An introduction to relativistic spin hydrodynamics

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Date: 2025-04-07T00:00:00+00:00

Abstract

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Full Text

Preamble

An Introduction to Relativistic Spin Hydrodynamics

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(Dated: April 7, 2025)

Spin polarization and spin transport are common phenomena in many quantum systems. Relativistic spin hydrodynamics provides an effective low-energy framework to describe these processes in quantum many-body systems. The fundamental symmetry underlying relativistic spin hydrodynamics is angular momentum conservation, which naturally leads to inter-conversion between spin and orbital angular momenta. This inter-conversion is a key feature of relativistic spin hydrodynamics, closely related to entropy production and introducing ambiguity in the construction of constitutive relations. In this article, we present a pedagogical introduction to relativistic spin hydrodynamics. We demonstrate how to derive the constitutive relations by applying local thermodynamic laws and explore several distinctive aspects of spin hydrodynamics. These include the pseudo-gauge ambiguity, the behavior of the system in the presence of strong vorticity, and the challenges of modeling the freeze-out of spin in heavy-ion collisions. We also outline some future prospects for spin hydrodynamics.

Introduction

Spin is a fundamental property of particles arising from quantum mechanics, playing a central role in numerous phenomena within the quantum regime. As a form of angular momentum, spin naturally couples to rotation, allowing it to become polarized by rotational motion. Similarly, for a charged particle with nonzero spin, or a neutral particle with a non-trivial charge form factor, spin can couple to an external magnetic field as well. Additionally, for a particle in motion (i.e., with finite momentum), its spin may couple to acceleration, electric fields, or gradients of external potentials, such as a chemical potential and temperature. In the case of massless particles, the spin state is specified by its helicity state, meaning it is intrinsically slaved to the particle's motion. As a result, spin can be manipulated by rotating fields, magnetic fields, electric fields, and a number of other external influences. Conversely, detecting the spin of a particle provides invaluable insights into the environment or the underlying dynamics of the system.

In heavy-ion collision physics, the primary interest lies in the creation of deconfined quark-gluon matter, commonly referred to as the quark-gluon plasma (QGP) [1–5]. To uncover the properties of QGP in heavy-ion collision experiments, it is essential to design specific hadronic observables that are sensitive to particular features of the QGP. Since charged particles are typically the easiest to detect, many observables rely on the charge of the hadrons. For instance, the total multiplicity of detected charged hadrons reflects the initial energy of the QGP. Meanwhile, the anisotropy in the momentum-space distribution of charged hadrons corresponds to the initial anisotropy in the spatial distribution of partons, leading to the well-known harmonic flow parameters [6]. By measuring these hadronic observables, people have revealed several novel properties of the hot and dense matter created in heavy-ion collisions. One significant finding is that the QGP must be extremely hot, with a typical temperature reaching 300-500 MeV at RHIC and LHC, indicating an extremely large energy density.

Additionally, the QGP medium is found to be strongly interacting, with a very small shear viscosity to entropy density ratio η/s. This low ratio is required to explain the observed harmonic flow parameters [4, 5]. In fact, the η/s of QGP is the lowest among all known fluids.

Since 2017, it has been established that the spin degree of freedom can also be used to probe the properties of the QGP [7]. This is achieved through measurements of the spin polarization of spinful hadrons, such as hyperons and vector mesons [8–10]. Notably, it has been observed that the Λ and $\bar{\Lambda}$ hyperons can exhibit significant spin polarization at collision energies of tens of GeV [7, 11–14]. Similarly, the ϕ and J/ψ mesons have been found to exhibit considerable spin alignment [15, 16]. These discoveries open new avenues for studying QGP through the spin degree of freedom. For instance, we now understand that the so-called global spin polarization (i.e., the total amount of spin polarization with respect to the reaction plane) of hyperons arises from angular momentum conservation through the formation of fluid vortices within the QGP: In non-central heavy-ion collisions, the system possesses substantial orbital angular momentum, which subsequently induces strong fluid vorticity in the QGP [17–19], thereby polarizing the spins of quarks via spin-rotation coupling [20–35]. However, to fully understand the spin polarization phenomena, a dynamical theory of spin polarization and spin transport in the hot medium is essential, analogous to the necessity of a dynamical theory of the bulk medium for understanding harmonic flows. Naturally, such a dynamical theory of spin transport could be derived from either kinetic theory or hydrodynamic theory. In recent years, both spin kinetic theory and spin hydrodynamics have made significant advancements. In this article, we will focus on spin hydrodynamics, while we refer readers to Refs. [36] for a review of spin kinetic theory and to Refs. [37–49] for a review of spin polarization phenomena in heavy-ion collisions. Besides, we will focus only on spin polarization in hot and dense medium rather than in systems created in, e.g., electron-ion collisions [50].

Throughout this article, we use the natural units $c = \hbar = k_B = 1$ and the metric convention $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

II. Relativistic Hydrodynamics as an Effective Theory

Before we go into the discussion of spin hydrodynamics, let us first briefly review the general structure of relativistic hydrodynamics from the perspective of effective field theory. Hydrodynamic theory describes the low-energy behavior of interacting many-body systems, where only conserved charge densities exhibit their dynamics. Since conserved charge densities do not vanish, they simply redistribute themselves in space according to their equations of motion (EOMs). When expressed in a manner of spatial gradient expansion, these EOMs constitute the hydrodynamic equations.

Let us consider the hydrodynamic theory of a system with spacetime translation symmetry and a global $U(1)$ symmetry. The corresponding conserved charge densities are the energy density $\varepsilon(x)$, momentum density $\pi_i(x)$, $i = 1-3$, and the $U(1)$ charge density $n(x)$. We want to derive the dynamical equations for these conserved charge densities. Sometimes it is more convenient to work with the potential variables conjugate to the charge densities. They are the temperature $T(x)$ (or its inverse $\beta(x) = 1/T(x)$), the fluid velocity $u^\mu(x)$ normalized as $u^\mu u^\nu \eta_{\mu\nu} = 1$, and the chemical potential of $n(x)$, $\mu(x)$. These conserved charge densities (and equivalently their conjugates) are hydrodynamic variables in hydrodynamics. Our starting point is the conservation laws: $\partial_\mu \Theta^{\mu\nu} = 0$, $\partial_\mu J^\mu = 0$, where $\Theta^{\mu\nu}$ is the energy-momentum tensor and J^μ is the $U(1)$ current. As an effective field theory, we want to express $\Theta^{\mu\nu}$ and J^μ in terms of the conserved charged density (or equivalently, their conjugates) and their various gradient orders. We assume spatial isotropy of the system, i.e., there is no external forces breaking the $SO(3)$ symmetry. The building blocks are the fluid velocity u^μ and various quantities that can be classified into different representations of $SO(3)$ in the rest frame of the fluid. Up to first order in gradients, these quantities are:

Scalar: $\varepsilon, n, D\varepsilon, Dn, \theta \equiv \nabla \cdot u = \partial \cdot u$
Vector: $Du^\mu, \nabla^\mu \varepsilon, \nabla^\mu n, \omega_{\mu\nu} \equiv -\frac{1}{2}(\nabla_\mu u_\nu - \nabla_\nu u_\mu)$
Tensor: $\sigma_{\mu\nu} \equiv \frac{1}{2}[\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\Delta_{\mu\nu}\theta]$

where $D \equiv u \cdot \partial$ is the co-moving time derivative, θ is the expansion rate of the fluid, $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$ is the spatial gradient operator with $\Delta^{\mu\nu} \equiv \eta^{\mu\nu} - u^\mu u^\nu$ the spatial projector, $\sigma_{\mu\nu}$ is the shear tensor which is traceless, and $\omega_{\mu\nu}$ is the vorticity tensor. Note that the co-moving time derivatives will be eventually replaced by spatial gradients upon using the EOMs at leading order. Note that the vorticity tensor transforms the same way as a three-vector under proper three-rotations (i.e., a three-rotation R with $\det R = 1$) as it can be substituted by a three pseudo-vector $\omega^\mu \equiv -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\omega_{\rho\sigma}$. Consequently, we can write down the most general structure decomposition up to $O(\partial)$ for $\Theta^{\mu\nu}$ and J^μ as follows:

$$\begin{aligned}\Theta^{\mu\nu} = {} & (a_0 + b^0_\varepsilon D\varepsilon + b^0_n Dn + b^0_u\theta)u^\mu u^\nu + c_0(u^\mu\nabla^\nu\varepsilon + u^\nu\nabla^\mu\varepsilon) + d_0(u^\mu\nabla^\nu n + u^\nu\nabla^\mu n) \\ & + e_0(u^\mu Du^\nu + u^\nu Du^\mu) + f_0(u^\mu\omega^\nu + u^\nu\omega^\mu) + (g_0 + h^0_\varepsilon D\varepsilon + h^0_n Dn + h^0_u\theta)\Delta^{\mu\nu} \\ & + i_0\sigma^{\mu\nu} + j_0(u^\mu\nabla^\nu\varepsilon - u^\nu\nabla^\mu\varepsilon) + k_0(u^\mu\nabla^\nu n - u^\nu\nabla^\mu n) + l_0(u^\mu Du^\nu - u^\nu Du^\mu) \\ & + m_0(u^\mu\omega^\nu - u^\nu\omega^\mu) + n_0\epsilon^{\mu\nu\rho\sigma}u_\rho\nabla_\sigma\varepsilon + o_0\epsilon^{\mu\nu\rho\sigma}u_\rho\nabla_\sigma n + p_0\epsilon^{\mu\nu\rho\sigma}u_\rho Du_\sigma + q_0\omega^{\mu\nu} + O(\partial^2),\end{aligned}$$

$$J^\mu = (A_0 + B^0_\varepsilon D\varepsilon + B^0_n Dn + B^0_u\theta)u^\mu + C_0\nabla^\mu\varepsilon + D_0\nabla^\mu n + E_0 Du^\mu + F_0\omega^\mu + O(\partial^2),$$

with all the coefficients (playing roles of the Wilson coefficients in effective field theory, as the short-distance physics are encoded in these coefficients) functions of ε and n. They are constructed by decomposing first with respect to u^μ and then with respect to different representations of $SO(3)$. In these decompositions, the terms with f_0, m_0, n_0, o_0, p_0 in $\Theta^{\mu\nu}$ and F_0 in J^μ as coefficients transform differently from $\Theta^{\mu\nu}$ and J^μ under parity (P), respectively, meaning that they can appear only when the system contains parity violating contents. Under time reversal transformation (T), all the terms of first-order gradients on the right-hand sides of $\Theta^{\mu\nu}$ and J^μ except for terms with coefficients $f_0, m_0, n_0, o_0, p_0, F_0$ transform differently from $\Theta^{\mu\nu}$ and J^μ, respectively. This means that these terms must be dissipative (that is to say, these terms are responsible for entropy generation in the fluid), while terms with coefficients $f_0, m_0, n_0, o_0, p_0, F_0$ can appear without generating entropy, i.e., they could arise in ideal hydrodynamics despite they are at first order in gradients. The terms with coefficients $f_0, m_0, n_0, o_0, p_0, F_0$ are thus especially interesting. In fact, some of them have been intensively studied and it was found that they contain very rich quantum phenomena (usually dubbed chiral anomalous transports) closely related to chiral anomaly of the system if the underlying physics is governed by quantum gauge theory. Recently, such chiral anomalous transports have been active subjects in condensed matter physics, astrophysics, and heavy-ion collision physics; see Refs. [41, 51–56] for recently reviews with a focus on heavy-ion collision physics. Similarly, we could also examine the balance of right-hand and left-hand sides of Eqs. (4)-(5) under charge conjugation (C) transformation. The terms with coefficients b_n^0, C_0, E_0, F_0 must vanish if there is no environmental charge-conjugation violation (Naturally, the presence of a nonzero charge density n violates the C symmetry and allows these terms to present). The antisymmetric terms in $\Theta^{\mu\nu}$ are also particularly interesting. To reveal their meaning, we consider the angular momentum conservation law:

$$\partial_\mu M^{\mu\nu\rho} = 0, \quad M^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu} + \Sigma^{\mu\nu\rho},$$

where $\Sigma^{\mu\nu\rho}$ is the spin tensor. We can re-write Eq. (6) in the following form:

$$\partial_\mu \Sigma^{\mu\nu\rho} = \Theta^{\rho\nu} - \Theta^{\nu\rho}.$$

Thus the antisymmetric part of $\Theta^{\mu\nu}$ provides a source for the generation of spin (one may more clearly see this by integrating Eq. (8) over space). This will be the focus of this article and we will come back to it from the next section. In the rest of this section, for the purpose of demonstrating the construction of the hydrodynamic theory, we will simply assume that the system does not possess a spin tensor, so that $\Theta^{\mu\nu}$ is symmetric, $\Theta^{\mu\nu} = \Theta^{\nu\mu}$, and will assume that there is no environmental parity violation, so that terms with coefficients $f_0, m_0, n_0, o_0, p_0, F_0$ must vanish.

Thus, the most general decomposition of $\Theta^{\mu\nu}$ and J^μ up to the first order in gradients into different components with respect to u^μ, and subsequently, for the components orthogonal to u^μ, with respect to different irreducible tensor structures under $SO(3)$ are as follows:

$$\Theta^{\mu\nu} = (a_0 + b_\varepsilon^0 D\varepsilon + b_n^0 Dn + b_u^0 \theta) u^\mu u^\nu + c_0 (u^\mu \nabla^\nu \varepsilon + u^\nu \nabla^\mu \varepsilon) + d_0 (u^\mu \nabla^\nu n + u^\nu \nabla^\mu n) + e_0 (u^\mu D u^\nu + u^\nu D u^\mu) + (g_0 + h_\varepsilon^0 D\varepsilon$$

$$J^\mu = (A_0 + B^0_\varepsilon D\varepsilon + B^0_n Dn + B^0_u \theta)u^\mu + C_0 \nabla^\mu \varepsilon + D_0 \nabla^\mu n + E_0 D u^\mu + O(\partial^2).$$

Up to this point, the expressions (9)-(10) are merely a parametrization of $\Theta^{\mu\nu}$ and J^μ, and such a parametrization is ambiguous at $O(\partial)$ order (and higher orders in gradients). To see this, we consider to re-express $\Theta^{\mu\nu}$ and J^μ in terms of a redefinition of the hydrodynamic variables ε', n', u'^μ which differ from ε, n, u^μ by $O(\partial)$-order shifts:

$$\varepsilon' = \varepsilon + \delta\varepsilon, \quad n' = n + \delta n, \quad u'^\mu = u^\mu + \delta u^\mu,$$

where $\delta\varepsilon, \delta n, \delta u^\mu$ are order-$O(\partial)$ quantities and $u_\mu \delta u^\mu = O(\partial^2)$ so that $u'^2 = 1$ is kept at $O(\partial)$. This can be seen by noticing that $u'^2 = u^2 + 2u_\mu \delta u^\mu + \delta u^2$ which leads to $2u_\mu \delta u^\mu + \delta u^2 = O(\partial^2)$ and therefore $u_\mu \delta u^\mu$ must be $O(\partial^2)$. In terms of the primed variables, we have

$$\Theta^{\mu\nu} = \Big[a_0 + \Big(\frac{\partial a_0}{\partial \varepsilon}\Big)\delta\varepsilon + \Big(\frac{\partial a_0}{\partial n}\Big)\delta n\Big] u'^\mu u'^\nu + c_0(u^\mu \nabla^\nu \varepsilon + u^\nu \nabla^\mu \varepsilon) + d_0(u^\mu \nabla^\nu n + u^\nu \nabla^\mu n) + e_0(u^\mu D u^\nu + u^\nu D u^\mu) + (g_0 -$$

$$J^\mu = \Big[A_0 + \Big(\frac{\partial A_0}{\partial \varepsilon}\Big)\delta\varepsilon + \Big(\frac{\partial A_0}{\partial n}\Big)\delta n\Big] u'^\mu - A_0 \delta u^\mu + C_0 \nabla^\mu \varepsilon + D_0 \nabla^\mu n + E_0 D u^\mu + O(\partial^2),$$

where a_0, g_0, A_0 are functions of ε', n' and all the second order terms are omitted. By observing the expressions in the three square brackets, one can realize that, by suitably choosing $\delta\varepsilon$ and δn, one can eliminate the first-order terms in two of the three square brackets. For example, one can solve out $\delta\varepsilon$ and δn by requiring the first-order terms in the square brackets in $\Theta^{\mu\nu}$ to vanish. But it is much more convenient to eliminate the first-order terms in the coefficients of $u'^\mu u'^\nu$ in $\Theta^{\mu\nu}$ and u'^μ in J^μ. Similarly, by suitably choosing δu^μ, one can eliminate either the second line in $\Theta^{\mu\nu}$ (such a choice is called Landau-Lifshitz frame for u^μ) or the second line in J^μ (such a choice is called Eckart frame for u^μ). Therefore, we could always choose the following simpler forms for $\Theta^{\mu\nu}$ and J^μ (Landau-Lifshitz frame):

$$\Theta^{\mu\nu} = a_0 u^\mu u^\nu + (g_0 + h^0_\varepsilon D\varepsilon + h^0_n Dn + h^0_u \theta)\Delta^{\mu\nu} + i_0 \sigma^{\mu\nu} + O(\partial^2),$$

$$J^\mu = A_0 u^\mu + C_0 \nabla^\mu \varepsilon + D_0 \nabla^\mu n + E_0 D u^\mu + O(\partial^2).$$

Contracting with u_μ, we can identify that $a_0 = u_\mu u_\nu \Theta^{\mu\nu}$ which is the local energy density ε and $A_0 = u \cdot J$ which is the local $U(1)$ charge density n.

[Sometimes, this is also considered as the matching condition because this means that $u_\mu u_\nu \Theta^{\mu\nu} = u_\mu u_\nu \Theta^{\mu\nu}_{(0)}$ with $\Theta^{\mu\nu}_{(0)}$ and $J^\mu_{(0)}$ the zeroth order energy-momentum tensor and charge current.] Let us first consider the zeroth order terms which, as we have already discussed, correspond to ideal hydrodynamics:

$$\Theta^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu + g_0 \Delta^{\mu\nu}, \quad J^\mu_{(0)} = n u^\mu.$$

In the rest frame of the fluid, $u^\mu = (1, 0)$, it becomes $\Theta^{\mu\nu}_{(0)} = \mathrm{diag}(\varepsilon, -g_0, -g_0, -g_0)$ which identifies $-g_0$ to be the thermodynamic pressure P. At zeroth order, the conservation laws read

$$(\varepsilon + P) D u^\mu - \nabla^\mu P = 0, \quad D\varepsilon + (\varepsilon + P)\theta = 0, \quad Dn + n\theta = 0.$$

To close these equations, we need to know the thermodynamic relation among P, ε, n, that is, the equation of state, $P = P(\varepsilon, n)$.

Let us then consider the first-order terms which correspond to dissipative hydrodynamics. From Eqs. (18)-(20), we notice that we could replace $D\varepsilon$ and Dn in the first order terms by $-(\varepsilon + P)\theta$ and $-n\theta$ and Du^μ by $\nabla^\mu P/(\varepsilon + P)$. This allows us to re-write energy-momentum tensor and charge current as:

$$\Theta^{\mu\nu} = \varepsilon u^\mu u^\nu - (P + h_0 \theta)\Delta^{\mu\nu} + i_0 \sigma^{\mu\nu} + O(\partial^2),$$

$$J^\mu = n u^\mu + C_0' \nabla^\mu \varepsilon + D_0' \nabla^\mu n + O(\partial^2),$$

with $h_0 = (\varepsilon + P) h^0_\varepsilon + n h^0_n - h^0_u$, $C_0' = C_0 + E_0 (\partial P/\partial \varepsilon)_n/(\varepsilon + P)$, and $D_0' = D_0 + E_0 (\partial P/\partial n)_\varepsilon/(\varepsilon + P)$. Further constraints can be imposed from the laws of local thermodynamics. For a fluid at rest, we have the first law of thermodynamics as

$$T ds + \mu dn = d\varepsilon,$$

with s the entropy density. To proceed, we propose the covariant generalization of the second one (the Gibbs-Duhem relation):

$$s^\mu = P\beta^\mu + \Theta^{\mu\nu}\beta_\nu - \alpha J^\mu,$$

with $\beta^\mu = \beta u^\mu$ ($\beta = 1/T$), $\alpha = \mu/T$, and s^μ the entropy current such that $u \cdot s = s$. The divergence of s^μ (multiplied by T) can be directly calculated,

$$T\partial_\mu s^\mu = \Theta^{\mu\nu}_{(1)} \nabla_\mu u_\nu - T J^\mu_{(1)} \nabla_\mu \alpha.$$

The second law of local thermodynamics requires that $T\partial_\mu s^\mu \geq 0$ for any configurations of velocity field u^μ, temperature T, and chemical potential μ, which imposes the following constraints:

$$h_0 = -\zeta \leq 0, \quad i_0 = 2\eta \geq 0, \quad C_0' \nabla^\mu \varepsilon + D_0' \nabla^\mu n = \sigma \nabla^\mu \alpha,$$

where ζ and η are the bulk and shear viscosities, and σ the charge conductivity. This also shows that the coefficients C_0' and D_0' are fixed in such a way that $C_0' \nabla^\mu \varepsilon + D_0' \nabla^\mu n = \sigma \nabla^\mu \alpha$. The EOMs of first-order dissipative hydrodynamics then read

$$(\varepsilon + P - \zeta\theta) Du^\mu - \nabla^\mu (P - \zeta\theta) + 2\eta \Delta^\mu{}_\nu \partial_\rho \sigma^{\nu\rho} = 0,$$

$$D\varepsilon + (\varepsilon + P - \zeta\theta)\theta - 2\eta \sigma_{\mu\nu} \sigma^{\mu\nu} = 0,$$

$$Dn + n\theta + \sigma \nabla^2 \alpha = 0.$$

The first equation is the relativistic Navier-Stokes equation. The above procedure can continue to higher order in gradients and will give us higher-order hydrodynamics. We, however, will not discuss these more complicated situation. The readers can find discussions in Refs. [57–61].

III. Construction of Relativistic Spin Hydrodynamics

With the above preparation, we now discuss the construction of relativistic spin hydrodynamics, in which the conservation of angular momentum is explicitly encoded within a (quasi)-hydrodynamic framework. The fundamental conservation laws are energy-momentum conservation (1) and angular momentum conservation (8). Before delving into the detailed construction, we note that if we assign the spin density $S^{\mu\nu} = u_\rho \Sigma^{\rho\mu\nu}$ as a dynamic variable in our framework, equation Eq. (8) means that it is generally not conserved. This reflects the fact that spin angular momentum can be transformed into orbital angular momentum, thus disqualifying it as a true hydrodynamic mode. Consequently, spin hydrodynamics will not be a strict hydrodynamic theory for gapless modes. Instead, it should be categorized as quasi-hydrodynamics, where the low-energy dynamic variables comprise true hydrodynamic modes and some gapped modes (quasi-hydrodynamic modes) whose gap in the low-momentum region is parametrically small compared to other microscopic modes (the hard modes of the system) [62]. This results in a spectrum separation; for physics at energy scales comparable to these modes, we can only consider the quasi-hydrodynamic modes alongside the true hydrodynamic modes. The so-called generalized hydrodynamics [63] and Hydro+ [64] near the QCD critical point fall into this category. The spin hydrodynamics we are going to discuss also belongs to this type of theory.

This framework requires that spin excitations, despite being gapped, remain low-energy excitations compared to other microscopic modes [62]. For instance, if the system contains massive fermions, the spins of these fermions are hard to relax, because the spin-orbit coupling is inversely suppressed by the mass of the fermions compared to typical energy transfer [65–67]. Thus, these spins are quasi-conserved and we can formulate a quasi-hydrodynamic theory for it, which is called the spin hydrodynamics.

We consider a charge neutral system such as the quark gluon plasma or usual electric plasma in which some of constituent particles are spinful particles. The symmetry we are considering are the spacetime translation symmetry and Lorentz symmetry. They lead to the energy-momentum conservation and angular momentum conservation as given by Eq. (1) and Eq. (8). Now the spin tensor $\Sigma^{\mu\rho\sigma}$ plays the role of the charge current J^μ and we could write it as $\Sigma^{\mu\rho\sigma} = S^{\rho\sigma}u^\mu +$ higher order terms with the spin density $S^{\rho\sigma}$ playing similar role of charge density n in Eq. (22).

To proceed, we need to choose suitable power counting scheme for all the (quasi-)hydrodynamic variables. If we are considering the QGP in heavy ion collisions, from the measurements of global spin polarization of hyperons, we can know that the spin density in QGP should be small because the hyperon spin polarization is only about a few percent. Thus, it is reasonable to assume the spin density $S^{\rho\sigma}$ to be parametrically smaller than the true hydrodynamic modes described by variables ε and u^μ. Thus, let us take the following power counting scheme:

$$\varepsilon, P, T, u^\mu \sim O(1), \quad S^{\rho\sigma} \sim O(\partial).$$

Analogous to that chemical potential μ is conjugate to charge density n, we can introduce the spin potential $\mu^{\rho\sigma}$ to be conjugate thermodynamically to spin density $S^{\rho\sigma}$ and propose the first law for local thermodynamics as (analogous to Eq. (23)):

$$Tds + \frac{1}{2}\mu_{\mu\nu}dS^{\mu\nu} = d\varepsilon, \quad \frac{1}{2}\mu_{\mu\nu}S^{\mu\nu} = \varepsilon + P.$$

Following the discussions about the fluid local frame, we realize that the same discussions are still valid for the symmetric part of the energy-momentum tensor and thus we still choose the definition of u^μ such that it is the eigenvector of the symmetric part of the energy-momentum tensor, $\Theta_s^{\mu\nu}$ (we still call it the Landau-Lifshitz frame):

$$\Theta_s^{\mu\nu}u_\nu = \varepsilon u^\mu.$$

Since the EOM for spin density involves only the antisymmetric part of the energy-momentum tensor, $\Theta_a^{\mu\nu}$, the symmetric part, $\Theta_s^{\mu\nu}$, still takes the same tensor decomposition upto the first order in gradients as Eq. (21):

$$\Theta_s^{\mu\nu} = \varepsilon u^\mu u^\nu - (P - \zeta\theta)\Delta^{\mu\nu} + 2\eta\sigma^{\mu\nu} + O(\partial^2).$$

To determine the form of $\Theta_a^{\mu\nu}$, we again utilize the second law of local thermodynamics. The covariant entropy current reads (an analogue of Eq. (25))

$$s^\mu = P\beta^\mu + \Theta^{\mu\nu}\beta_\nu - \alpha_{\rho\sigma}\Sigma^{\mu\rho\sigma},$$

with $\alpha_{\rho\sigma} = \mu_{\rho\sigma}/T$. The production rate of entropy then reads

$$T\partial_\mu s^\mu = \Theta_{s(1)}^{\mu\nu}\partial_{(\mu}u_{\nu)} + \Theta_a^{\mu\nu}[\mu_{\mu\nu} + T\partial_{[\mu}\beta_{\nu]}] + O(\partial^3).$$

The semi-positiveness of the first term on the right-hand side is already guaranteed when both bulk and shear viscosities are semi-positive. The requirement of the semi-positiveness of the second term gives the constitutive relation for $\Theta_a^{\mu\nu}$ at $O(\partial)$ order [68]:

$$\Theta_a^{\mu\nu} = q^\mu u^\nu - q^\nu u^\mu + \phi^{\mu\nu},$$

$$q^\mu = \lambda[\beta\nabla^\mu T + Du^\mu - 2\mu^{\mu\nu}u_\nu],$$

$$\phi^{\mu\nu} = \eta_s \Delta^\mu{}_\rho \Delta^\nu{}_\sigma(\mu^{\rho\sigma} - T\varpi^{\rho\sigma}).$$

The quantity $\varpi_{\mu\nu} = \frac{1}{2}(\partial_\nu\beta_\mu - \partial_\mu\beta_\nu)$ is called the thermal vorticity tensor. The quantities λ and η_s must be semi-positive to guarantee the semi-positiveness of entropy production. They are called boost heat conductivity and rotational viscosity, respectively [68]. With these constitutive relations, we obtain the spin hydrodynamic equations up to $O(\partial^2)$ order:

$$(\varepsilon+P-\zeta\theta)Du^\mu - \nabla^\mu(P-\zeta\theta) + 2\eta\Delta^\mu{}_\nu\partial_\rho\sigma^{\nu\rho} + q\cdot\partial u^\mu - \Delta^\mu{}_\nu Dq^\nu - q^\mu\theta + \Delta^\mu{}_\rho\partial_\nu\phi^{\nu\rho} = 0,$$

$$D\varepsilon + (\varepsilon + P - \zeta\theta)\theta - 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \partial\cdot q + q^\mu Du_\mu + \phi_{\mu\nu}\omega^{\mu\nu} = 0,$$

$$DS^{\rho\sigma} + S^{\rho\sigma}\theta + 2\Theta_a^{\rho\sigma} = 0.$$

In this section, we present a detailed derivation of the constitutive relations up to first order in gradients for relativistic spin hydrodynamics. For related discussions following a similar approach, see, for example, Refs.[62, 69–78]. Other methodologies for deriving and analyzing the constitutive relations of spin hydrodynamics have also discussed in literature, including utilizing the hydrostatic partition function with constraints from the entropy current and Onsager relations [79, 80], using local equilibrium and non-equilibrium statistical operators [73, 77, 81–83], and employing kinetic theories [76, 84–95]. Relativistic spin hydrodynamics has become a vibrant area of research, attracting intense discussions in recent years. In the following section, we will explore some of these developments; further insights can be found in, e.g., Refs. [96–110].

IV. Discussions

We have developed spin hydrodynamics based on local thermodynamic laws. Spin hydrodynamics exhibits several novel features that differ significantly from those found in conventional relativistic hydrodynamics for other types of conservation laws (e.g., the energy-momentum conservation and baryon number conservation). In this section, we aim to explore and discuss some of these intriguing characteristics.

A. Pseudo-gauge ambiguity

The definition of a conserved current is not unique. One example is the magnetization current and dipole charge density. Let $J^\mu = (\rho, \mathbf{J})$ represent the conserved conduction electric current. For a polarizable and magnetizable material, the total charge density and electric current are given by $\tilde{\rho} = \rho + \nabla \cdot \mathbf{P}$ and $\tilde{\mathbf{J}} = \mathbf{J} + \nabla \times \mathbf{M}$, respectively, where $\mathbf{P}$ is the electric dipole density, and $\mathbf{M}$ is the magnetization density. In covariant form, we have:

$$\tilde{J}^\mu = J^\mu + \partial_\nu M^{\mu\nu} \quad \text{with} \quad M^{\mu\nu} = -M^{\nu\mu}.$$

Obviously, the total current $\tilde{J}^\mu$ is conserved if the conduction current J^μ is conserved, and the total electric charge remains unchanged provided the surface dipole density vanishes. A transformation of a conserved current that preserves both the original conservation law and the total conserved charge is called a pseudo-gauge transformation. The above example demonstrates that the total current and the conduction current differ by a pseudo-gauge transformation (with the magnetization $M^{\mu\nu}$ serving as the pseudo-gauge field). This example also highlights that a pseudo-gauge transformation is not a true gauge transformation, as it alters the physical content of the transformed current. Further insight about the pseudo-gauge transformation can be gained by examining the Maxwell equation:

$$\partial_\mu F^{\mu\nu} = \tilde{J}^\nu.$$

One could subtract $-\partial_\rho M^{\nu\rho}$ from both sides and find $\partial_\mu H^{\mu\nu} = J^\nu$, where the new field strength tensor is defined as $H^{\mu\nu} \equiv F^{\mu\nu} + M^{\mu\nu}$. This demonstrates that, without imposing additional constraints, the two sets of fields, $(F^{\mu\nu}, \tilde{J}^{\mu})$ and $(H^{\mu\nu}, J^{\mu})$, describe the same physical laws, and one can freely choose which set to use. (If further constraints are imposed—such as the Bianchi equation, $\partial_{[\mu}F_{\nu\rho]} = 0$, which is not preserved under a general pseudo-gauge transformation—then only certain pseudo-gauges that respect the Bianchi equation are permitted.)

Similarly, let us consider angular momentum conservation (note the analogy with Eq. (48), where $\Sigma^{\mu\nu\rho}$ and $\Theta^{\rho\nu} - \Theta^{\nu\rho}$ play roles analogous to $H^{\mu\nu}$ and J^{μ} in Eq. (48)):

$$\partial_\mu \Sigma^{\mu\nu\rho} = \Theta^{\rho\nu} - \Theta^{\nu\rho},$$

which is preserved under the transformation

$$\Sigma^{\mu\rho\sigma} \to \tilde{\Sigma}^{\mu\rho\sigma} \equiv \Sigma^{\mu\rho\sigma} - \Phi^{\mu\rho\sigma}, \quad \Theta^{\mu\nu} \to \tilde{\Theta}^{\mu\nu} \equiv \Theta^{\mu\nu} + \partial_\lambda \Phi^{\lambda\mu\nu},$$

with $\Phi^{\lambda\mu\nu} = -\Phi^{\lambda\nu\mu}$ being an arbitrary local field. However, this transformation violates the conservation law of the energy-momentum tensor. It can be modified into the following form:

$$\Sigma^{\mu\rho\sigma} \to \tilde{\Sigma}^{\mu\rho\sigma} \equiv \Sigma^{\mu\rho\sigma} - \Phi^{\mu\rho\sigma},$$

$$\Theta^{\mu\nu} \to \tilde{\Theta}^{\mu\nu} \equiv \Theta^{\mu\nu} + \frac{1}{2}(\Phi^{\lambda\mu\nu} - \Phi^{\mu\lambda\nu} - \Phi^{\nu\lambda\mu}),$$

which preserves both Eq. (49) and Eq. (1). Given a spacelike hypersurface Ξ, the total energy-momentum and total angular momentum across Ξ are

$$P^{\nu} = \int_{\Xi} d\Xi_\mu \Theta^{\mu\nu}, \quad M^{\rho\sigma} = \int_{\Xi} d\Xi_\mu M^{\mu\rho\sigma} = \int_{\Xi} d\Xi_\mu (x^{\rho}\Theta^{\mu\sigma} - x^{\sigma}\Theta^{\mu\rho} + \Sigma^{\mu\rho\sigma}).$$

One can check that P^{μ} and $M^{\rho\sigma}$ are invariant under pseudo-gauge transformation (52) and (53) if the pseudo-gauge field $\Phi^{\mu\rho\sigma}$ vanishes at the boundary of Ξ.

One consequence of the existence of the pseudo-gauge transformation is the freedom to choose the symmetry properties of the spin tensor. To illustrate this, we consider an example in which we aim to transform a general spin tensor $\Sigma^{\mu\rho\sigma} = -\Sigma^{\mu\sigma\rho}$ into a completely antisymmetric form. We can choose $\Phi^{\mu\rho\sigma} = \Sigma^{(\mu\rho)\sigma} - \frac{1}{2}\Sigma^{\sigma\mu\rho}$. After applying the pseudo-gauge transformation, this yields:

$$\Sigma^{\mu\rho\sigma} \to \tilde{\Sigma}^{\mu\rho\sigma} = \frac{1}{3}(\Sigma^{\mu\rho\sigma} + \Sigma^{\rho\sigma\mu} + \Sigma^{\sigma\mu\rho}),$$

$$\Theta^{\mu\nu} \to \tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \frac{1}{2}(\Sigma^{\mu\nu\lambda} - \Sigma^{\nu\mu\lambda} + \Sigma^{\lambda\nu\mu}).$$

Note that such-obtained $\tilde{\Sigma}^{\mu\rho\sigma}$ is totally antisymmetric, so we can parameterize it as $\tilde{\Sigma}^{\mu\rho\sigma} = -\epsilon^{\mu\rho\sigma\nu}\tilde{S}_\nu$, with $\tilde{S}_\mu$ the corresponding spin (pseudo)vector. The spin density tensor is thus $\tilde{S}^{\mu\nu} = -\epsilon^{\mu\nu\rho\sigma}u_\rho\tilde{S}_\sigma$. The main difference between this spin density tensor and the one used in Section III is that $\tilde{S}^{\mu\nu}$ contains three degrees of freedom corresponding to the three spatial spin vectors, whereas $S^{\mu\nu}$ has six degrees of freedom, with three for spatial spin and three for boost. Thus, in some cases, it is more convenient to use $\tilde{\Sigma}^{\mu\rho\sigma}$ to construct the spin hydrodynamics. By following a procedure similar to that adopted in Section III, we can derive the constitutive relations in this context. In doing so, we decompose $\tilde{S}^\mu$ into $\tilde{S}^\mu = \sigma^\mu + n_5 u^\mu$, where σ^μ represents the spatial spin with the condition $\sigma \cdot u = 0$, and n_5 is a pseudoscalar field (hence the subscript 5). We also decompose $\tilde{\Theta}^{\mu\nu}$ into:

$$\tilde{\Theta}^{\mu\nu} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \tilde{\Theta}^{\mu\nu}_{s(1)} + \tilde{q}^\mu u^\nu - \tilde{q}^\nu u^\mu + \tilde{\phi}^{\mu\nu},$$

where, as we did in Section III, we have assumed Landau-Lifshitz frame $\tilde{\Theta}^{\mu\nu}_{s(1)}u_\nu = \varepsilon u^\mu$, so that $\tilde{\Theta}^{\mu\nu}_{s(1)}$ is purely transverse to u^μ. It is important to note that, although we use the same symbols ε, P, and u^μ as in Section III, their actual values may differ since the energy-momentum tensors and spin tensors in these two cases are different (but connected by the pseudo-gauge transformations (56) and (57)). We adopt a power counting scheme similar to the one we chose in Section III: $\varepsilon, P, T, u^\mu \sim O(1)$, $\tilde{S}^\mu, \tilde{q}^\mu, \tilde{\phi}^{\mu\nu} \sim O(\partial)$.

Using the same form for the entropy current and first law for local thermodynamics presented in Section III, one can then find the divergence of the entropy current to be

$$T\partial_\mu s^\mu = \tilde{\Theta}^{\mu\nu}_{s(1)}\partial_{(\mu}u_{\nu)} + \tilde{\Theta}^{\mu\nu}_a[\tilde{\mu}_{\mu\nu} + T\partial_{[\mu}\beta_{\nu]}] + O(\partial^3).$$

We note that in deriving this result, we have utilized the fact that contracting the equation of motion (49) with u_ρ reveals that $\tilde{q}^\mu$ is not an independent current, but is determined by $\tilde{S}^\mu$ through the following relation:

$$\tilde{q}^\mu = \epsilon^{\mu\nu\rho\sigma}u_\nu\nabla_\rho\tilde{S}_\sigma.$$

This is because, when the spin tensor is completely antisymmetric, the components responsible for boost are gauged away, meaning that the corresponding torque for the boost in the antisymmetric part of the energy-momentum tensor cannot be an independent current either. Due to this relation, we can show that $n_5 = \tilde{S} \cdot u$ is actually an $O(\partial^3)$ quantity (and thus does not appear on the right-hand side of Eq. (63)). In fact, by direct calculation, one can find that the higher order terms that neglected in Eq. (63) contains only one term $\propto n_5$,

$$n_5 \epsilon^{\mu\nu\rho\sigma} u_\sigma [\partial_\mu(\beta\tilde{\mu}_{\nu\rho}) + \nabla_\mu u_\rho(\partial_\nu \beta + D\beta_\nu)],$$

which infers that $n_5 \propto \epsilon^{\mu\nu\rho\sigma} u_\sigma [\partial_\mu(\beta\tilde{\mu}_{\nu\rho}) + \nabla_\mu u_\rho(\partial_\nu \beta + D\beta_\nu)] \sim O(\partial^3)$ and thus can be neglected [62]. Therefore, from Eq. (63), we derive the constitutive relations for the spin hydrodynamics with a completely antisymmetric spin tensor as follows [62]:

$$\tilde{\Theta}^{\mu\nu}_{s(1)} = \zeta\theta\Delta^{\mu\nu} + 2\eta\sigma^{\mu\nu},$$

$$\tilde{\Theta}^{\mu\nu}_{a} = \tilde{q}^{\mu} u^{\nu} - \tilde{q}^{\nu} u^{\mu} + \tilde{\phi}^{\mu\nu},$$

$$\tilde{\phi}^{\mu\nu}_{(1)} = \eta_s \Delta^{\mu}{}_{\rho} \Delta^{\nu}{}_{\sigma} (\mu^{\rho\sigma} - T\varpi^{\rho\sigma}).$$

Although these relations take the same form as those obtained in Section III, it is important to note that they apply specifically to the pseudo-gauge of a totally antisymmetric spin tensor. These relations are particularly convenient for describing the evolution of spatial spin degrees of freedom.

Thus, we see that choosing different forms for the spin tensor (loosely referred to as different pseudo-gauges) leads to different forms for the constitutive relations. In an extreme case, one might even select $\Phi^{\mu\rho\sigma} = \Sigma^{\mu\rho\sigma}$, which completely eliminates the spin tensor and renders the energy-momentum tensor totally symmetric (this choice is commonly referred to as the Belinfante gauge [111–113]). While this may seem to eliminate all information about spin in hydrodynamics, the energy density, viscous tensors, and heat current remain influenced by spin, meaning that the dynamics of spin are still embedded within those quantities. For discussions regarding the transformation from canonical to Belinfante gauges, see Refs. [69, 71, 75, 99, 114, 115]. Additionally, other pseudo-gauges have been employed and discussed in the context of spin hydrodynamics [85, 86, 88, 116–119].

B. Spin hydrodynamics for strong vorticity

The power counting scheme employed in the previous discussions is motivated by the observation that, at global equilibrium, the spin potential $\mu_{\mu\nu}$ is determined by the thermal vorticity $\varpi_{\mu\nu} = (\partial_\nu \beta_\mu - \partial_\mu \beta_\nu)/2$, which is naturally assumed to be an $O(\partial)$ quantity. However, this assumption may not hold true because global equilibrium allows for arbitrarily large rotations (vorticity). When the vorticity is large, the assignment $\varpi_{\mu\nu} \sim O(\partial)$ becomes inadequate; instead, it is more appropriate to consider that $\varpi_{\mu\nu} \sim O(1)$. We will explore this situation in this subsection, following closely discussions in Ref. [72]. Before going into the details, it is useful to compare spin hydrodynamics with magnetohydrodynamics (MHD), where the magnetic field is treated as an $O(1)$ quantity; See Ref. [120] for a review of relativistic MHD.

MHD describes the coupled evolution of fluid energy-momentum (or temperature and velocity) and the electromagnetic field in the low-energy and long-wavelength regime. The fundamental equations consist of the conservation laws for the energy-momentum tensor and Maxwell's equations. Due to the screening effect, electric fields within the fluid are gapped and are parametrically small compared to the magnetic field. This renders the electric field not an active mode in the low-energy, long-wavelength regime. In contrast, there is no screening of the magnetic field, allowing it to exhibit its own dynamics even in this regime. Consequently, the magnetic field can be large and is treated as an $O(1)$ quantity, despite the fact that $\mathbf{B} = \nabla \times \mathbf{A}$ involves one spatial gradient. The presence of an $O(1)$ magnetic field breaks the $SO(3)$ symmetry in the constitutive relations for $\Theta^{\mu\nu}$, introducing anisotropy even in ideal hydrodynamics. Specifically, we can define a normalized vector $b^\mu = B^\mu/B$, where $B = \sqrt{-B_\mu B^\mu}$, satisfying $b^2 = -1$ and $b \cdot u = 0$, as an additional building block for the hydrodynamic constitutive relations. For example, for a parity-even and charge neutral fluid, the energy-momentum tensor can be decomposed into

$$\Theta^{\mu\nu} = \varepsilon u^\mu u^\nu - P_\perp \Xi^{\mu\nu} + P_\parallel b^\mu b^\nu + \Theta^{\mu\nu}_{(1)},$$

where $\Xi^{\mu\nu} = \Delta^{\mu\nu} + b^\nu b^\nu$ is a projector that is transverse to both u^μ and b^μ. The terms $P_\perp$ and $P_\parallel$ represent the pressures in the directions transverse and parallel to the magnetic field, respectively. Note that when we allow an environmental parity violation (e.g., when there is a density imbalance between right- and left-hand particles in the fluid) and finite charge density, an term $u^{(\mu} b^{\nu)}$ can appear at zeroth order. The term $\Theta^{\mu\nu}_{(1)}$ (which is assumed to be symmetric since the spin degree of freedom is typically disregarded in MHD) denotes a collection of terms that are at least of order $O(\partial)$ in the gradient expansion and consistent with Onsager relations. For a parity-even fluid, all such terms are expressed as $\Theta^{\mu\nu}_{(1)} = \sum_{i=1}^{7} \lambda_i \eta_i^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma$, where λ_i are the corresponding transport coefficients [121–123]. The tensors $\eta_i^{\mu\nu\rho\sigma}$ are:

$$\eta_1^{\mu\nu\rho\sigma} = b^\mu b^\nu b^\rho b^\sigma,$$

$$\eta_2^{\mu\nu\rho\sigma} = \Xi^{\mu\nu} \Xi^{\rho\sigma},$$

$$\eta_3^{\mu\nu\rho\sigma} = -\Xi^{\mu\nu} b^\rho b^\sigma - \Xi^{\rho\sigma} b^\mu b^\nu,$$

$$\eta_4^{\mu\nu\rho\sigma} = -2[b^{(\mu}\Xi^{\nu)\rho}b^\sigma + b^{(\mu}\Xi^{\nu)\sigma}b^\rho],$$

$$\eta_5^{\mu\nu\rho\sigma} = 2\Xi^{\rho(\mu}\Xi^{\nu)\sigma} - \Xi^{\mu\nu}\Xi^{\rho\sigma},$$

$$\eta_6^{\mu\nu\rho\sigma} = -b^{(\mu}b^{\nu)\rho}b^\sigma - b^{(\mu}b^{\nu)\sigma}b^\rho,$$

$$\eta_7^{\mu\nu\rho\sigma} = \Xi^{\rho(\mu}b^{\nu)\sigma} + \Xi^{\sigma(\mu}b^{\nu)\rho},$$

where $b^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}u_\rho b_\sigma$ is a cross projector which appears only when charge-conjugation symmetry is violated (e.g., when a net charge density is presented).

Similar to the discussions above regarding MHD, we can consider a scenario for spin hydrodynamics where the vorticity is treated as zeroth order in gradients, while the gradients of other thermodynamic quantities are treated as first order. In line with MHD, this framework has been referred to as gyrohydrodynamics in Ref. [72]. To simplify the notation, we reuse b^μ to denote the unit vector along the vorticity:

$$b^\mu = \frac{\varpi^\mu}{\sqrt{-\varpi_\mu\varpi^\mu}} = \frac{\omega^\mu}{\sqrt{-\omega_\mu\omega^\mu}},$$

with $\varpi^\mu = -\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho\beta_\sigma/2 = \beta\omega^\mu$ the thermal vorticity vector. We choose the pseudo-gauge so that the spin tensor is totally antisymmetric. Using u^μ, b^μ as well as $g^{\mu\nu}, \epsilon^{\mu\nu\rho\sigma}$ as building blocks, we can decompose $\Theta^{\mu\nu}$ and $\Sigma^{\mu\nu\rho}$ into the following irreducible forms:

$$\Theta^{\mu\nu} = \varepsilon u^\mu u^\nu - P_\perp\Xi^{\mu\nu} + P_\parallel b^\mu b^\nu + P_\times b^{\mu\nu} + q^\mu u^\nu - u^\mu q^\nu + \Theta^{\mu\nu}_{s(1)} + \phi^{\mu\nu},$$

$$\Sigma^{\mu\nu\rho} = -\epsilon^{\mu\nu\rho\lambda}S_\lambda = -\epsilon^{\mu\nu\rho\lambda}(n_5 u_\lambda - S_\parallel b_\lambda + S_{\perp\lambda}),$$

where $P_{\perp,\parallel,\times}$ represent pressures (which will be counted as $O(1)$ quantities in gradient expansion) in different directions, whose physical meaning will become clear shortly. The quantity $S_\parallel = b\cdot S$ denotes the spin component in the direction of the vorticity, while $S_\perp^\mu = \Xi^{\mu\nu}S_\nu$ denotes the spin component transverse to the vorticity. As before, we have chosen the Landau-Lifshitz frame, with $q^\mu, \Theta^{\mu\nu}_{s(1)}, \phi^{\mu\nu}$, and $S_\perp^\mu$ all transverse to u^μ. Note again that, with this fully antisymmetric choice of spin tensor, the q^μ vector is no longer independent but is determined by S^μ through Eq. (64).

The power counting scheme is such that $S_{\parallel}$ is counted as order one, while $\phi^{\mu\nu} = -\phi^{\nu\mu}, S_{\perp}^{\mu}, n_5$, and q^{μ} (see Eq. (64)) are counted as being at least $O(\partial)$. Additionally, we will count S^{μ} as $O(\hbar)$ (since spin is totally quantum in nature) in comparison to other thermodynamic quantities, which can appear even at the classical level and are therefore assigned $O(\hbar^0)$. This allows for a double expansion in both ∂ and $\hbar$. For the entropy current, we can write $s^{\mu} = su^{\mu} + s_{(1)}^{\mu}$ and use still Eq. (33). It is straightforward to derive the divergence of the entropy current, and after some calculations, one finds that up to $O(\hbar\partial^2, \partial^3)$ [72]:

$$\partial_{\mu} s^{\mu} = [s - \beta(\varepsilon + P_{\perp})]\theta - (P_{\parallel} - P_{\perp} - \mu_{\parallel} S_{\parallel}) b^{\mu} b^{\nu} \partial_{\mu}\beta_{\nu} + P_{\times} b^{\mu\nu}(\partial_{\mu}\beta_{\nu} + \beta\varpi_{\mu\nu}) + \Theta_{s(1)}^{\mu\nu}\partial_{(\mu}\beta_{\nu)} + \phi^{\mu\nu}(\partial_{[\mu}\beta_{\nu]} + \beta\varpi_{\mu\nu}) + \partial_{\mu}[s_{(1}^{\mu}$$

The first line gives the zero-order contribution to the entropy production which is expected to vanish so that they represent the non-dissipative contributions. This gives

$$\varepsilon + P_{\perp} = Ts, \quad P_{\parallel} = P_{\perp} + \mu_{\parallel} S_{\parallel}, \quad P_{\times} = 0.$$

The first relation is simply the Gibbs-Duhem relation, indicating that $P_{\perp}$ can be interpreted as the thermodynamic pressure. The second relation shows that the pressure along the vorticity direction differs from the thermodynamic pressure by an amount due to spin polarization $\mu_{\parallel} S_{\parallel}$. This term is very similar to the $\mathbf{M} \cdot \mathbf{B}$ term in magnetohydrodynamic constitutive relation. The third relation shows that at leading order there is no spin torque.

At $O(\partial)$ order, the requirement of a semi-positive entropy production gives that

$$\Theta_{s(1)}^{\mu\nu} = T\eta^{\mu\nu\rho\sigma}\partial_{(\rho}\beta_{\sigma)} + T\xi^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} + \beta\varpi_{\rho\sigma}),$$

$$\phi^{\mu\nu} = T\gamma^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} + \beta\varpi_{\rho\sigma}) + T\xi'^{\mu\nu\rho\sigma}\partial_{(\rho}\beta_{\sigma)},$$

$$s_{(1)}^{\mu} = \beta\varpi^{\mu\nu} n_5,$$

where $\eta^{\mu\nu\rho\sigma}$ and $\gamma^{\mu\nu\rho\sigma}$ are the usual and rotational viscous tensors representing the response of the symmetric and antisymmetric parts of the energy-momentum tenor to fluid shear and expansion and the difference between vorticity and spin potential, and $\xi^{\mu\nu\rho\sigma}$ and $\xi'^{\mu\nu\rho\sigma}$ are two cross viscous tensors. Note that the cross viscous tensors are not independent from each other but inter-related according to Onsager's reciprocal principle, $\xi'^{\mu\nu\rho\sigma}(b) = \xi^{\rho\sigma\mu\nu}(-b)$. Decomposing these tensors into irreducible structures, one obtains a number of new transport coefficients (viscosities) that characterize the response of the fluid to gradients of fluid velocity and spin potential [72]:

$$\eta^{\mu\nu\rho\sigma} = \zeta_\perp \Xi^{\mu\nu}\Xi^{\rho\sigma} + \zeta_\parallel b^\mu b^\nu b^\rho b^\sigma + \zeta_\times (b^\mu b^\nu \Xi^{\rho\sigma} + \Xi^{\mu\nu} b^\rho b^\sigma) + 2\eta_\perp (\Xi^{\mu\rho}\Xi^{\nu\sigma} + \Xi^{\mu\sigma}\Xi^{\nu\rho} - \Xi^{\mu\nu}\Xi^{\rho\sigma})$$

$$+2\eta_\parallel (\Xi^{\mu(\rho} b^{\sigma)} b^\nu + \Xi^{\nu(\rho} b^{\sigma)} b^\mu) + 2\eta_{H\perp} (\Xi^{\mu\rho}\Xi^{\nu\sigma} - \Xi^{\mu\sigma}\Xi^{\nu\rho}) + 2\eta_{H\parallel} (b^\mu \Xi^{\nu[\rho} b^{\sigma]} - b^\nu \Xi^{\mu[\rho} b^{\sigma]}),$$

$$\gamma^{\mu\nu\rho\sigma} = \gamma_\perp (b^\mu b^\nu b^{[\rho} b^{\sigma]} - b^\nu b^\mu b^{[\rho} b^{\sigma]}) + \gamma_\parallel (b^\mu \Xi^{\nu[\rho} b^{\sigma]} + b^\nu \Xi^{\mu[\rho} b^{\sigma]}) + \zeta_{H\perp} \Xi^{\mu\nu} b^{\rho\sigma}$$

Note: Figure translations are in progress. See original paper for figures.

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