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Full Text

The Three-Dimensionality of Space Can Be Explained by the Properties of Photons

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Abstract

The three-dimensionality of space is usually introduced as a presupposed fact in modern physics, and the three-dimensionality of particle motion—that is, the three-dimensionality of particle momentum—is then a result of this fact. In this article, I propose a new logical possibility: that the three-dimensionality of photon momentum can be regarded as an inherent property of the photon

itself, and the three-dimensionality of space is a natural consequence of this property. Since H. Poincaré and A. Einstein, it has been recognized that the properties of spacetime are related to how people measure and define it. Both in daily life and in physics research, light plays a crucial role in how people gain experience and concepts about spacetime. The characteristics attributed to classical four-dimensional spacetime may simply reflect the inherent properties associated with photons.

1. Introduction

At least since ancient Greece, people have realized that the space in which we live is three-dimensional [1]. We learn as children that an object has height, width, and depth, and later that no more than three mutually perpendicular lines can be drawn from a point (this goes back at least to Galileo) [2].

From a deeper physical perspective, Kant first realized that there was an intimate connection between the inverse square law of gravitation and the existence of precisely three spatial dimensions [3, 4]. The existence of axial vector representations of quantities such as the magnetic vector **B** and the structure of electromagnetic fields is a consequence of the three-dimensional nature of space, as shown by Ehrenfest [5, 6]. The equipartition theorem in thermodynamics tells us that a system at equilibrium has an energy of $T/2$ per degree of freedom, and thus as a result of the three-dimensionality of space, $E = 3NT/2$ for an ideal monoatomic gas containing N atoms. In addition, Weyl pointed out that only in $(3 + 1)$-dimensional spacetime can Maxwell theory, in particular the laws for the propagation of light, be conformal-invariant [7, 8].

Although the three-dimensionality of space seems to be an unquestionable fact, the idea that there might be more than three spatial dimensions is not very startling to contemporary physicists. In order to unite gravity with electromagnetism, Kaluza and Klein proposed a $4+1$-dimensional universe [9-11]. To unite all of the fundamental interactions, superstring theory and M-theory were proposed to solve the problem of the non-renormalizability of quantum gravity, at the cost of requiring the dimension of spacetime to be ten (for superstrings) or eleven (for M-theory) [12-15].

By following the Kaluza–Klein approach of taking the extra dimensions to be a small compact manifold, one can argue that the resulting $3 + 1$-dimensional theory can reproduce the Standard Model at low energies. On the other hand, hints have emerged over the past few years from quantum gravity suggesting that the dimension of spacetime is dynamical and scale-dependent, and shrinks to $d \sim 2$ at very small distances, or equivalently, high energies [16, 17].

Physicists now have a better understanding of spatial dimensions, but to the ancient question of why the space we live in is three-dimensional, a straightforward explanation is still desired [18]. Ptkov, among others, suggested that one of the most fundamental questions of the 21st century is the search for an explanation of the dimensionality of the world [19]. In physics, explanation means figuring out how one or more physical facts lead to another. In the usual physical description, it is always assumed that three-dimensionality is a property of space itself, and the three-dimensionality of particle momentum is a following result of this property. However, if we look closely at the experiences which make us think that space is three-dimensional, we will find that they are directly or indirectly related to the momentum of photons. The orientation of celestial bodies is determined by photons from them; the different orientations correspond to different momentum directions of the photons. The three-dimensionality of the spatial structure of objects around us, including various experimental instruments, is also known from photons. The momentum of photons from different parts of these objects to the reference point has a three-dimensional structure. Photons are not only tools for us to observe the shape and position of objects, but they are also intermediate particles that transmit electromagnetic interactions. It is precisely these electromagnetic interactions that maintain the shapes of objects around us, ranging from daily necessities to experimental instruments. These considerations suggest a logical possibility: that the three-dimensionality of photon momentum is a property of photons themselves, and the three-dimensionality of space is a result of this property of photons. Henri Poincaré once proposed that "···the properties of space are merely those of the measuring instruments," and in addition, "···a ray of light is also one of our instruments" [20]. Special relativity provides an example of support for Poincaré' s view. The space and time coordinates in different inertial systems satisfy the Lorentz transformation, which depends on the fact that the propagation velocity of light in vacuum is independent of the choice of the inertial system [21].

2. The Three-Dimensionality of Photon Momentum

Let us argue the logical possibility that the three-dimensionality of photon momentum is a property of the photon itself. As a kind of massless particle, all photons have inherent momentum due to the fact that they cannot stand still in any inertial system. It seems that photons maintain their motion due to their own characteristics. This is different from non-zero mass particles, whose momentum can be zero in a particular reference frame. Thus, it is difficult to regard the characteristics of the momentum of these particles as the nature of these particles themselves. As a kind of spin-1 particle, the spin space of photons is expected to be three-dimensional. However, the description of the spin state of a photon with a given momentum direction $\hat{\mathbf{k}}$ requires only two base states, such as the right-handed polarization state $|S_{\hat{\mathbf{k}}} = +\hbar\rangle$ and the left-handed polarization state $|S_{\hat{\mathbf{k}}} = -\hbar\rangle$. We will show below that if we make two assumptions about the spin properties of photons, the momentum of photons will have those features we are familiar with, such as three-dimensionality, etc.

The two assumptions are as follows:

Assumption I (Correspondence Assumption) The spin state of a photon with definite momentum always belongs to a spin plane, which determines the direction of the photon' s momentum.

Assumption II (Symmetry Assumption) All the possible spin planes to which the spin state of a photon may belong can be defined by the two following orthonormal spin states:

$$|S_{\hat{\mathbf{k}}} = +\hbar\rangle = (1+\cos\beta)e^{-i\alpha}|a\rangle + \sin\beta|b\rangle + (1-\cos\beta)e^{i\alpha}|c\rangle, \quad (1a)$$
$$|S_{\hat{\mathbf{k}}} = -\hbar\rangle = (1-\cos\beta)e^{-i\alpha}|a\rangle - \sin\beta|b\rangle + (1+\cos\beta)e^{i\alpha}|c\rangle, \quad (1b)$$

where $|a\rangle$, $|b\rangle$ and $|c\rangle$ are the three orthonormal spin base states of the spin-1 particles, and α and β are two real numbers.

Assumption I indicates that there is a correspondence between the spin plane and the momentum direction of a photon. This correspondence can explain why the description of the spin state of a photon with a given momentum direction $\hat{\mathbf{k}}$ only requires two base states. We write this correspondence as

$$f(|S_{\hat{\mathbf{k}}} = +\hbar\rangle, |S_{\hat{\mathbf{k}}} = -\hbar\rangle) = \hat{\mathbf{k}} \quad (2)$$

The above equation means that if the spin state of a photon belongs to the spin plane determined by $|S_{\hat{\mathbf{k}}} = +\hbar\rangle$ and $|S_{\hat{\mathbf{k}}} = -\hbar\rangle$, the photon will have momentum along the direction $\hat{\mathbf{k}}$ through some mechanism that we still do not know. For photons, when the momentum direction is reversed, the original left- and right-handed polarization states are interchanged. Thus, the mapping function f has the property

$$f(|S_{\hat{\mathbf{k}}} = -\hbar\rangle, |S_{\hat{\mathbf{k}}} = +\hbar\rangle) = -\hat{\mathbf{k}} = -f(|S_{\hat{\mathbf{k}}} = +\hbar\rangle, |S_{\hat{\mathbf{k}}} = -\hbar\rangle). \quad (3)$$

That is to say, the direction of the photon momentum corresponds to the "directional" spin plane.

For a photon whose momentum is in the direction of $+\hat{\mathbf{k}}$ or $-\hat{\mathbf{k}}$, the base states of its spin can also be chosen as other two orthonormal states which are linear combinations of $|S_{\hat{\mathbf{k}}} = +\hbar\rangle$ and $|S_{\hat{\mathbf{k}}} = -\hbar\rangle$:

$$|1\rangle = C_{11}|S_{\hat{\mathbf{k}}} = +\hbar\rangle + C_{12}|S_{\hat{\mathbf{k}}} = -\hbar\rangle, \quad (4a)$$
$$|2\rangle = C_{21}|S_{\hat{\mathbf{k}}} = +\hbar\rangle + C_{22}|S_{\hat{\mathbf{k}}} = -\hbar\rangle. \quad (4b)$$

The orthonormality of $|1\rangle$ and $|2\rangle$ requires that C_{ij} $(i,j=1,2)$ are the matrix elements of a $U(2)$ matrix C. Since the spin plane determined by the base states $|1\rangle$ and $|2\rangle$ corresponds to the $+\hat{\mathbf{k}}$ or $-\hat{\mathbf{k}}$ direction, we have

$$f(C_{11}|S_{\hat{\mathbf{k}}} = +\hbar\rangle + C_{12}|S_{\hat{\mathbf{k}}} = -\hbar\rangle, C_{21}|S_{\hat{\mathbf{k}}} = +\hbar\rangle + C_{22}|S_{\hat{\mathbf{k}}} = -\hbar\rangle) = g(C_{11}, C_{12}, C_{21}, C_{22})\hat{\mathbf{k}}, \quad (5)$$

where $g(C_{11}, C_{12}, C_{21}, C_{22}) = \pm 1$. Let us analyze the function $g(C_{11}, C_{12}, C_{21}, C_{22})$. When $C_{11} = C_{22} = 1$ and $C_{12} = C_{21} = 0$, we have

$$g(1, 0, 0, 1) = 1. \quad \text{(6a)}$$

When $C_{11} = C_{22} = 0$ and $C_{12} = C_{21} = 1$, we have

$$g(0, 1, 1, 0) = -1. \quad \text{(6b)}$$

In addition, the base states $|1\rangle$ and $|2\rangle$ should satisfy $f(|1\rangle, |2\rangle) = -f(|2\rangle, |1\rangle)$, so we have

$$g(C_{11}, C_{12}, C_{21}, C_{22}) = -g(C_{21}, C_{22}, C_{11}, C_{12}). \quad \text{(6c)}$$

Obviously, the function

$$g(C_{11}, C_{12}, C_{21}, C_{22}) = C_{11}C_{22} - C_{12}C_{21} \quad (7)$$

can satisfy all the Eqs. (6a-c). The right side of Eq. (7) is the determinant of the matrix C. In choosing the base states, if we multiply all the base states by a common phase factor $e^{i\theta}$ (θ is a real number), it does not have any effect on the description of the physical states. Therefore, we can always absorb the appropriate phase factors into the definition of the base states so that the determinant of matrix C is real, i.e., $\det C = \pm 1$ (see Appendix A for more discussion). Equations (6a, b) are the corresponding special cases. When two rows of the matrix C are interchanged, the value of its determinant changes sign. This corresponds to Eq. (6c). Combining Eqs. (2), (5) and (7), we have

$$f(C_{11}|S_{\hat{\mathbf{k}}} = +\hbar\rangle + C_{12}|S_{\hat{\mathbf{k}}} = -\hbar\rangle, C_{21}|S_{\hat{\mathbf{k}}} = +\hbar\rangle + C_{22}|S_{\hat{\mathbf{k}}} = -\hbar\rangle) = (C_{11}C_{22} - C_{12}C_{21})\hat{\mathbf{k}} = (C_{11}C_{22} - C_{12}C_{21})f(|S$$

From the above analysis, we always require that any two sets of base states of a same spin plane are associated with a $SU(2)$ matrix.

Let us consider Assumption II. Because $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha=\alpha_0+2\pi, \beta=\beta_0} = |S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha=\alpha_0, \beta=\beta_0}$ and $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha=\alpha_0+2\pi, \beta=\beta_0} = |S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha=\alpha_0, \beta=\beta_0}$, we limit the range of α to $[0, 2\pi)$. For β, we similarly have $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha=\alpha_0, \beta=\beta_0+2\pi} = |S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha=\alpha_0, \beta=\beta_0}$ and $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha=\alpha_0, \beta=\beta_0+2\pi} = |S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha=\alpha_0, \beta=\beta_0}$. And because $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha=\alpha_0, \beta=\beta_0+\pi} = -|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha=\alpha_0+\pi, \beta=\pi-\beta_0}$ and $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha=\alpha_0, \beta=\beta_0+\pi} = -|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha=\alpha_0+\pi, \beta=\pi-\beta_0}$, we limit the range of β to $[0, \pi]$.

Define three new base states:

$$|x\rangle = -e^{i\pi/4}(|a\rangle - |c\rangle), \quad |y\rangle = e^{i\pi/4}(|a\rangle + |c\rangle), \quad |z\rangle = e^{i\pi/4}|b\rangle. \quad (9)$$

With these three base states, Eqs. (1a, b) can be rewritten as

$$|S_{\hat{\mathbf{k}}} = +\hbar\rangle = (i\sin\alpha - \cos\alpha\cos\beta)e^{-i\pi/4}|x\rangle + (-i\cos\alpha - \sin\alpha\cos\beta)e^{-i\pi/4}|y\rangle + \sin\beta e^{-i\pi/4}|z\rangle, \quad (10a)$$
$$|S_{\hat{\mathbf{k}}} = -\hbar\rangle = (i\sin\alpha + \cos\alpha\cos\beta)e^{-i\pi/4}|x\rangle + (-i\cos\alpha + \sin\alpha\cos\beta)e^{-i\pi/4}|y\rangle - \sin\beta e^{-i\pi/4}|z\rangle. \quad (10b)$$

Substituting Eqs. (10a, b) into Eq. (2), we have

$$\hat{\mathbf{k}} = f((i\sin\alpha - \cos\alpha\cos\beta)e^{-i\pi/4}|x\rangle + (-i\cos\alpha - \sin\alpha\cos\beta)e^{-i\pi/4}|y\rangle + \sin\beta e^{-i\pi/4}|z\rangle, (i\sin\alpha + \cos\alpha\cos\beta)e^{-i\pi/4}|x$$

Let us analyze the structure of the photon momentum space according to Eq. (11). For any two spin planes in the photon spin space, if there is a spin state in one of the spin planes orthogonal to all the spin states in the other spin plane, we say that the two spin planes are orthogonal. From the base states $|x\rangle$, $|y\rangle$ and $|z\rangle$, we can obtain three mutually orthogonal spin planes: $(|x\rangle, |y\rangle)$, $(|y\rangle, |z\rangle)$ and $(|z\rangle, |x\rangle)$. With respect to these three spin planes, we can prove the following three equations:

$$f(|x\rangle, |y\rangle) = f(|S_{\hat{z}} = +\hbar\rangle, |S_{\hat{z}} = -\hbar\rangle), \quad (12a)$$
$$f(|y\rangle, |z\rangle) = f(|S_{\hat{x}} = +\hbar\rangle, |S_{\hat{x}} = -\hbar\rangle), \quad (12b)$$
$$f(|z\rangle, |x\rangle) = f(|S_{\hat{y}} = +\hbar\rangle, |S_{\hat{y}} = -\hbar\rangle), \quad (12c)$$

where

$$|S_{\hat{x}} = +\hbar\rangle \equiv |S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha=0,\beta=\pi/2} = e^{-i\pi/4}|y\rangle + e^{-i\pi/4}|z\rangle, \quad (13a)$$
$$|S_{\hat{x}} = -\hbar\rangle \equiv |S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha=0,\beta=\pi/2} = e^{-i\pi/4}|y\rangle - e^{-i\pi/4}|z\rangle, \quad (13b)$$
$$|S_{\hat{y}} = +\hbar\rangle \equiv |S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha=\pi/2,\beta=\pi/2} = e^{-i\pi/4}|x\rangle + e^{-i\pi/4}|z\rangle, \quad (13c)$$
$$|S_{\hat{y}} = -\hbar\rangle \equiv |S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha=\pi/2,\beta=\pi/2} = e^{-i\pi/4}|x\rangle - e^{-i\pi/4}|z\rangle, \quad (13d)$$
$$|S_{\hat{z}} = +\hbar\rangle \equiv |S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha=0,\beta=0} = e^{-i\pi/4}|x\rangle - e^{-i\pi/4}|y\rangle, \quad (13e)$$
$$|S_{\hat{z}} = -\hbar\rangle \equiv |S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha=0,\beta=0} = e^{-i\pi/4}|x\rangle - e^{-i\pi/4}|y\rangle. \quad (13f)$$

We give the proof of Eq. (12a) below, and the proofs of Eqs. (12b, c) are given in Appendix B.

According to Eqs. (13e, f) and (8), we have

$$f(|S_{\hat{z}} = +\hbar\rangle, |S_{\hat{z}} = -\hbar\rangle) = f(-e^{-i\pi/4}|x\rangle - e^{-i\pi/4}|y\rangle, -e^{-i\pi/4}|x\rangle - e^{-i\pi/4}|y\rangle) = [(-e^{-i\pi/4})(-e^{-i\pi/4}) - (-e^{-i\pi/4})(-$$

The three spin planes appearing on the right side of Eqs. (12a-c) are obtained from the two base states defined by Eqs. (10a, 10b) with different given values of α and β, so that they all correspond to directions in which the actual photon momentum can be. Denoting these three directions of the photon momentum as $\hat{\mathbf{k}}_x$, $\hat{\mathbf{k}}_y$, and $\hat{\mathbf{k}}_z$, respectively, we have

$$f(|x\rangle,|y\rangle)=\hat{\mathbf{k}}_z,\quad f(|y\rangle,|z\rangle)=\hat{\mathbf{k}}_x,\quad f(|z\rangle,|x\rangle)=\hat{\mathbf{k}}_y.\quad (15)$$

To simplify the expression, we write Eqs. (10a, b) as follows:

$$|S_{\hat{\mathbf{k}}}=+\hbar\rangle=d_{11}|x\rangle+d_{12}|y\rangle+d_{13}|z\rangle,\quad (16a)$$
$$|S_{\hat{\mathbf{k}}}=-\hbar\rangle=d_{21}|x\rangle+d_{22}|y\rangle+d_{23}|z\rangle,\quad (16b)$$

where d_{ij} $(i=1,2;j=1,2,3)$ represent the parameter expressions before the corresponding base states in Eqs. (10a, b) (see Appendix C for the expressions of d_{ij} $(i=1,2;j=1,2,3)$). Thus, Eq. (11) can be written as

$$\hat{\mathbf{k}}=f(d_{11}|x\rangle+d_{12}|y\rangle+d_{13}|z\rangle,d_{21}|x\rangle+d_{22}|y\rangle+d_{23}|z\rangle).\quad (17)$$

Let' s analyze the relationship between $\hat{\mathbf{k}}$ and the three directions $\hat{\mathbf{k}}_x$, $\hat{\mathbf{k}}_y$, and $\hat{\mathbf{k}}_z$. When $d_{11}=d_{21}=0$, according to Eqs. (8) and (15), we have

$$\hat{\mathbf{k}}=f(d_{12}|y\rangle+d_{13}|z\rangle,d_{22}|y\rangle+d_{23}|z\rangle)=(d_{12}d_{23}-d_{13}d_{22})f(|y\rangle,|z\rangle)=(d_{12}d_{23}-d_{13}d_{22})\hat{\mathbf{k}}_x.\quad (18a)$$

Similarly, when $d_{12}=d_{22}=0$ and $d_{13}=d_{23}=0$, we have

$$\hat{\mathbf{k}}=f(d_{11}|x\rangle+d_{13}|z\rangle,d_{21}|x\rangle+d_{23}|z\rangle)=(d_{13}d_{21}-d_{11}d_{23})f(|z\rangle,|x\rangle)=(d_{13}d_{21}-d_{11}d_{23})\hat{\mathbf{k}}_y,\quad (18b)$$

$$\hat{\mathbf{k}}=f(d_{11}|x\rangle+d_{12}|y\rangle,d_{21}|x\rangle+d_{22}|y\rangle)=(d_{11}d_{22}-d_{12}d_{21})f(|x\rangle,|y\rangle)=(d_{11}d_{22}-d_{12}d_{21})\hat{\mathbf{k}}_z,\quad (18c)$$

respectively. Suppose the relation between $\hat{\mathbf{k}}$ and the three directions $\hat{\mathbf{k}}_x$, $\hat{\mathbf{k}}_y$, and $\hat{\mathbf{k}}_z$ is linear, and thus we have

$$\hat{\mathbf{k}}=g_1(d_{11},d_{12},d_{13},d_{21},d_{22},d_{23})(d_{12}d_{23}-d_{13}d_{22})\hat{\mathbf{k}}_x+g_2(d_{11},d_{12},d_{13},d_{21},d_{22},d_{23})(d_{13}d_{21}-d_{11}d_{23})\hat{\mathbf{k}}_y+g_3(d_{11},d_1$$

According to Eqs. (18a-c), we have

$$g_1(0, d_{12}, d_{13}, 0, d_{22}, d_{23}) = g_2(d_{11}, 0, d_{13}, d_{21}, 0, d_{23}) = g_3(d_{11}, d_{12}, 0, d_{21}, d_{22}, 0) = 1. \quad (20)$$

In addition, according to Eq. (3), we have

$$\begin{aligned}
&f(d_{11}|x\rangle + d_{12}|y\rangle + d_{13}|z\rangle, d_{21}|x\rangle + d_{22}|y\rangle + d_{23}|z\rangle) \\
&= g_1(d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23})(d_{12}d_{23} - d_{13}d_{22})\hat{\mathbf{k}}_x + g_2(d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23})(d_{13}d_{21} - d_{11}d_{23})\hat{\mathbf{k}}_y + g_3(d_{11}, \\
&= -f(d_{21}|x\rangle + d_{22}|y\rangle + d_{23}|z\rangle, d_{11}|x\rangle + d_{12}|y\rangle + d_{13}|z\rangle) \\
&= g_1(d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23})(d_{12}d_{23} - d_{13}d_{22})\hat{\mathbf{k}}_x + g_2(d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23})(d_{13}d_{21} - d_{11}d_{23})\hat{\mathbf{k}}_y + g_3(d_{11},
\end{aligned}$$

Obviously, taking

$$g_i(d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23}) \equiv 1, \quad (i = 1, 2, 3), \quad (22)$$

can satisfy Eqs. (20) and (21). Substituting Eq. (22) into Eq. (19) and using Eqs. (17) and (15), we have

$$\hat{\mathbf{k}} = (d_{12}d_{23} - d_{13}d_{22})\hat{\mathbf{k}}_x + (d_{13}d_{21} - d_{11}d_{23})\hat{\mathbf{k}}_y + (d_{11}d_{22} - d_{12}d_{21})\hat{\mathbf{k}}_z = f(d_{11}|x\rangle + d_{12}|y\rangle + d_{13}|z\rangle, d_{21}|x\rangle + d_{22}|y\rangle + d_{23}|$$

Applying Eq. (23) to Eq. (11), we obtain

$$\begin{aligned}
\hat{\mathbf{k}} &= [\sin\beta e^{-i\pi/4}(-i\cos\alpha + \sin\alpha\cos\beta)e^{-i\pi/4} - \sin\beta e^{-i\pi/4}(-i\cos\alpha - \sin\alpha\cos\beta)e^{-i\pi/4}]f(|y\rangle, |z\rangle) \\
&\quad + [\sin\beta e^{-i\pi/4}(i\sin\alpha + \cos\alpha\cos\beta)e^{-i\pi/4} - \sin\beta e^{-i\pi/4}(i\sin\alpha - \cos\alpha\cos\beta)e^{-i\pi/4}]f(|z\rangle, |x\rangle) \\
&\quad + [(i\sin\alpha - \cos\alpha\cos\beta)e^{-i\pi/4}(-i\cos\alpha + \sin\alpha\cos\beta)e^{-i\pi/4} - (-i\cos\alpha - \sin\alpha\cos\beta)e^{-i\pi/4}(i\sin\alpha + \cos\alpha\cos \\
&= \sin\beta\cos\alpha f(|y\rangle, |z\rangle) + \sin\beta\sin\alpha f(|z\rangle, |x\rangle) + \cos\beta f(|x\rangle, |y\rangle) \\
&= \sin\beta\cos\alpha\hat{\mathbf{k}}_x + \sin\beta\sin\alpha\hat{\mathbf{k}}_y + \cos\beta\hat{\mathbf{k}}_z. \quad (24)
\end{aligned}$$

If we compare the selected momentum directions $\hat{\mathbf{k}}_x$, $\hat{\mathbf{k}}_y$ and $\hat{\mathbf{k}}_z$ to the three unit direction vectors $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ in the Cartesian coordinate system, the three components in Eq. (24) are exactly the three Cartesian components of a unit vector represented by the spherical coordinates α and β. Thus, the parameters α and β, which come from Eqs. (1a, b), are the azimuth and polar angles in the spherical coordinate system, respectively.

We have derived the expressions of the correspondence between the spin plane of a photon and the direction of photon momentum based on Assumption I and Assumption II. It can be shown that this correspondence is one-to-one. That is to say, the same spin plane corresponds to the same momentum direction; the same momentum direction also corresponds to the same spin plane (see Appendix D for the proof). From the structure of the momentum space represented by Eq. (24), it can be seen that Assumption II is equivalent to assuming that all the spin planes to which the photon spin state may belong can be obtained through all the $SO(3)$ rotations from one of these spin planes. This is the reason why Assumption II is referred to as the Symmetry Assumption.

Based on this one-to-one correspondence we can prove that it is impossible to decompose all photon momenta into two selected directions (see Appendix E for the proof).

The base states $|x\rangle$, $|y\rangle$ and $|z\rangle$ determine the three momentum directions of photons, $\hat{\mathbf{k}}_x$, $\hat{\mathbf{k}}_y$ and $\hat{\mathbf{k}}_z$, by Eq. (15). We take other three mutually perpendicular directions $\hat{\mathbf{k}}_{x'}$, $\hat{\mathbf{k}}_{y'}$ and $\hat{\mathbf{k}}_{z'}$ in the momentum space of the photon, satisfying

$$\begin{pmatrix} \hat{\mathbf{k}}_{x'} \\ \hat{\mathbf{k}}_{y'} \\ \hat{\mathbf{k}}_{z'} \end{pmatrix} = M \begin{pmatrix} \hat{\mathbf{k}}_{x} \\ \hat{\mathbf{k}}_{y} \\ \hat{\mathbf{k}}_{z} \end{pmatrix}, \quad (25)$$

where M is a 3×3 real orthogonal matrix, and $\det M = 1$. Take a new set of base states $|x'\rangle$, $|y'\rangle$ and $|z'\rangle$, satisfying

$$\begin{pmatrix} |x'\rangle \\ |y'\rangle \\ |z'\rangle \end{pmatrix} = N \begin{pmatrix} |x\rangle \\ |y\rangle \\ |z\rangle \end{pmatrix}, \quad (26)$$

where N also is a 3×3 real orthogonal matrix, and $\det N = 1$. If

$$\hat{\mathbf{k}}_{x'} = f(|y'\rangle, |z'\rangle), \quad \hat{\mathbf{k}}_{y'} = f(|z'\rangle, |x'\rangle), \quad \hat{\mathbf{k}}_{z'} = f(|x'\rangle, |y'\rangle) \quad (27)$$

hold, we can prove that (see Appendix F for the proof)

$$M = N. \quad (28)$$

That is, under rotation in momentum space, $(|x\rangle, |y\rangle, |z\rangle)^T$ and $(\hat{\mathbf{k}}_x, \hat{\mathbf{k}}_y, \hat{\mathbf{k}}_z)^T$ (superscript T stands for transpose) transform in the same way.

According to Eqs. (9) and (13a, b), we have

$$|S_{\hat{x}} = +\hbar\rangle \equiv |S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha=0,\beta=\pi/2} = e^{-i\pi/4}|y\rangle + e^{-i\pi/4}|z\rangle, \quad (29a)$$
$$|S_{\hat{x}} = -\hbar\rangle \equiv |S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha=0,\beta=\pi/2} = e^{-i\pi/4}|y\rangle - e^{-i\pi/4}|z\rangle. \quad (29b)$$

Take the three directions $(\alpha = 0, \beta = \pi/2)$, $(\alpha = \pi/2, \beta = \pi/2)$ and $(\alpha = 0, \beta = 0)$ as the directions of the x, y and z axes of a Cartesian coordinate system, respectively. The base states $|a\rangle$, $|b\rangle$ and $|c\rangle$ are exactly the commonly used three base states $|S_{\hat{z}} = +\hbar\rangle$, $|S_{\hat{z}} = 0\hbar\rangle$ and $|S_{\hat{z}} = -\hbar\rangle$ of a spin-1 particle, respectively. In quantum mechanics, it is a well-known result that the base states $|x\rangle$, $|y\rangle$ and $|z\rangle$ defined by Eq. (9) transform in the same way as the unit space vectors $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ under rotation of space. Here, again, we see the identity of the photon momentum direction and the space direction. This identity allows us to calculate the transformation rules of $|S_{\hat{z}} = +\hbar\rangle$, $|S_{\hat{z}} = 0\hbar\rangle$ and $|S_{\hat{z}} = -\hbar\rangle$ under space rotation by using Eq. (28) (see Appendix G for the proof).

3. Conclusion

In summary, we have demonstrated such a logical possibility: that the three-dimensionality of photon momentum is a property of the photon itself. Based on two assumptions regarding the relationship between photon momentum and spin, we have proven that the momentum space of the photon has the same structure as familiar three-dimensional Euclidean space. In this way, we have provided a direct physical answer to the age-old question of why space is three-dimensional: the photons we use to observe spatial relationships between objects have three-dimensional momentum. From the perspective of interaction, photons are the intermediate particles that transmit the electromagnetic interaction. This makes objects (rigid bodies) whose shapes are maintained by electromagnetic interactions three-dimensional. We can imagine a world with only electromagnetic interactions. In this world, the three-dimensionality of photon momentum makes both rigid bodies and the spatial relationships between them three-dimensional. Such three-dimensional spatial relationships will be presented in the most natural way when observing the world using photons as a tool. When gravity exists, since the spin of the graviton (a hypothetical massless particle that transmits the gravitational interaction proposed by physicists) is 2, we expect that the structure of the intrinsic momentum space of the graviton will be different from that of the photon. This may be the physical reason why three-dimensional Euclidean space is not applicable to gravity, whether in terms of its flatness (general relativity) or its dimensionality (superstring theory). Such a perspective may provide some inspiration for people to deeply understand the great difficulties encountered in the quantization of gravity.

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Conflict of Interest

The authors declare no conflict of interest.

Keywords

spatial dimension, photon momentum, photon spin

Appendix A

The $U(2)$ matrix can be expressed generally as

$$C = e^{i\phi} \begin{pmatrix} a & -b^* e^{i\varphi} \\ b & a^* e^{i\varphi} \end{pmatrix}, \quad \text{(A1)}$$

satisfying $|a|^2 + |b|^2 = 1$. Therefore, Eqs. (4a, b) in the main text can be expressed as

$$|1\rangle = e^{i\phi}(a|S_{\hat{\mathbf{k}}} = +\hbar\rangle - b^* e^{i\varphi}|S_{\hat{\mathbf{k}}} = -\hbar\rangle), \quad \text{(A2a)}$$
$$|2\rangle = e^{i\phi}(b|S_{\hat{\mathbf{k}}} = +\hbar\rangle + a^* e^{i\varphi}|S_{\hat{\mathbf{k}}} = -\hbar\rangle). \quad \text{(A2b)}$$

Define two new base states:

$$|1'\rangle = e^{-i\phi}|1\rangle, \quad |2'\rangle = e^{-i\phi}|2\rangle. \quad \text{(A3a)}$$

We have

$$|1'\rangle = a|S_{\hat{\mathbf{k}}} = +\hbar\rangle - b^* e^{i\varphi}|S_{\hat{\mathbf{k}}} = -\hbar\rangle, \quad \text{(A3b)}$$
$$|2'\rangle = b|S_{\hat{\mathbf{k}}} = +\hbar\rangle + a^* e^{i\varphi}|S_{\hat{\mathbf{k}}} = -\hbar\rangle. \quad \text{(A3c)}$$

This is equivalent to the requirement $\det C' = 1$. Similarly, if we define two new base states:

$$|1''\rangle = e^{-i\varphi/2}|1'\rangle, \quad |2''\rangle = e^{-i\varphi/2}|2'\rangle, \quad \text{(A4a)}$$

then we have

$$|1''\rangle = ae^{-i\varphi/2}|S_{\hat{\mathbf{k}}} = +\hbar\rangle - b^* e^{i\varphi/2}|S_{\hat{\mathbf{k}}} = -\hbar\rangle, \quad \text{(A4b)}$$
$$|2''\rangle = be^{-i\varphi/2}|S_{\hat{\mathbf{k}}} = +\hbar\rangle + a^* e^{i\varphi/2}|S_{\hat{\mathbf{k}}} = -\hbar\rangle. \quad \text{(A4c)}$$

This is equivalent to the requirement $\det C'' = \pm 1$.

Appendix B. Proofs for Eqs. (12b, c) in the main text

According to Eqs. (13a, b) and (8), we have

$$f(|S_{\hat{x}}=+\hbar\rangle,|S_{\hat{x}}=-\hbar\rangle)=f(e^{-i\pi/4}|y\rangle+e^{-i\pi/4}|z\rangle,e^{-i\pi/4}|y\rangle-e^{-i\pi/4}|z\rangle)$$
$$=\frac{1}{2}[(e^{-i\pi/4})(-e^{-i\pi/4})-(e^{-i\pi/4})(e^{-i\pi/4})]f(|y\rangle,|z\rangle)=f(|y\rangle,|z\rangle). \quad \text{(B1)}$$

Similarly, according to Eqs. (13c, d) and (8), we have

$$f(|S_{\hat{y}}=+\hbar\rangle,|S_{\hat{y}}=-\hbar\rangle)=f(e^{-i\pi/4}|x\rangle+e^{-i\pi/4}|z\rangle,e^{-i\pi/4}|x\rangle-e^{-i\pi/4}|z\rangle)$$
$$=\frac{1}{2}[(e^{-i\pi/4})(-e^{-i\pi/4})-(e^{-i\pi/4})(e^{-i\pi/4})]f(|x\rangle,|z\rangle)=f(|z\rangle,|x\rangle). \quad \text{(B2)}$$

Appendix C

According to Eqs. (10a, b) and (16a, b) in the main text, we have

$$d_{11}=\frac{1}{\sqrt{2}}(i\sin\alpha-\cos\alpha\cos\beta),\quad d_{12}=\frac{1}{\sqrt{2}}(-i\cos\alpha-\sin\alpha\cos\beta),\quad d_{13}=\sin\beta,$$
$$d_{21}=\frac{1}{\sqrt{2}}(i\sin\alpha+\cos\alpha\cos\beta),\quad d_{22}=\frac{1}{\sqrt{2}}(-i\cos\alpha+\sin\alpha\cos\beta),\quad d_{23}=-\sin\beta.$$

Appendix D

Proposition: The correspondence given by the following equation from the spin planes of a photon to the photon momentum directions is one-to-one:

$$f(|S_{\hat{\mathbf{k}}}=+\hbar\rangle,|S_{\hat{\mathbf{k}}}=-\hbar\rangle)=\hat{\mathbf{k}}. \quad \text{(D1)}$$

Proof: If the two sets of base states $|S_{\hat{\mathbf{k}}}=+\hbar\rangle$ and $|S_{\hat{\mathbf{k}}}=-\hbar\rangle$ determine the same spin plane, thus, according to the analysis in the main text, we have

$$\begin{pmatrix}|S_{\hat{\mathbf{k}}}=+\hbar\rangle\\|S_{\hat{\mathbf{k}}}=-\hbar\rangle\end{pmatrix}=C\begin{pmatrix}|S_{\hat{\mathbf{k}}}=+\hbar\rangle\\|S_{\hat{\mathbf{k}}}=-\hbar\rangle\end{pmatrix}, \quad \text{(D2)}$$

where C_{ij} are the elements of a $SU(2)$ matrix C. According to Eq. (8) in the main text and $\det C=1$, we have

$$f(|S_{\hat{\mathbf{k}}}=+\hbar\rangle,|S_{\hat{\mathbf{k}}}=-\hbar\rangle)=f(C_{11}|S_{\hat{\mathbf{k}}}=+\hbar\rangle+C_{12}|S_{\hat{\mathbf{k}}}=-\hbar\rangle,C_{21}|S_{\hat{\mathbf{k}}}=+\hbar\rangle+C_{22}|S_{\hat{\mathbf{k}}}=-\hbar\rangle)=(C_{11}C_{22}-C_{12}C_2$$

Therefore, the same spin plane corresponds to the same direction of momentum.

Below we prove that the same direction of momentum corresponds to the same spin plane. Suppose $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_1,\beta_1}$ and $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_1,\beta_1}$ represent the same direction of momentum as $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_2,\beta_2}$ and $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_2,\beta_2}$, and thus according to Eq. (D1) we have

$$\hat{\mathbf{k}}(\alpha_1, \beta_1) = \hat{\mathbf{k}}(\alpha_2, \beta_2), \quad \text{(D4a)}$$
$$\sin\beta_1 \cos\alpha_1 = \sin\beta_2 \cos\alpha_2, \quad \text{(D4b)}$$
$$\sin\beta_1 \sin\alpha_1 = \sin\beta_2 \sin\alpha_2, \quad \text{(D4c)}$$
$$\cos\beta_1 = \cos\beta_2. \quad \text{(D4d)}$$

We need to prove that $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_1,\beta_1}$, $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_1,\beta_1}$ and $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_2,\beta_2}$, $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_2,\beta_2}$ are the same spin plane.

According to Eqs. (D4c) and (D4d), we have $\beta_1 = \beta_2$ or $\beta_1 = \pi - \beta_2$.

1) If $\beta_1 = \beta_2$, according to Eqs. (D4a) and (D4b), we have $\sin\alpha_1 = \sin\alpha_2$ and $\cos\alpha_1 = \cos\alpha_2$. Since $\alpha_1, \alpha_2 \in [0, 2\pi)$, we have $\alpha_1 = \alpha_2$. Therefore, $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_1,\beta_1} = |S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_2,\beta_2}$ and $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_1,\beta_1} = |S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_2,\beta_2}$. Obviously, they represent the same spin plane.
2) If $\beta_1 = \pi - \beta_2$, then $\cos\beta_1 = -\cos\beta_2$ and $\sin\beta_1 = \sin\beta_2$. According to Eqs. (D4a) and (D4b), we have $\cos\alpha_1 = -\cos\alpha_2$ and $\sin\alpha_1 = -\sin\alpha_2$. Since $\alpha_1, \alpha_2 \in [0, 2\pi)$, we have $\alpha_1 = \alpha_2 + \pi$. According to Eqs. (10a, b) in the main text, we have

$$|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_2,\beta_2} = (i\sin\alpha_2 - \cos\alpha_2\cos\beta_2)e^{-i\pi/4}|x\rangle + (-i\cos\alpha_2 - \sin\alpha_2\cos\beta_2)e^{-i\pi/4}|y\rangle + \sin\beta_2 e^{-i\pi/4}|z\rangle,$$
$$|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_2,\beta_2} = (i\sin\alpha_2 + \cos\alpha_2\cos\beta_2)e^{-i\pi/4}|x\rangle + (-i\cos\alpha_2 + \sin\alpha_2\cos\beta_2)e^{-i\pi/4}|y\rangle - \sin\beta_2 e^{-i\pi/4}|z\rangle.$$

Now we show that there is a $SU(2)$ matrix C' satisfying

$$\begin{pmatrix} |S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_1,\beta_1} \\ |S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_1,\beta_1} \end{pmatrix} = C' \begin{pmatrix} |S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_2,\beta_2} \\ |S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_2,\beta_2} \end{pmatrix}. \quad \text{(D5)}$$

That is,

$$|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_1,\beta_1} = C'_{11}|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_2,\beta_2} + C'_{12}|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_2,\beta_2}, \quad \text{(D6a)}$$
$$|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_1,\beta_1} = C'_{21}|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_2,\beta_2} + C'_{22}|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_2,\beta_2}. \quad \text{(D6b)}$$

Thus, we have

$$C'_{11} = \frac{1}{2}(e^{i(\alpha_1-\alpha_2)} + e^{-i(\alpha_1-\alpha_2)}), \quad \text{(D7a)}$$
$$C'_{12} = \frac{1}{2}(e^{i(\alpha_1+\alpha_2)} - e^{-i(\alpha_1+\alpha_2)}), \quad \text{(D7b)}$$
$$C'_{21} = \frac{1}{2}(e^{i(\alpha_1+\alpha_2)} - e^{-i(\alpha_1+\alpha_2)}), \quad \text{(D7c)}$$
$$C'_{22} = \frac{1}{2}(e^{i(\alpha_1-\alpha_2)} + e^{-i(\alpha_1-\alpha_2)}). \quad \text{(D7d)}$$

The solutions of Eqs. (D6a, b, c, d) are

$$C' = \begin{pmatrix} 0 & -e^{i(\alpha_1+\alpha_2)} \\ e^{-i(\alpha_1+\alpha_2)} & 0 \end{pmatrix}. \quad \text{(D8)}$$

It is easy to verify that $\det C' = 1$ and $C'^{\dagger}C' = I$. Namely, C' is a $SU(2)$ matrix. Therefore, $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_1,\beta_1}$, $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_1,\beta_1}$ and $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_2,\beta_2}$, $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_2,\beta_2}$ represent the same spin plane.

3) If $\beta_1 = \beta_2 = 0$ or π, then α_1 and α_2 can take any values in the range of $[0, 2\pi)$. Similarly, we can prove that the $SU(2)$ matrix

$$C' = \begin{pmatrix} e^{i(\alpha_1-\alpha_2)} & 0 \\ 0 & e^{-i(\alpha_1-\alpha_2)} \end{pmatrix} \quad \text{(D9)}$$

satisfying $\det C' = 1$. Therefore, $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_1,\beta_1}$, $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_1,\beta_1}$ and $|S_{\hat{\mathbf{k}}} = +\hbar\rangle_{\alpha_2,\beta_2}$, $|S_{\hat{\mathbf{k}}} = -\hbar\rangle_{\alpha_2,\beta_2}$ represent the same spin plane.

Therefore, the same direction of momentum corresponds to the same spin plane.

In summary, the correspondence from the spin planes of a photon to the directions of photon momentum given by Eq. (D1) in the main text is one-to-one.

Appendix E

Proposition: It is impossible to decompose all photon momenta into two selected directions.

Proof: Suppose it is possible to decompose all photon momenta into two selected directions $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$, and thus we have

$$\hat{\mathbf{k}}(\alpha, \beta) = g_1(\alpha, \beta)\hat{\mathbf{k}}_1 + g_2(\alpha, \beta)\hat{\mathbf{k}}_2. \quad \text{(E1)}$$

Expand both sides of Eq. (E1) using Eq. (24) in the main text. According to the one-to-one correspondence proved in Appendix D, we have

$$\sin\beta\cos\alpha = g_1(\alpha,\beta)\sin\beta_1\cos\alpha_1 + g_2(\alpha,\beta)\sin\beta_2\cos\alpha_2, \quad \text{(E2a)}$$
$$\sin\beta\sin\alpha = g_1(\alpha,\beta)\sin\beta_1\sin\alpha_1 + g_2(\alpha,\beta)\sin\beta_2\sin\alpha_2, \quad \text{(E2b)}$$
$$\cos\beta = g_1(\alpha,\beta)\cos\beta_1 + g_2(\alpha,\beta)\cos\beta_2. \quad \text{(E2c)}$$

Given $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$, Eqs. (E2a, b, c) are equations about $g_1(\alpha,\beta)$ and $g_2(\alpha,\beta)$. Because $\sin\beta\cos\alpha$, $\sin\beta\sin\alpha$ and $\cos\beta$ are three linearly independent functions, these equations have no solution. We can also prove this by solving $g_1(\alpha,\beta)$ and $g_2(\alpha,\beta)$ from Eqs. (E2a, b) and substituting them into Eq. (E2c).

Therefore, it is impossible to decompose all photon momenta into two selected directions.

Appendix F

Proposition: If

$$\begin{pmatrix}\hat{\mathbf{k}}_{x'}\\ \hat{\mathbf{k}}_{y'}\\ \hat{\mathbf{k}}_{z'}\end{pmatrix} = M\begin{pmatrix}\hat{\mathbf{k}}_{x}\\ \hat{\mathbf{k}}_{y}\\ \hat{\mathbf{k}}_{z}\end{pmatrix}, \quad \text{(F1)}$$

$$\begin{pmatrix}|x'\rangle\\ |y'\rangle\\ |z'\rangle\end{pmatrix} = N\begin{pmatrix}|x\rangle\\ |y\rangle\\ |z\rangle\end{pmatrix}, \quad \text{(F2)}$$

where M and N are both 3×3 real orthogonal matrices and $\det M = \det N = 1$, and

$$\hat{\mathbf{k}}_{x'} = f(|y'\rangle,|z'\rangle), \quad \hat{\mathbf{k}}_{y'} = f(|z'\rangle,|x'\rangle), \quad \hat{\mathbf{k}}_{z'} = f(|x'\rangle,|y'\rangle), \quad \text{(F3)}$$

then

$$M = N. \quad \text{(F4)}$$

Proof: Denote the elements of M and N as M_{ij} and N_{ij} ($i,j = 1,2,3$), respectively. According to Eq. (F1), we have

$$\hat{\mathbf{k}}_{x'} = M_{11}\hat{\mathbf{k}}_x + M_{12}\hat{\mathbf{k}}_y + M_{13}\hat{\mathbf{k}}_z. \quad \text{(F5)}$$

According to Eq. (F2), we have

$$|y'\rangle = N_{21}|x\rangle + N_{22}|y\rangle + N_{23}|z\rangle, \quad |z'\rangle = N_{31}|x\rangle + N_{32}|y\rangle + N_{33}|z\rangle. \quad \text{(F6)}$$

Substitute Eqs. (F5), (F6) and (15) into the first equation of (F3). We have

$$\begin{aligned}\hat{\mathbf{k}}_{x'} &= f(N_{21}|x\rangle + N_{22}|y\rangle + N_{23}|z\rangle, N_{31}|x\rangle + N_{32}|y\rangle + N_{33}|z\rangle) \\ &= (N_{21}N_{32} - N_{22}N_{31})f(|x\rangle,|y\rangle) + (N_{22}N_{33} - N_{23}N_{32})f(|y\rangle,|z\rangle) + (N_{23}N_{31} - N_{21}N_{33})f(|z\rangle,|x\rangle) \\ &= (N_{21}N_{32} - N_{22}N_{31})\hat{\mathbf{k}}_z + (N_{22}N_{33} - N_{23}N_{32})\hat{\mathbf{k}}_x + (N_{23}N_{31} - N_{21}N_{33})\hat{\mathbf{k}}_y. \quad \text{(F7)}\end{aligned}$$

According to Eq. (23) in the main text, we have

$$\begin{aligned}M_{11} &= N_{22}N_{33} - N_{23}N_{32}, \\ M_{12} &= N_{23}N_{31} - N_{21}N_{33}, \\ M_{13} &= N_{21}N_{32} - N_{22}N_{31}. \quad \text{(F8)}\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}M_{21} &= N_{32}N_{13} - N_{33}N_{12}, \\ M_{22} &= N_{33}N_{11} - N_{31}N_{13}, \\ M_{23} &= N_{31}N_{12} - N_{32}N_{11}, \quad \text{(F9)}\end{aligned}$$

$$\begin{aligned}M_{31} &= N_{12}N_{23} - N_{13}N_{22}, \\ M_{32} &= N_{13}N_{21} - N_{11}N_{23}, \\ M_{33} &= N_{11}N_{22} - N_{12}N_{21}. \quad \text{(F10)}\end{aligned}$$

Because N is a 3×3 real orthogonal matrix and $\det N = 1$, it can be expressed with Euler angles as

$$N = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\cos\psi\sin\phi - \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ \sin\psi\cos\phi + \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\cos\psi\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{pmatrix}. \quad \text{(F11)}$$

Using the expressions of matrix elements in Eq. (F11), we can obtain through direct calculations

$$\begin{aligned}N_{22}N_{33} - N_{23}N_{32} &= N_{11}, \\ N_{23}N_{31} - N_{21}N_{33} &= N_{12}, \\ N_{21}N_{32} - N_{22}N_{31} &= N_{13}. \quad \text{(F12)}\end{aligned}$$

Thus, according to Eq. (F8), we have

$$M_{11} = N_{11}, \quad M_{12} = N_{12}, \quad M_{13} = N_{13}. \quad \text{(F13)}$$

Similarly, we can obtain through direct calculations that for all $i, j = 1, 2, 3,$

$$M_{ij} = N_{ij}. \quad \text{(F14)}$$

Therefore, the proposition is proved.

Appendix G

If the rotation in momentum space equates to the rotation in position space, then the proposition in Appendix F means that the base states transform in the same way under rotation in position space (represented by the matrix M). Here we use this to calculate the transformation of $|S_z = +\hbar\rangle$, $|S_z = 0\hbar\rangle$, $|S_z = -\hbar\rangle$ under space rotation.

According to the analysis in the main text, the base states $|x\rangle$, $|y\rangle$, $|z\rangle$ are related to $|S_z = +\hbar\rangle$, $|S_z = 0\hbar\rangle$, $|S_z = -\hbar\rangle$ by Eq. (9). According to Eq. (F2) in Appendix F and Eq. (9) in the main text, we have

$$\begin{aligned}
|S_{z'} = +\hbar\rangle \equiv |S_{\hat{z}} = +\hbar\rangle' &= \frac{1}{\sqrt{2}}(|x'\rangle - |y'\rangle) = \frac{1}{\sqrt{2}}(N_{11}|x\rangle + N_{12}|y\rangle + N_{13}|z\rangle - N_{21}|x\rangle - N_{22}|y\rangle - N_{23}|z\rangle) \\
&= \frac{1}{\sqrt{2}}[(N_{11} - N_{21})|x\rangle + (N_{12} - N_{22})|y\rangle + (N_{13} - N_{23})|z\rangle] \\
&= \frac{1}{2}[(N_{11} - N_{21} - iN_{12} + iN_{22})|S_z = +\hbar\rangle + \sqrt{2}(N_{13} - N_{23})|S_z = 0\hbar\rangle + (N_{11} - N_{21} + iN_{12} - iN_{22})
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|S_{z'} = 0\hbar\rangle \equiv |S_{\hat{z}} = 0\hbar\rangle' &= |z'\rangle = N_{31}|x\rangle + N_{32}|y\rangle + N_{33}|z\rangle \\
&= \frac{1}{\sqrt{2}}[(N_{31} - iN_{32})|S_z = +\hbar\rangle + \sqrt{2}N_{33}|S_z = 0\hbar\rangle + (N_{31} + iN_{32})|S_z = -\hbar\rangle], \quad \text{(G2)}
\end{aligned}$$

$$\begin{aligned}
|S_{z'} = -\hbar\rangle \equiv |S_{\hat{z}} = -\hbar\rangle' &= \frac{1}{\sqrt{2}}(|x'\rangle + |y'\rangle) = \frac{1}{\sqrt{2}}(N_{11}|x\rangle + N_{12}|y\rangle + N_{13}|z\rangle + N_{21}|x\rangle + N_{22}|y\rangle + N_{23}|z\rangle) \\
&= \frac{1}{2}[(N_{11} + N_{21} - iN_{12} - iN_{22})|S_z = +\hbar\rangle + \sqrt{2}(N_{13} + N_{23})|S_z = 0\hbar\rangle + (N_{11} + N_{21} + iN_{12} + iN_{22})
\end{aligned}$$

Substitute the expressions of the matrix elements in Eq. (F11) in Appendix F, and we have

$$|S_{z'} = +\hbar\rangle = \frac{1}{2}(1+\cos\theta)e^{i\phi}|S_z = +\hbar\rangle + \frac{1}{\sqrt{2}}\sin\theta e^{i\phi}|S_z = 0\hbar\rangle + \frac{1}{2}(1-\cos\theta)e^{i\phi}|S_z = -\hbar\rangle, \quad \text{(G4)}$$

$$|S_{z'} = 0\hbar\rangle = -\frac{1}{\sqrt{2}}\sin\theta e^{-i\psi}|S_z = +\hbar\rangle + \cos\theta|S_z = 0\hbar\rangle + \frac{1}{\sqrt{2}}\sin\theta e^{i\psi}|S_z = -\hbar\rangle, \quad \text{(G5)}$$

$$|S_{z'} = -\hbar\rangle = \frac{1}{2}(1-\cos\theta)e^{-i\phi}|S_z = +\hbar\rangle - \frac{1}{\sqrt{2}}\sin\theta e^{-i\phi}|S_z = 0\hbar\rangle + \frac{1}{2}(1+\cos\theta)e^{-i\phi}|S_z = -\hbar\rangle. \quad \text{(G6)}$$

Equations (G4), (G5) and (G6) are just the well-known transformation formulas of the base states $|S_z = +\hbar\rangle$, $|S_z = 0\hbar\rangle$, $|S_z = -\hbar\rangle$ for a spin-1 particle under space rotation (see J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, Chapter 3 (China Edition, Beijing World Publishing Corporation, 2013)).

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Note: Figure translations are in progress. See original paper for figures.