the infinitesimal flux rate $\partial \phi(t,z)/\partial t$, are approximated as

$$\begin{cases} q(t,z) = Q_{ab}(t)\omega^q_{ab}(z), & \phi(t,z) = \Phi_{ab}(t)\omega^\phi_{ab}(z), \\ \dfrac{\partial q(t,z)}{\partial t} = f^q_{ab}(t)\omega^q_{ab}(z), & \dfrac{\partial \phi(t,z)}{\partial t} = f^\phi_{ab}(t)\omega^\phi_{ab}(z), \end{cases} \tag{2.7}$$

where one-forms $\omega^q_{ab}(z)$ and $\omega^\phi_{ab}(z)$ satisfy

$$\int_{I_{ab}} \omega^q_{ab}(z) = \int_{I_{ab}} \omega^\phi_{ab}(z) = 1, \quad dQ_{ab}(t) = f^q_{ab}(t), \quad d\Phi_{ab}(t) = f^\phi_{ab}(t). \tag{2.8}$$

Step 2: Approximations of zero-forms on the domain I_{ab}. The co-energy variables $e_q(t,z)$ and $e_\phi(t,z)$ are approximated on I_{ab} according to the following expressions:

$$\begin{cases} e_q(t,z) = e_q^a(t)\omega_a^q(z) + e_q^b(t)\omega_b^q(z), \\ e_\phi(t,z) = e_\phi^a(t)\omega_a^\phi(z) + e_\phi^b(t)\omega_b^\phi(z), \end{cases} \tag{2.11}$$

where zero-forms $\omega_a^q(z), \omega_b^q(z)$ and $\omega_a^\phi(z), \omega_b^\phi(z)$ satisfy the boundary value conditions:

$$\omega_a^q(a) = 1, \quad \omega_a^q(b) = 0, \quad \omega_b^q(a) = 0, \quad \omega_b^q(b) = 1,$$

as well as the compatibility of the forms, given by

$$-d\omega_a^\phi = d\omega_b^\phi = \omega_{ab}^q, \quad -d\omega_a^q = d\omega_b^q = \omega_{ab}^\phi. \tag{2.12}$$

Substituting (2.8), (2.11) and (2.12) into (2.5) gives

$$\begin{cases} f_{ab}^q(t)\omega_{ab}^q(z) = e_\phi^a(t)\omega_a^q(z) - e_\phi^b(t)\omega_b^q(z), \\ f_{ab}^\phi(t)\omega_{ab}^\phi(z) = e_q^a(t)\omega_a^\phi(z) - e_q^b(t)\omega_b^\phi(z). \end{cases} \tag{2.13}$$

By integrating these identities over the interval I_{ab} and utilizing expression (2.9), we obtain

$$f_{ab}^q(t) = e_\phi^a(t) - e_\phi^b(t), \quad f_{ab}^\phi(t) = e_q^a(t) - e_q^b(t). \tag{2.14}$$

Step 3: Approximates of Hamiltonian and dynamics. On the one hand, the power transmitted along the transmission line between points a and b is given by

$$\frac{dH_{ab}(t)}{dt} = \int_{I_{ab}} \left(\frac{\partial p(t,z)}{\partial t} e_q(t,z) + \frac{\partial \phi(t,z)}{\partial t} e_\phi(t,z) \right).$$

By substituting expressions (2.8) and (2.11) into the above equation and applying proposition 1 from [**?**], we arrive at

$$\frac{dH_{ab}(t)}{dt} = [\alpha_{ab} e_q^a(t) + (1-\alpha_{ab}) e_q^b(t)] f_{ab}^q(t) + [(1-\alpha_{ab}) e_\phi^a(t) + \alpha_{ab} e_\phi^b(t)] f_{ab}^\phi(t),$$

where

$$\alpha_{ab} = \int_{I_{ab}} \omega_a^q(z) * \omega_{ab}^q(z). \tag{2.15}$$

Since $f^q_{ab}(t)$ and $f^\phi_{ab}(t)$ correspond to the infinitesimal charge rate $\partial q(t,z)/\partial t$ and the infinitesimal flux rate $\partial \phi(t,z)/\partial t$, respectively, we define new discrete co-energy variables as follows:

$$\begin{cases} \tilde{e}^{ab}_q(t) := \alpha_{ab} e^a_q(t) + (1-\alpha_{ab}) e^b_q(t), \\ \tilde{e}^{ab}_\phi(t) := (1-\alpha_{ab}) e^a_\phi(t) + \alpha_{ab} e^b_\phi(t), \end{cases}$$

such that $dH(t)/dt = f^q_{ab}(t)\tilde{e}^{ab}_q(t) + f^\phi_{ab}(t)\tilde{e}^{ab}_\phi(t)$.

On the other hand, if we define

$$C_{ab} = \int_{I_{ab}} \frac{\omega^q_{ab}(z) * \omega^q_{ab}(z)}{C(z)}, \quad L_{ab} = \int_{I_{ab}} \frac{\omega^\phi_{ab}(z) * \omega^\phi_{ab}(z)}{L(z)},$$

then the energy of the considered segment of the transmission line can be approximated by

$$H_{ab}(t) = \frac{1}{2}\left(C_{ab} Q^2_{ab}(t) + L_{ab}\Phi^2_{ab}(t)\right). \tag{2.16}$$

The discrete co-energy variables are again derived through the following expression:

$$\tilde{e}^{ab}_q(t) = \frac{\partial H_{ab}(t)}{\partial Q_{ab}(t)}, \quad \tilde{e}^{ab}_\phi(t) = \frac{\partial H_{ab}(t)}{\partial \Phi_{ab}(t)}.$$

Therefore, the discrete dynamics of (2.1) on the interval I_{ab} is derived from equations (2.10), (2.14)-(2.16), and is given by

$$\begin{cases} C_{ab}[\alpha_{ab} e^a_q(t) + (1-\alpha_{ab}) e^b_q(t)] = e^a_\phi(t) - e^b_\phi(t), \\ L_{ab}[(1-\alpha_{ab}) e^a_\phi(t) + \alpha_{ab} e^b_\phi(t)] = e^a_q(t) - e^b_q(t). \end{cases} \tag{2.17}$$

Remark 2.1. Setting $h_{ab} = |I_{ab}|$ as the length of the interval I_{ab} and dividing both sides of the identities in (2.17) by h_{ab}, we obtain

$$\begin{cases} C_{ab}[\alpha_{ab} e^a_q(t) + (1-\alpha_{ab}) e^b_q(t)] = -\dfrac{e^b_\phi(t) - e^a_\phi(t)}{h_{ab}}, \\ L_{ab}[(1-\alpha_{ab}) e^a_\phi(t) + \alpha_{ab} e^b_\phi(t)] = -\dfrac{e^b_q(t) - e^a_q(t)}{h_{ab}}. \end{cases}$$

In the above equation, the convex combinations $C_{ab}[\alpha_{ab} e^a_q(t) + (1-\alpha_{ab}) e^b_q(t)]$ and $L_{ab}[(1-\alpha_{ab}) e^a_\phi(t) + \alpha_{ab} e^b_\phi(t)]$ can be interpreted as approximations of $q(t,z)$

and $\phi(t,z)$ on I_{ab}, respectively. Similarly, $\frac{e_q^b(t)-e_q^a(t)}{h_{ab}}$ and $\frac{e_\phi^b(t)-e_\phi^a(t)}{h_{ab}}$ represent approximations of the derivatives $de_q(t,z)$ and $de_\phi(t,z)$ on I_{ab}. This approach is analogous to the finite difference scheme proposed in [**?**]. For brevity, (2.17) will be retained for further discussion.

The discrete energy associated with the considered segment of the transmission line is given by

$$H_{ab}(t) = \frac{1}{2}\left(C_{ab}(\alpha_{ab}e_q^a(t) + (1-\alpha_{ab})e_q^b(t))^2 + L_{ab}((1-\alpha_{ab})e_\phi^a(t) + \alpha_{ab}e_\phi^b(t))^2\right) \tag{2.18}$$

and it possesses the following noteworthy property.

Proposition 2.1. The discrete energy defined in (2.18) satisfies

$$\frac{dH_{ab}(t)}{dt} = e_q^a(t)e_\phi^a(t) - e_q^b(t)e_\phi^b(t). \tag{2.19}$$

Proof. Starting from (2.17), we have

$$\frac{dH_{ab}(t)}{dt} = [\alpha_{ab}e_q^a(t)+(1-\alpha_{ab})e_q^b(t)][e_\phi^a(t)-e_\phi^b(t)]+[(1-\alpha_{ab})e_\phi^a(t)+\alpha_{ab}e_\phi^b(t)][e_q^a(t)-e_q^b(t)].$$

By simple calculation, (2.19) is derived from the above identity.

2.3. Semi-Discretization of the Whole Transmission Line

To achieve the semi-discretization of the entire transmission line, for any positive integer n, we insert $n-1$ points S_j (with $j=1,\cdots,n-1$) into the spatial domain I. This results in a partition of I into segments $I_{S_{j-1}S_j}$ for $j=1,\cdots,n$, with $S_0=0$, $S_n=S$ and $I_{S_{j-1}S_j}=[S_{j-1},S_j]$.

On each segment $I_{S_{j-1}S_j}$, we select

$$C_j = \int_{I_{S_{j-1}S_j}} C(z)dz, \quad L_j = \int_{I_{S_{j-1}S_j}} L(z)dz, \quad \alpha_j = \int_{I_{S_{j-1}S_j}} \omega_{S_{j-1}}^q(z)*\omega_{S_{j-1}S_j}^q(z)dz,$$

with $\tilde{\alpha}_j = 1-\alpha_j$ for $j=0,1,\cdots,n$. Subsequently, we obtain the semi-discretized approximation of (2.5) on $I_{S_{j-1}S_j}$ as follows:

$$\begin{cases} C_j[\alpha_j e_{q,j-1}(t) + \tilde{\alpha}_j e_{q,j}(t)] = e_{\phi,j-1}(t) - e_{\phi,j}(t), \\ L_j[\tilde{\alpha}_j e_{\phi,j-1}(t) + \alpha_j e_{\phi,j}(t)] = e_{q,j-1}(t) - e_{q,j}(t), \qquad j = 0, 1, \cdots, n. \\ e_{q,0}(t) = 0, \quad e_{q,n}(t) = Re_{\phi,n}(t), \end{cases} \tag{2.20}$$

The final two equations in (2.20) are derived directly from the boundary conditions specified in (2.1). Furthermore, the discrete energy is given by

$$H_n(t) = \frac{1}{2}\sum_{j=1}^{n}\left(C_j(\alpha_j e_{q,j-1}(t) + \tilde{\alpha}_j e_{q,j}(t))^2 + L_j(\tilde{\alpha}_j e_{\phi,j-1}(t) + \alpha_j e_{\phi,j}(t))^2\right) \tag{2.21}$$

Remark 2.2. The discrete scheme (2.20) represents a generalization of two well-known semi-discretization approaches. Specifically, when $C(z)$ and $L(z)$ are set to constant values, and $\alpha_j = 1$, the scheme reduces to the finite difference method on staggered grids as presented in [**?**]. On the other hand, if $C(z) = L(z) = 1$ and α_j is set to $\frac{1}{2}$, then (2.20) corresponds to the order-reduced finite difference scheme described in [**?**, **?**].

Now, let us define some notations and outline certain assumptions.

Assumption 2.1. Let $h_j := |I_{S_{j-1}S_j}| = S_j - S_{j-1}$ denote the length of each segment, and let $\Delta_n := \max_{1\le j\le n} h_j$ be the maximum segment length. We assume that $\Delta_n < 1$ for all $n \in \mathbb{N}^+$ and that $\Delta_n = O(n^{-1})$, meaning there exists a positive constant C such that $\Delta_n \le Cn^{-1}$ for all $n \in \mathbb{N}^+$.

Assumption 2.2. We assume the existence of positive constants c and C such that the coefficients α_j satisfy

$$c \le \min_{1\le j\le n} \alpha_j \le \max_{1\le j\le n} \alpha_j \le C < 1$$

for all $j = 1, 2, \cdots, n$ and $n \in \mathbb{N}^+$.

Assumption 2.3. Suppose there exist positive constants m and M such that $m \le C(z), L(z) \le M$ for all relevant z. This implies, in conjunction with Assumption 2.1, that $\max_{1\le j\le n} C_j = O(n^{-1})$, $\min_{1\le j\le n} C_j = O(n^{-1})$, $\max_{1\le j\le n} L_j = O(n^{-1})$ and $\min_{1\le j\le n} L_j = O(n^{-1})$.

3. Uniform Exponential Stability of (2.20)

Firstly, based on (2.18) and Proposition 2.1, the discrete energy $H_n(t)$ defined in (2.21) satisfies the following balance equation and dissipative property.

Proposition 3.1. The balance equation and dissipative property, given by

$$\frac{dH_n(t)}{dt} = e_{q,0}(t)e_{\phi,0}(t) - e_{q,n}(t)e_{\phi,n}(t) = -R|e_{\phi,n}(t)|^2 \tag{3.1}$$

hold for all positive integer n.

Proof. Indeed, we can express the discrete energy $H_n(t)$ as the sum of local contributions:

$$H_n(t) = \sum_{j=1}^{n} H_{S_{j-1}S_j}(t),$$

where each local energy $H_{S_{j-1}S_j}(t)$ is given by:

$$H_{S_{j-1}S_j}(t) = \frac{1}{2}\left(C_j(\alpha_j e_{q,j-1}(t) + \tilde{\alpha}_j e_{q,j}(t))^2 + L_j(\tilde{\alpha}_j e_{\phi,j-1}(t) + \alpha_j e_{\phi,j}(t))^2\right).$$

By applying Proposition 2.1 to each $H_{S_{j-1}S_j}(t)$, we obtain

$$\frac{dH_{S_{j-1}S_j}(t)}{dt} = e_{q,j-1}(t)e_{\phi,j-1}(t) - e_{q,j}(t)e_{\phi,j}(t).$$

Therefore, summing over all segments gives

$$\frac{dH_n(t)}{dt} = \sum_{j=1}^{n} \frac{dH_{S_{j-1}S_j}(t)}{dt} = \sum_{j=1}^{n} \left(e_{q,j-1}(t)e_{\phi,j-1}(t) - e_{q,j}(t)e_{\phi,j}(t)\right) = e_{q,0}(t)e_{\phi,0}(t) - e_{q,n}(t)e_{\phi,n}(t).$$

Finally, substituting the boundary conditions from (2.21) yields the dissipative property stated in (3.1). This completes the proof of the proposition.

Secondly, to analyze (2.20) from a frequency domain perspective, it is necessary to reformulate the equation into suitable state equations defined on a particular state space. To this end, we define the state space $X_n = \mathbb{C}^{2n}$ endowed with the inner product:

$$\langle X_n, Y_n\rangle_n = \sum_{j=1}^{n} C_j(\alpha_j x_{j-1} + \tilde{\alpha}_j x_j)(\alpha_j y_{j-1} + \tilde{\alpha}_j y_j) + \sum_{j=1}^{n} L_j(\tilde{\alpha}_j x_{n+j} + \alpha_j x_{n+j+1})(\tilde{\alpha}_j y_{n+j} + \alpha_j y_{n+j+1}), \tag{3.2}$$

where $X_n = (x_1, \cdots, x_{2n})$ and $Y_n = (y_1, \cdots, y_{2n}) \in X_n$ are elements of X_n. Note that in the definition of the inner product, additional terms like x_0, y_0, x_{2n+1}

and y_{2n+1} appear. These are assigned specific values: $x_0 = y_0 = 0$, $x_{2n+1} = kx_n$ and $y_{2n+1} = ky_n$ with $k = R^{-1} > 0$.

Now, we recast (2.20) into a vectorial form for clarity and convenience. To do so, we introduce the vectors $W_n(t) = (e_{q,1}(t), \cdots, e_{q,n-1}(t))^\top$ and $V_n(t) = (e_{\phi,0}(t), \cdots, e_{\phi,n}(t))^\top$ as the unknown variables of (2.20). Additionally, we define the $n \times n$ matrices D_n, $\tilde{D}_n$ and M_n as follows:

$$D_n = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix}, \quad \tilde{D}_n = \begin{pmatrix} \tilde{\alpha}_1 & & & \\ & \tilde{\alpha}_2 & & \\ & & \ddots & \\ & & & \tilde{\alpha}_n \end{pmatrix}, \quad M_n = \begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & \ddots \\ & & & 1 \end{pmatrix}. \tag{3.3}$$

We further introduce the $n \times n$ diagonal matrices $C_n = \mathrm{diag}\{C_1, C_2, \cdots, C_n\}$, $L_n = \mathrm{diag}\{L_1, L_2, \cdots, L_n\}$ and $E_n = \mathrm{diag}\{0, 0, \cdots, 1\}$ to facilitate the vectorial representation of (2.20). Consequently, the equation (2.20) can be rewritten in the concise form:

$$H_n \Phi_n \dot{X}_n(t) = \Psi_n X_n(t) \iff \dot{X}_n(t) = A_n X_n(t), \tag{3.4}$$

where $A_n = \Phi_n^{-1} H_n^{-1} \Psi_n$, and the vector $X_n(t)$ combines both $W_n(t)$ and $V_n(t)$ as

$$X_n(t) = \begin{pmatrix} W_n(t) \\ V_n(t) \end{pmatrix}.$$

The matrices involved in this representation are defined as follows:

$$H_n = \begin{pmatrix} C_n & 0 \\ 0 & L_n \end{pmatrix}, \quad \Phi_n = \begin{pmatrix} D_n & \tilde{D}_n \\ \tilde{D}_n & D_n \end{pmatrix}, \quad \Psi_n = \begin{pmatrix} 0 & -M_n^\top \\ M_n & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & R^{-1} E_n \end{pmatrix}.$$

It is noteworthy that Φ_n serves as a weighting operator, reflecting the influence of α_j and $\tilde{\alpha}_j$. If we substitute the weighting operator Φ_n with the identity operator I, then the equation (3.4) simplifies to the classical finite difference scheme of (2.1).

Furthermore, the inner product defined in (3.2) within the space X_n can be elegantly expressed as

$$\langle X_n, Y_n \rangle_{X_n} = \langle \Phi_n X_n, H_n \Phi_n Y_n \rangle, \quad \forall X_n, Y_n \in X_n,$$

where $X_n = (x_1, \cdots, x_{2n})$, $Y_n = (y_1, \cdots, y_{2n}) \in X_n$ under the conditions $x_0 = y_0 = 0$, $x_{2n+1} = kx_n$, and $y_{2n+1} = ky_n$ with $k = R^{-1} > 0$. Here $\langle \cdot, \cdot \rangle$ represents the standard inner product in $\mathbb{C}^{2n}$.

Lemma 3.1. The operator A_n is dissipative and generates a C_0-semigroup of contractions $T_n(t)$ on the state space X_n.

Proof. By setting $A_n X_n = Y_n$ with the boundary conditions $x_0 = y_0 = 0$, $x_{2n+1} = kx_n$ and $y_{2n+1} = ky_n$, for $j = 1, \cdots, n$ and any $X_n \in X_n$, we obtain

$$C_j(\alpha_j y_{j-1} + \tilde{\alpha}_j y_j) = x_{n+j} - x_{n+j+1}, \quad L_j(\tilde{\alpha}_j y_{n+j} + \alpha_j y_{n+j+1}) = x_{j-1} - x_j.$$

Subsequently, utilizing the definition of the inner product $\langle \cdot, \cdot \rangle_n$, we derive:

$$2\mathrm{Re}\langle A_n X_n, X_n \rangle_n = \langle A_n X_n, X_n \rangle_n + \langle X_n, A_n X_n \rangle_n = -2k|x_n|^2. \tag{3.5}$$

Equation (3.5) directly implies that the operator A_n is dissipative. The second statement is self-evident, thus completing the proof of this lemma.

Lemma 3.2. The intersection of the spectral set $\sigma(A_n)$ of the operator A_n with the imaginary axis is empty.

Proof. Given $X_n \in X_n$ and $s \in \mathbb{R}$, by setting $A_n X_n = isX_n$ with the boundary conditions $x_0 = 0$ and $x_{2n+1} = kx_n$, we obtain from (3.4)

$$isC_j(\alpha_j x_{j-1} + \tilde{\alpha}_j x_j) = x_{n+j} - x_{n+j+1}, \quad isL_j(\tilde{\alpha}_j x_{n+j} + \alpha_j x_{n+j+1}) = x_{j-1} - x_j. \tag{3.6}$$

From (3.5) and the fact that $A_n X_n = isX_n$, we have

$$0 = \mathrm{Re}\langle A_n X_n, X_n \rangle_n = -k|x_n|^2. \tag{3.7}$$

Setting $j = n$ in (3.6) and using (3.7) along with $x_{2n+1} = kx_n = 0$, we get

$$isC_j\alpha_j x_{n-1} - x_{2n} = 0, \quad -x_{n-1} + isL_j\tilde{\alpha}_j x_{2n} = 0.$$

This implies that $x_{n-1} = x_{2n} = 0$ since the determinant of the coefficient matrix

$$\begin{vmatrix} isC_j\alpha_j & -1 \\ -1 & isL_j\tilde{\alpha}_j \end{vmatrix} = -s^2\alpha_j\tilde{\alpha}_j C_j L_j - 1$$

is nonzero. Similarly, by setting $j = n - 1$ in (3.6) and using the fact that $x_{n-1} = x_{2n} = 0$, we can deduce that $x_{n-2} = x_{2n-1} = 0$. By induction on j, we can show that $x_j = 0$ for all $j = 1, \cdots, 2n$. This implies that $X_n = 0$, which contradicts the assumption that X_n is a nonzero eigenvector corresponding to an eigenvalue is. Therefore, is does not belong to the spectral set $\sigma(A_n)$ of the operator A_n. Thus, we have completed the proof of Lemma 3.2.

Now, we are ready to present the uniform stability standard, as outlined in [**?**], which will be instrumental in proving Theorem 3.2.

Theorem 3.1. Let $h^* > 0$ and consider a family of semigroups of contractions $(S_h(t))$ on the Hilbert space $(\tilde{X}_h)$. Let $(\tilde{A}_h)$ denote the corresponding infinitesimal generators. The family $(S_h(t))$ is uniformly exponentially stable if and only if the following two conditions are met:

(i) For all $h \in (0, h^*)$, $i\mathbb{R}$ is contained in the resolvent set $\rho(\tilde{A}_h)$ of $\tilde{A}_h$.

(ii) $\sup_{h\in(0,h^*),\beta\in\mathbb{R}} \|(i\beta I - \tilde{A}_h)^{-1}\|_{\mathcal{L}(\tilde{X}_h)} < \infty$.

By using the preceding results, we can now arrive at the main result of this section.

Theorem 3.2. Let h^* be $h^* = \max_{n\geq 1} \Delta_n$. Then the semigroups $T_n(t)$ generated by the operators A_n are uniformly exponentially stable, i.e., there exist two positive constants M and ω independent of Δ_n and t such that

$$\|T_n(t)\|_n \leq Me^{-\omega t}.$$

Proof. Since (i) of Theorem 3.1 has already been established in Lemma 3.2, our focus in this proof is to demonstrate that condition (ii) of Theorem 3.1 also holds. To do so, we employ a proof by contradiction.

Suppose, to the contrary, that condition (ii) is false. Then, for any $n \in \mathbb{N}$, there exist $\beta_n \in \mathbb{R}$, $m_n \in \mathbb{N}$, and $X_{m_n} \in X_{m_n}$ such that

$$\|X_{m_n}\|_{m_n} = 1 \quad \text{and} \quad \|Y_{m_n}\|_{m_n} \leq n^{-2}, \tag{3.8}$$

where $Y_{m_n} = (i\beta_n - A_{m_n})X_{m_n}$, with the property that $m_n \to \infty$ as $n \to \infty$.

Firstly, the equation $Y_{m_n} = (i\beta_n - A_{m_n})X_{m_n}$ is equivalent to the following system

$$\begin{cases} C_j[\alpha_j(i\beta_n x_{j-1} - y_{j-1}) + \tilde{\alpha}_j(i\beta_n x_j - y_j)] = x_{m_n+j} - x_{m_n+j+1}, \\ L_j[\tilde{\alpha}_j(i\beta_n x_{m_n+j} - y_{m_n+j}) + \alpha_j(i\beta_n x_{m_n+j+1} - y_{m_n+j+1})] = x_{j-1} - x_j, \end{cases}$$

where $j = 1, 2, \cdots, 2m_n$, with the initial conditions $x_0 = y_0 = 0$, $x_{2m_n+1} = kx_{m_n}$, and $y_{2m_n+1} = ky_{m_n}$. By regrouping the terms in the above equations, we obtain

$$\begin{cases} i\beta_n\alpha_jC_jx_{j-1} - x_{m_n+j} = C_j(\alpha_jy_{j-1} + \tilde{\alpha}_jy_j) - i\beta_n\tilde{\alpha}_jC_jx_j - x_{m_n+j+1}, \\ -x_{j-1} + i\beta_n\tilde{\alpha}_jL_jx_{m_n+j} = L_j(\tilde{\alpha}_jy_{m_n+j} + \alpha_jy_{m_n+j+1}) - i\beta_n\alpha_jL_jx_{m_n+j+1} - x_j. \end{cases} \tag{3.9}$$

Since the determinant of the coefficient matrix

$$D_j := \begin{vmatrix} i\beta_nC_j\alpha_j & -1 \\ -1 & i\beta_nL_j\tilde{\alpha}_j \end{vmatrix} = -\beta_n^2\alpha_j\tilde{\alpha}_jC_jL_j - 1$$

is always nonzero, we can uniquely solve the system (3.9) to obtain

$$\begin{cases} x_{j-1} = \frac{i\beta_n\tilde{\alpha}_jL_j}{D_j}[C_j(\alpha_jy_{j-1} + \tilde{\alpha}_jy_j) - i\beta_n\tilde{\alpha}_jC_jx_j - x_{m_n+j+1}], \\ x_{m_n+j} = \frac{1+i\beta_n\alpha_jC_j}{D_j}[L_j(\tilde{\alpha}_jy_{m_n+j} + \alpha_jy_{m_n+j+1}) - i\beta_n\alpha_jL_jx_{m_n+j+1} - x_j]. \end{cases}$$

For any $\beta_n \in \mathbb{R}$, we derive the following relations:

$$|D_j| \geq 1, \quad \left|\frac{C_jL_j\alpha_j\tilde{\alpha}_j\beta_n^2}{D_j}\right| < 1, \tag{3.10}$$

and

$$\sqrt{C_j}|\alpha_jy_{j-1} + \tilde{\alpha}_jy_j| = O(n^{-2}), \quad \sqrt{L_j}|\tilde{\alpha}_jy_{m_n+j} + \alpha_jy_{m_n+j+1}| = O(n^{-2}), \tag{3.11}$$

$$\sqrt{C_j}|\alpha_jx_{j-1} + \tilde{\alpha}_jx_j| = O(1), \quad \sqrt{L_j}|\tilde{\alpha}_jx_{m_n+j} + \alpha_jx_{m_n+j+1}| = O(1). \tag{3.12}$$

The relations (3.11) and (3.12) are straightforward consequences of $\|Y_{m_n}\|_{m_n} = O(n^{-2})$. For (3.13), we utilize the inequality $a^2 + b^2 \geq 2ab$. By Assumptions (2.1)-(2.3), it follows that $\sqrt{\alpha_j\tilde{\alpha}_jC_jL_j} = O(1)$. The inequality (3.14) can be proven similarly.

Secondly, (3.8) and (3.5) imply that

$$k|x_{m_n}|^2 = |\mathrm{Re}\langle(i\beta_n - A_{m_n})X_{m_n}, X_{m_n}\rangle_{m_n}| = |\mathrm{Re}\langle Y_{m_n}, X_{m_n}\rangle_{m_n}| \leq \|Y_{m_n}\|_{m_n} = O(n^{-2}).$$

Thus, by considering $x_{2m_n+1} = kx_{m_n}$, we derive

$$|x_{m_n}|^2 = O(n^{-2}), \quad |x_{2m_n+1}|^2 = O(n^{-2}). \tag{3.15}$$

Substituting (3.15) into (3.10) with $j = m_n$, we obtain

$$|x_{m_n-1}|^2 = O(n^{-2}), \quad |x_{2m_n}|^2 = O(n^{-2}), \tag{3.16}$$

since the coefficients of $C_j\alpha_j(y_{j-1} + \tilde{\alpha}_j y_j)$, $L_j(\tilde{\alpha}_j y_{m_n+j} + \alpha_j y_{m_n+j+1})$, x_{m_n} and x_{2m_n} in (3.10) are all of the forms of $1/D_j$, $|C_j L_j \alpha_j \tilde{\alpha}_j \beta_n^2 / D_j|$, respectively. Similarly, for $j = 1, 2, \cdots, m_n - 1$, it can be readily proven by induction and using an analogous approach as from (3.15) to (3.16) that

$$|x_{m_n-j}|^2 = O(n^{-2}), \quad |x_{2m_n+1-j}|^2 = O(n^{-2}). \tag{3.17}$$

Finally, by the definition of the norm $\|\cdot\|_{m_n}$ and the Assumptions 2.1 and 2.3, we have

$$\|X_{m_n}\|_{m_n}^2 = \sum_{j=1}^{m_n} C_j|\alpha_j x_{j-1} + \tilde{\alpha}_j x_j|^2 + \sum_{j=1}^{m_n} L_j|\tilde{\alpha}_j x_{m_n+j} + \alpha_j x_{m_n+j+1}|^2 \leq 4C_{m_n} \sum_{j=1}^{m_n}(|x_j|^2 + |x_{m_n+j}|^2) + C_{m_n}|x_{2m_n+1}|^2$$

which contradicts the assumption that $\|X_{m_n}\|_{m_n} = 1$.

4.1. PDE Timoshenko Beam

The Timoshenko beam equation, which describes the dynamics of a beam under transverse loading and rotational effects, is given by

$$\begin{cases} \rho \ddot{w}(t,z) = K(w'(t,z) - \phi(t,z))', \\ I_\rho \ddot{\phi}(t,z) = EI\phi''(t,z) + K(w'(t,z) - \phi(t,z)), \end{cases} \tag{4.1}$$

where - $w(t,z)$ represents the transverse displacement of the beam at time t and spatial position $z \in [0, S]$; - $\phi(t,z)$ is the rotation angle of a filament of the beam at time t and spatial position z; - $\rho(z)$, $I_\rho(z)$, $EI(z)$ and $K(z)$ are the material properties of the beam: mass per unit length, rotary moment of inertia of a cross-section, the product of Young's modulus of elasticity and the moment of inertia of a cross-section, and the shear modulus, respectively.

Here, the dot $\dot{}$ and the prime $'$ denote derivatives with respect to time and spatial variables, respectively.

Additionally, we assume that the beam is clamped at the left-hand side ($z = 0$), meaning that both the transverse displacement and the rotation angle are zero there. At the right-hand side ($z = S$), we apply a damping force proportional to the velocity of the transverse displacement. Therefore, the boundary conditions are

$$\dot{w}(t,0) = 0, \quad \dot{\phi}(t,0) = 0, \tag{4.2}$$

$$EI(S)\phi'(t,S) = -k_1\dot{\phi}(t,S), \quad K(S)(w'(t,S) + \phi(t,S)) = -k_2\dot{w}(t,S). \tag{4.3}$$

To formulate the Timoshenko beam model as a port-Hamiltonian system, we introduce the following physical notations that represent key dynamical quantities:

$$\begin{cases} x_1(t,z) = w_z(t,z) - \phi(t,z), & \text{shear displacement} \\ x_2(t,z) = \rho w_t(t,z), & \text{transverse momentum} \\ x_3(t,z) = \phi_z(t,z), & \text{angular displacement gradient} \\ x_4(t,z) = I_\rho \phi_t(t,z), & \text{angular momentum} \end{cases}$$

By utilizing equations (4.1) through (4.4), we derive the time derivatives of the variables $x_i(t,z)$ for $i = 1,2,3,4$, which are given by

$$\begin{cases} \dot{x}_1(t,z) = [\beta_2 x_2(t,z)]' - \beta_4 x_4(t,z), \\ \dot{x}_2(t,z) = [\beta_1 x_1(t,z)]', \\ \dot{x}_3(t,z) = [\beta_4 x_4(t,z)]', \\ \dot{x}_4(t,z) = [\beta_3 x_3(t,z)]' + \beta_1 x_1(t,z), \end{cases}$$

with boundary conditions

$$x_1(t,S) = -K_2 x_2(t,S), \quad x_2(t,0) = 0, \quad x_3(t,S) = -K_1 x_4(t,S), \quad x_4(t,0) = 0, \tag{4.5}$$

where $\beta_1 = K$, $\beta_2 = \rho^{-1}$, $\beta_3 = EI$ and $\beta_4 = I_\rho^{-1}$, $K_1 = k_1\beta_2(S)/\beta_1(S)$ and $K_2 = k_2\beta_4(S)/\beta_3(S)$.

Let us define the vector $x(t,z) = (x_1(t,z), x_2(t,z), x_3(t,z), x_4(t,z))^\top$ and introduce the diagonal matrix $L(z) = \operatorname{diag}\{\beta_1(z), \beta_2(z), \beta_3(z), \beta_4(z)\}$. Additionally, we define the matrices P_1 and P_0 as follows:

$$P_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad P_0 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Utilizing these definitions and the time derivative expressions derived in (4.5), we can rewrite the Timoshenko beam equations without boundary conditions into the form

$$\dot{x}(t,z) = P_1[L(z)x(t,z)]' + P_0[L(z)x(t,z)]. \tag{4.6}$$

The Hamiltonian/energy associated with either (4.6) or (4.5) is defined as

$$H(t) = \int_0^S x^\top(t,z)L(z)x(t,z)dz. \tag{4.7}$$

According to Exercise 9.2 in [**?**], it is shown that if both k_1 and k_2 are positive, then the system described by (4.5) is exponentially stable with respect to the energy $H(t)$. For a more comprehensive understanding of the well-posedness and exponential stability of (4.5), please also refer to theorem 2.1 in [**?**].

4.2. Semi-Discretization of Timoshenko Beam

In this subsection, we apply the concepts introduced in subsections 2.2 and 2.3 to discretize the spatial variable of the Timoshenko beam in its port-Hamiltonian form (4.6). For this purpose, we treat the components of the flow variables $x(t,z)$ and the matrix $L(z)$ as one-forms, whereas the components $e_1(t,z), e_2(t,z), e_3(t,z), e_4(t,z)$ of effort variables $e(t,z) := L(z)x(t,z)$ are considered as zero-forms. The partitioning of the interval $[0,S]$ into $I = \cup_{j=1}^n I_{S_{j-1}S_j}$ remains valid in this section.

To approximate $x_1(t,z)$ and $x_3(t,z)$ on each interval $[S_{j-1}, S_j]$, we utilize the one-form finite elements $\omega^q_{S_{j-1}S_j}$. Specifically,

$$x_1(t,z) = X_{1,j}(t)\omega^q_{S_{j-1}S_j}(z), \quad x_3(t,z) = X_{3,j}(t)\omega^q_{S_{j-1}S_j}(z),$$

where $\dot{X}_{1,j}(t) = x_{1,j}(t)$ and $\dot{X}_{3,j}(t) = x_{3,j}(t)$. The one-form finite elements $\omega^\phi_{S_{j-1}S_j}$ are employed to approximate $x_2(t,z)$ and $x_4(t,z)$ within the interval $[S_{j-1}, S_j]$. Specifically, the approximations take the form

$$x_2(t,z) = X_{2,j}(t)\omega^\phi_{S_{j-1}S_j}(z), \quad x_4(t,z) = X_{4,j}(t)\omega^\phi_{S_{j-1}S_j}(z),$$

where $\dot{X}_{2,j}(t) = x_{2,j}(t)$ and $\dot{X}_{4,j}(t) = x_{4,j}(t)$.

However, the zero-form finite elements $\omega^q_{S_{j-1}}(z), \omega^q_{S_j}(z)$ and $\omega^\phi_{S_{j-1}}(z), \omega^\phi_{S_j}(z)$ are utilized to approximate the effort variables $e_1(t,z), e_3(t,z)$ and $e_2(t,z), e_4(t,z)$ respectively, within the interval $[S_{j-1}, S_j]$. More precisely, we set:

$$\begin{cases} e_i(t,z) = e_{i,j-1}(t)\omega^q_{S_{j-1}}(z) + e_{i,j}(t)\omega^q_{S_j}(z), & i = 1,3, \\ e_i(t,z) = e_{i,j-1}(t)\omega^\phi_{S_{j-1}}(z) + e_{i,j}(t)\omega^\phi_{S_j}(z), & i = 2,4, \end{cases}$$

where $e_{i,j}(t)$ will be determined later for $i = 1, \cdots, 4$ and $j = 0, 1, \cdots, n$. By utilizing the concepts outlined in Remark 2.1, we derive the following relationships on the interval $[S_{j-1}, S_j]$:

$$e_i'(t,z) \approx \frac{e_{i,j}(t) - e_{i,j-1}(t)}{h_j}, \quad j = 1, 2, \cdots, n,$$

where $h_j = S_j - S_{j-1}$ and

$$\int_{I_{S_{j-1}S_j}} e_i(t,z) \approx \beta_{i,j} = \begin{cases} \alpha_j e_{i,j-1}(t) + \tilde{\alpha}_j e_{i,j}(t), & i = 1,3, \\ \tilde{\alpha}_j e_{i,j-1}(t) + \alpha_j e_{i,j}(t), & i = 2,4, \end{cases}$$

for $i = 1, \cdots, 4$.

Consequently, we formulate the semi-discretization scheme for (4.5) as follows:

$$\begin{cases} \beta_{1,j}[\alpha_j \dot{e}_{1,j-1}(t) + \tilde{\alpha}_j \dot{e}_{1,j}(t)] = e_{2,j}(t) - e_{2,j-1}(t) - h_j[\tilde{\alpha}_j e_{4,j-1}(t) + \alpha_j e_{4,j}(t)], \\ \beta_{2,j}[\tilde{\alpha}_j \dot{e}_{2,j-1}(t) + \alpha_j \dot{e}_{2,j}(t)] = e_{1,j}(t) - e_{1,j-1}(t), \\ \beta_{3,j}[\alpha_j \dot{e}_{3,j-1}(t) + \tilde{\alpha}_j \dot{e}_{3,j}(t)] = e_{4,j}(t) - e_{4,j-1}(t), \\ \beta_{4,j}[\tilde{\alpha}_j \dot{e}_{4,j-1}(t) + \alpha_j \dot{e}_{4,j}(t)] = e_{3,j}(t) - e_{3,j-1}(t) + h_j[\alpha_j e_{1,j-1}(t) + \tilde{\alpha}_j e_{1,j}(t)], \end{cases} \tag{4.8}$$

with boundary conditions

$$e_{1,n}(t) = -K_1 e_{2,n}(t), \quad e_{2,0}(t) = 0, \quad e_{3,n}(t) = -K_2 e_{4,n}(t), \quad e_{4,0}(t) = 0, \tag{4.9}$$

where $j = 1, 2, \cdots, n$. Furthermore, the discrete energy H_n is defined as follows:

$$H_n(t) = \frac{1}{2}\sum_{j=1}^{n} \left(\beta_{1,j}[\alpha_j e_{1,j-1}(t) + \tilde{\alpha}_j e_{1,j}(t)]^2 + \beta_{2,j}[\tilde{\alpha}_j e_{2,j-1}(t) + \alpha_j e_{2,j}(t)]^2\right) + \frac{1}{2}\sum_{j=1}^{n} \left(\beta_{3,j}[\alpha_j e_{3,j-1}(t) + \tilde{\alpha}_j e_{3,j}(t)]\right. \tag{4.10}$$

The state space associated with equations (4.8) through (4.13) resides in the Hilbert space, denoted as $X_n = \mathbb{C}^{4n}$, with inner product $\langle Z_n, \tilde{Z}_n \rangle_n$ defined as

$$\langle Z_n,\tilde{Z}_n\rangle_n=\sum_{j=1}^{n}\beta_{1,j}[\alpha_j z_{1,j-1}+\tilde{\alpha}_j z_{1,j}][\alpha_j\tilde{z}_{1,j-1}+\tilde{\alpha}_j\tilde{z}_{1,j}]+\sum_{j=1}^{n}\beta_{2,j}[\tilde{\alpha}_j z_{2,j-1}+\alpha_j z_{2,j}][\tilde{\alpha}_j\tilde{z}_{2,j-1}+\alpha_j\tilde{z}_{2,j}]+\sum_{j=1}^{n}\beta_{3,j}[\alpha_j z_3$$

where $Z_n, \tilde{Z}_n \in X_n$, subject to the additional constraints $z_{2,0} = z_{4,0} = 0$, $z_{1,n} = -K_1 z_{2,n}$, and $z_{3,n} = -K_2 z_{4,n}$.

To express equations (4.8) through (4.13) in a vectorial form, we introduce several $n \times n$ matrices. Specifically, let M_n and E_n be defined as outlined in section 3. Furthermore, define $C_n = \text{diag}\{h_1, h_2, \cdots, h_n\}$, $B_n = \text{diag}\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, $\tilde{B}_n = \text{diag}\{\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\}$, $L_n = \text{diag}(L_1, L_2, L_3, L_4)$, with $L_i = \text{diag}\{\beta_{i,1}, \beta_{i,2}, \cdots, \beta_{i,n}\}$, $i = 1, \dots, 4$,

$$\Phi_n=\begin{pmatrix}B_n & \tilde{B}_n\\ \tilde{B}_n & B_n\end{pmatrix},\quad \Psi_n=\begin{pmatrix}0 & -M_n^\top\\ M_n & 0\end{pmatrix},\quad \Omega_n=\begin{pmatrix}0 & -C_n\tilde{B}_n\\ C_nB_n & 0\end{pmatrix}-\begin{pmatrix}K_1E_n & 0\\ 0 & K_2E_n\end{pmatrix}.$$

The state variables of equations (4.8) through (4.13) are consolidated into the vector $Y_n(t)$, defined as:

$$Y_n(t)=(y_{1,n}^\top(t),y_{2,n}^\top(t),y_{3,n}^\top(t),y_{4,n}^\top(t))^\top,$$

where each component vector $y_{i,n}(t)$ is specified as follows:

$$y_{1,n}(t)=(e_{1,0}(t),\cdots,e_{1,n-1}(t))^\top,\quad y_{2,n}(t)=(e_{2,1}(t),\cdots,e_{2,n}(t))^\top,$$

$$y_{3,n}(t)=(e_{3,0}(t),\cdots,e_{3,n-1}(t))^\top,\quad y_{4,n}(t)=(e_{4,1}(t),\cdots,e_{4,n}(t))^\top.$$

Subsequently, the system of equations (4.8) through (4.13) can be equivalently expressed in the compact form

$$\begin{cases}\dot{Y}_n(t)=A_nY_n(t),\\ Y_n(0)=(y_{1,h},y_{3,h},y_{2,h},y_{4,h})^\top\in X_n,\end{cases}\tag{4.11}$$

where the matrix $A_n = \Phi_n^{-1}L_n^{-1}(\Psi_n + \Omega_n)$, assuming that both Φ_n and L_n are invertible, as stated.

Furthermore, for any $Z_n, \tilde{Z}_n \in X_n$, the inner product in the space X_n can be reformulated as

$$\langle Z_n,\tilde{Z}_n\rangle_n=\langle\Phi_nZ_n,L_n\Phi_n\tilde{Z}_n\rangle,\tag{4.12}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^{4n}$.

In the context of this section, Assumptions 2.1 and 2.2 continue to hold. However, Assumption 2.3 is superseded by the following:

Assumption 4.1. Assume that there exist two positive constants c' and C' such that for all $i = 1, \cdots, 4$, the inequality $c' \leq \beta_i \leq C'$ holds. Furthermore, when combined with Assumption 2.1, it implies that both $\max_{1\leq j\leq n} \beta_{i,j} = O(n^{-1})$ and $\min_{1\leq j\leq n} \beta_{i,j} = O(n^{-1})$ are valid.

Furthermore, we require an additional assumption regarding h_j, which is crucial for our analysis.

Assumption 4.2. Assume that h_j satisfies the following inequality for all $j = 1, 2, \cdots, n$:

$$h_j^2 \leq \frac{2\alpha_j \tilde{\alpha}_j}{\beta_{1,j}\beta_{3,j}}. \tag{4.13}$$

Given these assumptions, we can now present the following result on dissipativity.

Lemma 4.1. The operator A_n is dissipative on the space X_n for all $n \in \mathbb{N}^+$. Consequently, A_n generates a family of semigroups of contractions $T_n(t)$ that ensures the existence and uniqueness of the solution $Y_n(t)$ to the system of equations (4.8)-(4.13). This solution satisfies the balance equations in their discrete version, which can be expressed as

$$\dot{H}_n(t) = -K_1|e_{2,n}(t)|^2 - K_2|e_{4,n}(t)|^2. \tag{4.14}$$

Proof. Given $Z_n \in X_n$ defined by (4.15), we can derive the following identity by considering the inner product involving the operator A_n:

$$\langle Z_n, A_n Z_n\rangle_n + \langle A_n Z_n, Z_n\rangle_n = \langle \Phi_n Z_n, (\Psi_n + \Omega_n) Z_n\rangle + \langle (\Psi_n + \Omega_n) Z_n, \Phi_n Z_n\rangle. \tag{4.15}$$

From the definition of Φ_n, Ψ_n and Ω_n, we have

$$\langle \Phi_n Z_n, (\Psi_n+\Omega_n) Z_n\rangle = \sum_{j=1}^{n} \beta_{1,j}[\alpha_j z_{1,j-1} + \tilde{\alpha}_j z_{1,j}][z_{2,j} - z_{2,j-1}] + \sum_{j=1}^{n} \beta_{2,j}[\tilde{\alpha}_j z_{2,j-1} + \alpha_j z_{2,j}][z_{1,j} - z_{1,j-1}] + \sum_{j=1}^{n} \beta_{3,j}[\alpha_j \tag{4.16}$$

By substituting equations (4.16) and (4.17) into (4.15), and leveraging the same calculation methodology as in (3.5), we arrive at the following result

$$\mathrm{Re}\langle A_n Z_n, Z_n\rangle_n = -K_1|z_{2,n}|^2 - K_2|z_{4,n}|^2, \tag{4.18}$$

where we have utilized the identities $z_{1,n} = -K_1 z_{2,n}$ and $z_{3,n} = -K_2 z_{4,n}$. This confirms that the balance equation (4.14) holds true, as can be seen by considering the energy functional $H_n(t) = \langle Y_n(t), Y_n(t)\rangle_n$ and utilizing equation (4.11).

To provide a precise frequency domain analysis, we establish a crucial Lemma 4.2 following.

Lemma 4.2. Define Γ_j by

$$\Gamma_j := \begin{vmatrix} i\alpha_j\beta_{1,j}\beta & i\tilde{\alpha}_j\beta_{2,j}\beta & -h_j\alpha_j & h_j\tilde{\alpha}_j \\ i\alpha_j\beta_{3,j}\beta & i\tilde{\alpha}_j\beta_{4,j}\beta & 0 & 0 \\ 0 & 0 & i\alpha_j\beta_{3,j}\beta & i\tilde{\alpha}_j\beta_{4,j}\beta \\ -h_j\alpha_j & h_j\tilde{\alpha}_j & i\alpha_j\beta_{1,j}\beta & i\tilde{\alpha}_j\beta_{2,j}\beta \end{vmatrix} = 1+(\beta_{1,j}\beta_{2,j}+\beta_{3,j}\beta_{4,j}-h_j^2\alpha_j\tilde{\alpha}_j\beta_{2,j}\beta_{3,j})\alpha_j\tilde{\alpha}_j\beta^2+\beta_{1,j}\beta_{2,j}\beta_{3,j}\beta_{4,j}$$

for any $\beta \in \mathbb{R}$ and $0 < h_j \le \Delta_n$. Then,

$$\Gamma_j \ge 1 + (\beta_{1,j}\beta_{2,j} + \beta_{3,j}\beta_{4,j})\alpha_j\tilde{\alpha}_j\beta^2 + \beta_{1,j}\beta_{2,j}\beta_{3,j}\beta_{4,j}\alpha_j^2\tilde{\alpha}_j^2\beta^4 \ge 1. \tag{4.19}$$

Furthermore, if we define $I_{i,j} = (\beta_{*,j})^i\beta^i\Gamma^{-1}$ for $i = 0,1,2,3,4$, where $(\beta_{*,j})^i$ is the product of i terms selected from $\{\beta_{1,j}, \beta_{2,j}, \beta_{3,j}, \beta_{4,j}\}$, then $I_{i,j}$ are uniformly bounded, and the upper bounds are all independent of β. Specifically, the following estimates hold:

$$|I_{i,j}| = O(1), \quad i = 0,1,2,3,4. \tag{4.20}$$

Proof. The inequality (4.19) holds due to Assumption 4.2, which implies that the middle term of the right-hand side of (4.18) satisfies

$$(\beta_{1,j}\beta_{2,j} + \beta_{3,j}\beta_{4,j} - h_j^2\alpha_j\tilde{\alpha}_j\beta_{2,j}\beta_{3,j})\alpha_j\tilde{\alpha}_j\beta^2 \ge [\beta_{1,j}\beta_{2,j} + \beta_{3,j}\beta_{4,j}]\alpha_j\tilde{\alpha}_j\beta^2.$$

This ensures the inequality $|I_{0,j}| \le 1$. Furthermore, by (4.19), Assumption 4.1, and Young's inequality, we can derive bounds for $|I_{1,j}|$ for $i = 1,2,3,4$. Specifically,

$$|I_{1,j}| \le \frac{2\beta_{*,j}|\beta|}{1 + (\beta_{1,j}\beta_{2,j} + \beta_{3,j}\beta_{4,j})\alpha_j\tilde{\alpha}_j\beta^2} \le \frac{(\beta_{*,j})^2\beta^2}{1 + \beta_{1,j}\beta_{2,j}\beta_{3,j}\beta_{4,j}\alpha_j^2\tilde{\alpha}_j^2\beta^4} \le \frac{(\beta_{*,j})^2}{\sqrt{2\beta_{1,j}\beta_{2,j}\beta_{3,j}\beta_{4,j}}} = O(1),$$

and similarly for $|I_{2,j}|, |I_{3,j}|, |I_{4,j}|$, which implies that (4.20) hold for $i = 1, 2, 3, 4$.

The dissipativity of A_n guarantees that the spectral set $\sigma(A_n)$ of A_n lies strictly within the open left half-plane of the complex plane. This is a strengthening of the basic fact that $\sigma(A_n)$ lies within the closed left half-plane, as dissipativity implies that the real part of every eigenvalue of A_n is strictly negative, ensuring that no eigenvalues lie on the imaginary axis. Thus, for any $n \in \mathbb{N}$, the spectral set $\sigma(A_n)$ is contained entirely within the open left half-plane of $\mathbb{C}$.

Lemma 4.3. For every $n \in \mathbb{N}$, $i\mathbb{R} \subset \rho(A_n)$.

Proof. If there exist $\beta \in \mathbb{R}$ and nonzero $Z_n \in X_n$ such that $i\beta Z_n = A_n Z_n$, then it follows from the definition of A_n that $A_n Z_n = i\beta Z_n$ is equivalent to

$$\begin{cases} i\beta\beta_{1,j}[\alpha_j z_{1,j-1} + \tilde{\alpha}_j z_{1,j}] = z_{2,j} - z_{2,j-1} - h_j[\tilde{\alpha}_j z_{4,j-1} + \alpha_j z_{4,j}], \\ i\beta\beta_{2,j}[\tilde{\alpha}_j z_{2,j-1} + \alpha_j z_{2,j}] = z_{1,j} - z_{1,j-1}, \\ i\beta\beta_{3,j}[\alpha_j z_{3,j-1} + \tilde{\alpha}_j z_{3,j}] = z_{4,j} - z_{4,j-1}, \\ i\beta\beta_{4,j}[\tilde{\alpha}_j z_{4,j-1} + \alpha_j z_{4,j}] = z_{3,j} - z_{3,j-1} + h_j[\alpha_j z_{1,j-1} + \tilde{\alpha}_j z_{1,j}]. \end{cases} \tag{4.21}$$

On the other hand, from (4.18), we derive the following:

$$0 = \mathrm{Re}\langle i\beta Z_n, Z_n\rangle_n = \mathrm{Re}\langle A_n Z_n, Z_n\rangle_n = -K_1|z_{2,n}|^2 - K_2|z_{4,n}|^2.$$

Given that $z_{1,n} = -K_1 z_{2,n}$ and $z_{3,n} = -K_2 z_{4,n}$, the only way for the above equation to hold true is if $z_{1,n} = z_{2,n} = z_{3,n} = z_{4,n} = 0$. Substituting these values into (4.21) with $j = n$, we obtain

$$\begin{cases} i\beta\beta_{1,n}\alpha_n z_{1,n-1} + z_{2,n-1} + h_n\tilde{\alpha}_n z_{4,n-1} = 0, \\ z_{1,n-1} + i\beta\beta_{2,n}\tilde{\alpha}_n z_{2,n-1} = 0, \\ i\beta\beta_{3,n}\alpha_n z_{3,n-1} + z_{4,n-1} = 0, \\ -h_n\alpha_n z_{1,n-1} + z_{3,n-1} + i\beta\beta_{4,n}\tilde{\alpha}_n z_{4,n-1} = 0. \end{cases} \tag{4.22}$$

The coefficients determinant of above equations is just Γ_n which is defined in Lemma 4.2 with $j = n$. Since Lemma 4.2 shows that the coefficients determinant Γ_j is positive, and by using Cramer's rule, $\Gamma_j \neq 0$ is equivalent to the unique solution to the equations (4.22) being zeros, i.e., $z_{1,n-1} = z_{2,n-1} = z_{3,n-1} = z_{4,n-1} = 0$. By induction and employing the same reasoning from (4.21) to (4.22), we can conclude that $z_{i,j} = 0$ for all $i = 1, 2, 3, 4$ and $j = 0, 1, \cdots, n$. Consequently, $Z_n = 0$, which contradicts the initial assumption that Z_n was nonzero.

Now, we are poised to present the main result of this section. Accordingly, we define h^* as

$$h^* = \max\left\{\Delta_n, \max_{1\le j\le n} \frac{2\alpha_j\tilde{\alpha}_j}{\beta_{1,j}\beta_{3,j}}\right\}. \tag{4.23}$$

Theorem 4.1. For the matrices A_n defined by (4.11), the associated family of C_0-semigroups $T_n(t)$ generated by A_n is uniformly exponentially stable. Specifically, there exist two constants $M > 0$ and $\omega > 0$, both independent of t and $\Delta_n \in (0, h^*)$, such that for all Δ_n in this range,

$$\|T_n(t)\| \le Me^{-\omega t}, \quad \forall t \ge 0. \tag{4.24}$$

Proof. Building upon Lemma 4.1, we know that for every $\Delta_n \in (0, h^*)$, the semigroup $T_n(t)$ is a C_0-semigroup of contractions. Moreover, Lemma 4.3 has already verified that A_n satisfies the first condition of Theorem 3.1.

Now, following a similar contradiction argument as in the proof of Theorem 3.2, we suppose that the second condition of Theorem 3.1 is false. If this supposition holds, then there exists a sequence $\{(\beta_k, \Delta_{n_k}, Z_k)\}_{k\in\mathbb{N}^+}$ with $n_k + 1 = \lfloor 1/\Delta_{n_k}\rfloor$ (where $\lfloor a\rfloor$ denotes the largest integer less than or equal to the real number a), $\beta_k \in \mathbb{R}$, $\Delta_{n_k} \in (0, h^*)$, and $Z_k \in X_{n_k}$ such that

$$\|Z_k\|_{n_k} = 1, \quad \|U_k\|_{n_k} \le k^{-2}, \quad U_k := (i\beta_k I_{n_k} - A_{n_k})Z_k.$$

The proof is structured into the following three steps for clarity and precision.

Step 1: $U_k = (i\beta_k I_{n_k} - A_{n_k})Z_k$ is equivalent to

$$\begin{cases} \beta_{1,j}[i\beta_k(\alpha_j z_{1,j-1} + \tilde{\alpha}_j z_{1,j}) - (\alpha_j u_{1,j-1} + \tilde{\alpha}_j u_{1,j})] = z_{2,j} - z_{2,j-1} - h_j(\tilde{\alpha}_j z_{4,j-1} + \alpha_j z_{4,j}), \\ \beta_{2,j}[i\beta_k(\tilde{\alpha}_j z_{2,j-1} + \alpha_j z_{2,j}) - (\tilde{\alpha}_j u_{2,j-1} + \alpha_j u_{2,j})] = z_{1,j} - z_{1,j-1}, \\ \beta_{3,j}[i\beta_k(\alpha_j z_{3,j-1} + \tilde{\alpha}_j z_{3,j}) - (\alpha_j u_{3,j-1} + \tilde{\alpha}_j u_{3,j})] = z_{4,j} - z_{4,j-1}, \\ \beta_{4,j}[i\beta_k(\tilde{\alpha}_j z_{4,j-1} + \alpha_j z_{4,j}) - (\tilde{\alpha}_j u_{4,j-1} + \alpha_j u_{4,j})] = z_{3,j} - z_{3,j-1} + h_j(\alpha_j z_{1,j-1} + \tilde{\alpha}_j z_{1,j}), \end{cases} \tag{4.25}$$

for $j = 1, \cdots, n_k$. To streamline the above formulas, we introduce the simplifications $z_{2,0} = z_{4,0} = 0$, $z_{1,n_k} = -K_1 z_{2,n_k}$, and $z_{3,n_k} = -K_2 z_{4,n_k}$ for Z_k. Similarly, we apply these settings to U_k.

Rearranging equations (4.25) results in

$$\begin{cases} i\beta_k\beta_{1,j}\alpha_j z_{1,j-1} + z_{2,j-1} + h_j\tilde{\alpha}_j z_{4,j-1} = b_{1,j}, \\ z_{1,j-1} + i\beta_k\beta_{2,j}\tilde{\alpha}_j z_{2,j-1} = b_{2,j}, \\ i\beta_k\beta_{3,j}\alpha_j z_{3,j-1} + z_{4,j-1} = b_{3,j}, \\ -h_j\alpha_j z_{1,j-1} + z_{3,j-1} + i\beta_k\beta_{4,j}\tilde{\alpha}_j z_{4,j-1} = b_{4,j}, \end{cases} \tag{4.26}$$

where

$$\begin{cases} b_{1,j} = \beta_{1,j}(\alpha_j u_{1,j-1} + \tilde{\alpha}_j u_{1,j}) + z_{2,j} - i\beta_k \beta_{1,j} \tilde{\alpha}_j z_{1,j} - h_j \alpha_j z_{4,j}, \\ b_{2,j} = \beta_{2,j}(\tilde{\alpha}_j u_{2,j-1} + \alpha_j u_{2,j}) - i\beta_k \beta_{2,j} \alpha_j z_{2,j} + z_{1,j}, \\ b_{3,j} = \beta_{3,j}(\alpha_j u_{3,j-1} + \tilde{\alpha}_j u_{3,j}) - i\beta_k \beta_{3,j} \tilde{\alpha}_j z_{3,j} + z_{4,j}, \\ b_{4,j} = \beta_{4,j}(\tilde{\alpha}_j u_{4,j-1} + \alpha_j u_{4,j}) + h_j \tilde{\alpha}_j z_{1,j} + z_{3,j} - i\beta_k \beta_{4,j} \alpha_j z_{4,j}. \end{cases} \tag{4.27}$$

It is straightforward to observe that the determinant of the coefficient matrix associated with equations (4.26) is precisely Γ_j, as defined in Lemma 4.2, where $\beta = \beta_k$. According to Lemma 4.2, Γ_j is guaranteed to be nonzero, implying that the system of equations (4.26) possesses a unique solution for $(z_{1,j-1}, z_{2,j-1}, z_{3,j-1}, z_{4,j-1})^\top$, given by

$$\begin{pmatrix} z_{1,j-1} \\ z_{2,j-1} \\ z_{3,j-1} \\ z_{4,j-1} \end{pmatrix} = \Gamma_j^{-1} \begin{pmatrix} a_{2,j} B_j & -h_j \alpha_j a_{2,j} & -h_j \tilde{\alpha}_j a_{2,j} a_{3,j} & h_j \tilde{\alpha}_j \\ a_{1,j} B_j + \tilde{\alpha}_j \alpha_j h_j^2 a_{3,j} & -h_j \tilde{\alpha}_j a_{2,j} a_{3,j} & h_j \tilde{\alpha}_j & -h_j \alpha_j a_{3,j} \\ -h_j \alpha_j a_{2,j} a_{3,j} & a_{4,j} A_j + \alpha_j \tilde{\alpha}_j h_j^2 a_{2,j} & -h_j \tilde{\alpha}_j a_{2,j} a_{3,j} & h_j \tilde{\alpha}_j a_{3,j} \\ h_j \tilde{\alpha}_j a_{3,j} & -h_j \alpha_j a_{3,j} & a_{3,j} A_j & -h_j \tilde{\alpha}_j a_{3,j} \end{pmatrix} \begin{pmatrix} b_{1,j} \\ b_{2,j} \\ b_{3,j} \\ b_{4,j} \end{pmatrix}, \tag{4.28}$$

where $a_{1,j} = i\beta_k \beta_{1,j} \alpha_j$, $a_{2,j} = i\beta_k \beta_{2,j} \tilde{\alpha}_j$, $a_{3,j} = i\beta_k \beta_{3,j} \alpha_j$, $a_{4,j} = i\beta_k \beta_{4,j} \tilde{\alpha}_j$, $A_j = a_{1,j} a_{2,j} - 1$, and $B_j = a_{3,j} a_{4,j} - 1$.

Step 2: It is easy to see that $\|U_k\|_{n_k} \le k^{-2}$ directly implies that

$$\beta_{i,j} |(\alpha_j u_{i,j-1} + \tilde{\alpha}_j u_{i,j})| = O(k^{-2}), \quad i = 1, 2, 3, 4, \tag{4.29}$$

for all $j = 1, 2, \cdots, n_k$. From (4.25) and (4.18), we deduce that

$$K_1 |z_{2,n_k}|^2 + K_2 |z_{4,n_k}|^2 = \mathrm{Re}\langle (i\beta_k - A_{n_k}) Z_k, Z_k \rangle_{n_k} = \mathrm{Re}\langle U_k, Z_k \rangle_{n_k} \le k^{-2}.$$

Combining this with the given conditions $z_{1,n_k} = -K_1 z_{2,n_k}$ and $z_{3,n_k} = -K_2 z_{4,n_k}$, we obtain

$$z_{i,n_k} = O(k^{-1}), \quad i = 1, 2, 3, 4. \tag{4.30}$$

Step 3: We assert the claim that

$$z_{i,j} = O(k^{-1}), \quad i = 1, 2, 3, 4, \quad j = 1, 2, \cdots, n_k. \tag{4.31}$$

In fact, from (4.27) and (4.28) with $j = n_k$, we derive

$$|z_{1,n_k-1}| = |\Gamma_{n_k}^{-1}|\cdot|a_{2,n_k}B_{n_k}(\beta_{1,n_k}(\alpha_{n_k}u_{1,n_k-1}+\tilde{\alpha}_{n_k}u_{1,n_k})+z_{2,n_k}-i\beta_k\beta_{1,n_k}\tilde{\alpha}_{n_k}z_{1,n_k}-h_{n_k}\alpha_{n_k}z_{4,n_k})+B_{n_k}(\beta_{2,n_k}(\tilde{\alpha}_n$$

The coefficients of $\beta_{1,n_k}|(\alpha_{n_k}u_{1,n_k-1}+\tilde{\alpha}_{n_k}u_{1,n_k})|$, $\beta_{2,n_k}|(\tilde{\alpha}_{n_k}u_{2,n_k-1}+\alpha_{n_k}u_{2,n_k})|$, $\beta_{3,n_k}|(\alpha_{n_k}u_{3,n_k-1}+\tilde{\alpha}_{n_k}u_{3,n_k})|$, $\beta_{4,n_k}|(\tilde{\alpha}_{n_k}u_{4,n_k-1}+\alpha_{n_k}u_{4,n_k})|$ along with $|z_{i,n_k}|$ $(i=1,2,3,4)$ all share the form of $|I_{i,n_k}|$ as defined in Lemma 4.2 with $j=n_k$ and $\beta=\beta_k$. According to Lemma 4.2 and equations (4.29)-(4.30), it follows that $z_{1,n_k-1}=O(k^{-1})$. Similarly, we can deduce that $|z_{i,n_k-1}|=O(k^{-1})$ for $i=2,3,4$. This establishes that (4.30) implies (4.31) for $j=n_k-1$ with the aid of (4.28)-(4.29). By extending this reasoning through induction, we can prove that (4.31) holds for all $j=1,2,\cdots,n_k$.

Finally, utilizing (4.31) alongside Assumptions 2.1 and 4.1, we obtain

$$\|Z_k\|_{n_k}^2 = \sum_{j=1}^{n_k}\beta_{1,j}|\alpha_j z_{1,j-1}+\tilde{\alpha}_j z_{1,j}|^2+\sum_{j=1}^{n_k}\beta_{2,j}|\tilde{\alpha}_j z_{2,j-1}+\alpha_j z_{2,j}|^2+\sum_{j=1}^{n_k}\beta_{3,j}|\alpha_j z_{3,j-1}+\tilde{\alpha}_j z_{3,j}|^2+\sum_{j=1}^{n_k}\beta_{4,j}|\tilde{\alpha}_j z_{4,j-1}+\alpha_j$$

which contradicts the fact that $\|Z_k\|_{n_k}=1$. This completes the proof of the theorem.

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5. Numerical Simulations

In this section, we present numerical simulations to demonstrate the validity of our theoretical findings. These simulations are performed under the assumption of uniform mesh size $h_j=h=S/n$, where $S=1$. The coefficients C_j and L_j in (2.20) are approximated as $C(jh)h\approx C_j$, $L(jh)h\approx L_j$, $j=0,\cdots,n$.

We present four figures to highlight the significance of the discrete schemes (2.20) and (3.4). In Figure 1 [Figure 1: see original paper], the blue points represent the maximal real parts of the eigenvalues of A_n obtained from (3.4) with $\alpha_j=3/4$ and the inclusion of the weighting operator Φ_n. In contrast, the red points represent the maximal real parts of the eigenvalues of A_n using the same α_j but without the weighting operator Φ_n, which essentially reduces to the classical finite difference scheme. Notably, the maximal real parts of the red points approach zero, indicating that the classical finite difference scheme fails to uniformly preserve the exponential stability of (2.1). This observation aligns with the conclusions presented in [?]. However, for the semidiscrete scheme (2.20) with the same step size, the maximal real parts of the eigenvalues approach a negative number, which is consistent with the statement of Theorem 3.1.

With $\alpha_j=3/4$, Figure 2 [Figure 2: see original paper] illustrates the distribution of the eigenvalues of A_n both with and without the weighting operator Φ_n. This figure reinforces the same conclusions drawn from Figure 1. Similarly, Figures

3 and 4 mirror Figures 1 and 2, respectively, but with $\alpha_j = 1/2$. For $\alpha_j = 1/2$, similar numerical simulations were previously reported in [**?**]. In these figures, we use $C(z) = \ln(1+z)$, $L(z) = \exp(z)$, $R = 5$.

Furthermore, numerical experiments indicate that the discrete schemes (2.20) and (3.4) with $\alpha_j = 1/2$ and identical parameters exhibit an optimal decay rate. Specifically, the maximal real parts observed in Figures 3 and 4 are notably smaller than those in Figures 1 and 2, respectively. These findings are consistent across various numerical simulations conducted with $\alpha_j \neq 1/2$. However, we are only drawing conclusions from numerical simulation results, which deserve rigorous theoretical verification in the future.

For the Timoshenko beam model, we have generated four additional figures, numbered Figure 5 [Figure 5: see original paper] through Figure 8 [Figure 8: see original paper], to showcase the numerical results obtained from the discrete schemes (4.8)-(4.11) or (4.11). For these simulations, we set $K_1 = K_2 = 1$, and choose $\beta_1 = \exp(z)$, $\beta_2 = 1 + 2z$, $\beta_3 = 2 + \sin z$, and $\beta_4 = 1 + z^2$. Figures 5 through 8 demonstrate the same effectiveness and conclusions as Figures 1 through 4.

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Note: Figure translations are in progress. See original paper for figures.