Studying Thermal and Dynamical Stability of Interacting Rényi and Tsallis Holographic Dark Energy Models in LTB Inhomogeneous Universe (Postprint)

Authors: Mohamed Abdelrashied, Ayman Aly and Mustafa Selim

Date: 2025-03-11T00:00:00+00:00

Abstract

This work aims to investigate the different stability conditions of two scenarios of the inhomogeneous Lemaitre–Tolman–Bond model of the universe with holographic dark energy. We considered the Rényi and Tsallis holographic models of interacting dark energy. These holographic models are investigated using the IR cutoff that equals the Hubble horizon. Various stability conditions of these models have been investigated to understand how much these models can tell us about the recent and future epochs of the universe in comparison with the cosmological constant model, or ΛCDM model. The conditions of violating the cosmological energy conditions have been studied. The evolution of the entropy and its first and second derivatives have been calculated and plotted for these holographic models. This gives an idea of how far these models satisfy the generalized second law of thermodynamics and hence have thermodynamical stability. The dynamical stability is studied for these evolved models, which give us glimpses of the dynamical stability at different phases of its evolution. We focus on investigating the stability in recent and near future times up to z ≤ -4. Further investigation of stability has been obtained by studying the evolved sound speed squared parameter for these models, which gave us a final and decisive evaluation of the stability of these models.

Full Text

Preamble

Research in Astronomy and Astrophysics, 25:025012 (14pp), 2025 February

https://doi.org/10.1088/1674-4527/ada423
CSTR: 32081.14.RAA.ada423

Studying Thermal and Dynamical Stability of Interacting Rényi and Tsallis Holographic Dark Energy Models in LTB Inhomogeneous Universe

Mohamed Abdelrashied[1], Ayman Aly[2], and Mustafa Selim[1]
[1] Physics Department, Faculty of Science, Damietta University, New Damietta, Egypt; mohamed.a.elrashied@gmail.com
[2] Physics Department, Faculty of Science, Damanhour University, Damanhour, Egypt

Received 2024 August 27; revised 2024 December 12; accepted 2024 December 20; published 2025 February 3

Abstract

This work investigates the stability conditions of two scenarios of the inhomogeneous Lemaitre-Tolman-Bondi (LTB) universe model with holographic dark energy. We consider the Rényi and Tsallis holographic models of interacting dark energy, using the Hubble horizon as the infrared cutoff. Various stability conditions are examined to assess how well these models describe the recent and future epochs of the universe compared to the cosmological constant (ΛCDM) model. The violation of cosmological energy conditions is studied, and the evolution of entropy and its first and second derivatives is calculated and plotted for these holographic models. This analysis reveals the extent to which these models satisfy the generalized second law of thermodynamics and achieve thermodynamic stability. Dynamical stability is also examined, providing insight into the stability at different evolutionary phases. We focus on investigating stability in recent and near-future times up to z   -4. Further stability analysis is obtained by studying the evolved sound speed squared parameter for these models, providing a final and decisive evaluation of their stability.

Key words: (cosmology:) cosmological parameters -(cosmology:) dark energy -(cosmology:) large-scale structure of universe -cosmology: theory

1. Introduction

The accelerated expansion of the observable universe has been confirmed over the last two decades through observations of distant supernovae (Riess et al. 1998; Perlmutter et al. 1999) and the cosmic microwave background (CMB, Ade et al. 2000; Huang et al. 2006). These observations provide significant evidence for a cosmic component called dark energy (DE), thought to be responsible for the accelerated expansion and representing approximately 70% of the matter-energy content of the universe. While the cosmological constant model (ΛCDM) is highly consistent with observations, this scenario faces fundamental difficulties. The first is the cosmological constant problem:

the vacuum energy density calculated from quantum field theory exceeds the observed value by about 120 orders of magnitude. The second is the coincidence problem: the dark energy density is not only small but also of the same order of magnitude as the matter density today.

These challenges have motivated cosmologists to seek alternative models. Some alternatives involve scalar field models such as quintessence and k-essence, while others include modified gravity theories and higher-derivative gravity. All these proposals remain active research topics. Another alternative suggests the universe is isotropic but inhomogeneous, rather than homogeneous as in standard cosmological models. The most common inhomogeneous model is the Lemaitre-Tolman-Bondi (LTB) model, which posits that we live near the center of a huge spherically symmetric matter underdensity on gigaparsec scales. While this idea faces difficulties due to the lack of observed large-scale underdensities, inhomogeneity can instead be assumed in the dark energy sector to explain accelerated expansion without abandoning dark energy entirely (Grande & Perivolaropoulos 2011). This scenario envisions the universe as composed of regions of matter underdensity (voids or bubbles), where dark energy has constant density within each bubble, as assumed in the ΛCDM model.

Adopting this scenario has important observational consequences for understanding large-scale phenomena that suggest a preferred cosmological direction and cannot be consistently explained using the perfectly homogeneous FRW universe. These phenomena include: (i) planarity and alignment of CMB multipole moments, (ii) large-scale alignment of optical polarization data, (iii) large-scale velocity bulk flows, and (iv) profiles of cluster halos. The inhomogeneous LTB model predicts that observers near the center of the assumed isotropic inhomogeneity will naturally detect a preferred direction in the alignment of low CMB multipole moments and bulk velocity flows. Some of these observational phenomena can be explained using appropriate models of interacting dark energy with dark matter.

Previous work by Sheykhi (2011) and Jahromi et al. (2018) demonstrated that the holographic dark energy (HDE) model with the Hubble horizon can explain the present state of the universe when considering interacting dark energy scenarios. This motivated us to use HDE models with the Hubble horizon as the infrared (IR) cutoff to replace the cosmological constant within proposed matter underdensity bubbles. The holographic principle, originally proposed by ' t Hooft (1993) and developed by Susskind (1995) and Thorn (1994), connects the physics of black holes and their event horizons to a quantum theory of gravity. In cosmology, HDE models have become an active research area as alternatives to ΛCDM. In the interacting scenario between dark energy and cold dark matter (CDM), the universe' s evolution becomes non-adiabatic, requiring generalized statistical formalisms for horizon entropy. Recent work has shown that Rényi and Tsallis generalized entropies generate suitable models for the current universe, motivating our adoption of these formalisms.

Previous studies (Abd Elrashied et al. 2019; Aly et al. 2020) considered non-interacting ordinary holographic and Tsallis holographic dark energy (THDE) models in the LTB inhomogeneous universe, finding acceptable evolution of cosmological parameters compared to Type Ia supernova data. The THDE model has been investigated in various cosmological contexts and compared with ΛCDM predictions and observations. Similarly, the Rényi holographic dark energy (RHDE) model has been studied with different IR cutoffs and in various cosmological scenarios (Komatsu 2017; Moradpour et al. 2018, 2017; Chunlen & Rangdee 2020). In this work, we extend previous studies by investigating the stability conditions of interacting RHDE and THDE models in the LTB universe, analyzing their thermodynamic and dynamical behavior in recent and future times.

The paper is organized as follows. In Section 2, we present the basic equations representing the interacting RHDE and THDE models in the LTB universe, specifying particular LTB models and their free parameters based on our previous work (Aly et al. 2020; Abd Elrashied et al. 2019). Section 3 investigates the four energy conditions for both holographic models, analyzing their violation in recent and near-future times and plotting results against redshift. Section 4 examines thermodynamic stability through the evolution of the entropy function and its derivatives. Section 5 discusses dynamical stability by defining critical points of the models' evolution and analyzing phase space portraits. Section 6 studies the evolution of the sound speed squared parameter for further stability investigation. Finally, Section 7 provides concluding remarks.

2. Basic Equations

We consider the interacting scenario between holographic dark energy and CDM, where the energy conservation equation is given by (Jawad et al. 2018):

$$\dot{\rho}_D + 3H(\rho_D + P_D) = -Q,$$
$$\dot{\rho}_m + 3H\rho_m = Q,$$

where Q is the interaction term representing coupling between dark sectors. Positive Q indicates energy transfer from dark matter to dark energy, while negative Q indicates the opposite. The evolution of cosmological models with dark sector interactions has been extensively studied for various HDE models (Som & Sil 2014; Nayak 2020; Landim 2022; Saha et al. 2023; Rodriguez-Benites et al. 2024). We adopt a simple interaction function:

$$Q = 3\xi H\rho_m,$$

where ξ is the coupling parameter. Late-time constraints on interacting dark energy (Benisty et al. 2024) reveal that accepted values of ξ lie in the range $\xi \in [0.33, 1]$.

2.1. Interacting Rényi Holographic Dark Energy in the LTB Universe

Quantum aspects of gravity motivate generalized definitions of entropy that introduce non-additivity and non-extensivity. Rényi entropy is among these generalized measures that has been widely investigated in cosmological contexts. We consider the interacting RHDE model in the LTB inhomogeneous universe to study its stability conditions. The Rényi entropy is given by:

$$S_R = \frac{1}{\delta}\ln(1+\delta S_{BH}),$$

where δ is the non-additivity parameter and S_{BH} is the Bekenstein-Hawking entropy. The RHDE density is:

$$\rho_D = \frac{3c^2H^2}{8\pi G(1+\delta\ln(H^2/H_0^2))},$$

where c^2 is a dimensionless constant and the IR cutoff is the reciprocal of the Hubble horizon, $1/H$. The energy conservation equations for the interacting HDE in LTB models become (Grande & Perivolaropoulos 2011):

$$\dot{\rho}_D + H(2\rho_D + P_t + P_r) = -Q,$$
$$\dot{\rho}_m + 3H\rho_m = Q,$$

where P_t and P_r are the transverse and radial components of dark energy pressure in the LTB inhomogeneous universe. Combining these equations yields the evolution equation for the RHDE density parameter $\Omega_D = \rho_D/\rho_{crit}$:

$$\Omega_D' = -3\Omega_D(1-\Omega_D)\left(1+\frac{P_{eff}}{\rho_D}\right) + \frac{3\xi\Omega_D(1-\Omega_D)}{1+\delta\ln(H^2/H_0^2)},$$

where prime denotes derivative with respect to $\ln a$ and P_{eff} is the effective pressure.

2.2. Interacting Tsallis Holographic Dark Energy in the LTB Universe

Another statistical generalization of holographic entropy is the THDE model, previously studied for specific LTB models by Abd Elrashied et al. (2019), where acceptable results for cosmological parameters were found comparable to late-universe observations. We investigate the stability conditions of the interacting THDE model in the LTB universe and compare its features with the RHDE model. The THDE density is given by:

$$\rho_D = BH^{4-2\delta},$$

where B is a parameter written as $B = \frac{c^{2\delta-4}}{(2\pi)^{\delta-2}}$, and m_P is the Planck mass. Taking the derivative and combining with the conservation equations yields the evolution equation:

$$\Omega'_D = -3\Omega_D(1-\Omega_D)\left(1+\frac{P_{eff}}{\rho_D}\right) + 3\xi\Omega_D(1-\Omega_D)(2-\delta).$$

2.3. Models of the Scale Function Ratio λ

We consider the same LTB models studied in Abd Elrashied et al. (2019) and Aly et al. (2020), specified by the scale function $R(r,t)$. For an on-center observer arbitrarily close to the center of the assumed void region with dark energy overdensity (Grande & Perivolaropoulos 2011), the scale function takes the form:

$$R(r,t) = R_0(r)\left[1+\frac{1}{2}\left(\frac{t}{t_0}\right)^{2/3} + \frac{1}{4}\left(\frac{t}{t_0}\right)^{4/3}\right],$$

with parameter ranges $q \in [0.5, 0.65]$ and $\eta_0 = 50$ (Ribeiro 2008; Wang et al. 2000). The Hubble parameter for the inhomogeneous LTB universe is:

$$H(r,t) = H_0(r)\left[\Omega_D(r,t) + \Omega_m(r,t) + \Omega_k(r,t)\right]^{1/2},$$

where $H_0 = H(r,0)$ and can be expressed as a function of Ω_D and the scale factor ratio $\lambda = R(r,t)/R_0(r)$.

3. Energy Conditions

Energy conditions play a crucial role in explaining cosmic acceleration and singularities in cosmological models, indicating stability across different phases and being vital for deriving black hole thermodynamics laws. They express the requirement that energy density should be positive in any region of the universe, generally constraining the energy-momentum tensor $T_{\mu\nu}$.

3.1. Strong Energy Condition (SEC)

The SEC requires $T_{\mu\nu}u^\mu u^\nu \geq \frac{1}{2}T^\lambda_\lambda u^\sigma u_\sigma$ for all timelike four-vectors u^μ. For a perfect fluid, this relates density and pressure as $\rho + 3P \geq 0$. The SEC is satisfied by matter and radiation in all cosmic phases but is violated by dark energy due to its negative pressure, making violation necessary for late-time acceleration. The SEC is also violated during cosmological inflation.

3.2. Weak Energy Condition (WEC)

The WEC states that observed matter-energy density must be positive for any observer, expressed as $T_{\mu\nu}u^\mu u^\nu \geq 0$ or simply $\rho \geq 0$ and $\rho + P \geq 0$. This condition is thought to be satisfied by all cosmic components at any evolutionary phase. Figures 1 and 2 show that both RHDE and THDE models satisfy the WEC at $\delta = \{3.5, 4.5, 5.5\}$ in the LTB universe.

3.3. Dominant Energy Condition (DEC)

The DEC requires $T_{\mu\nu}u^{\mu}u^{\nu} \geq 0$ and that $T_{\mu\nu}u^{\mu}$ is always non-spacelike, guaranteeing positive energy density and causality (energy cannot flow faster than light) (Hawking & Ellis 2023). For a perfect fluid, the DEC becomes $\rho \geq |P|$. Although satisfied by the standard ΛCDM model, the DEC is violated at present in both interacting RHDE and THDE models in the inhomogeneous LTB universe, as shown in Figure 3 [Figure 3: see original paper] for $\delta = 4.5$. Both holographic models exhibit identical evolutionary behavior with redshift for all chosen δ values. In the near cosmic future, the DEC will also be violated, though at $z \approx -0.3$ it is satisfied by the interacting THDE model when $\xi = -0.209$, where energy flows from the DE sector to the dark matter sector.

3.4. Null Energy Condition (NEC)

The NEC is crucial because its violation implies violation of both WEC and SEC, and it is vital for deriving black hole thermodynamics laws. Expressed as $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ for every null vector k^{μ}, for a perfect fluid it becomes $\rho + P \geq 0$. Figure 4 [Figure 4: see original paper] plots the evolution of $\rho + P$ with redshift, showing that the NEC is violated by both interacting RHDE cases and by THDE when $\xi = 0.290$. However, for THDE with $\xi = -0.209$, the NEC is satisfied at $z \approx -0.3$. This leads to SEC violation for both holographic interacting models in recent and near-future times, as shown in Figure 5 [Figure 5: see original paper]. The WEC is violated within the same cosmic time range for the RHDE model, while for THDE it is violated except at $z \approx -0.3$ when $\xi = -0.209$.

This NEC violation implies phantom-like behavior for these interacting HDE models in the LTB inhomogeneous universe, manifesting again in subsequent stability analyses where the equation of state parameter and sound speed squared parameters have negative values in present and near-future times.

4. Thermodynamics of RHDE and THDE Models

This section analyzes the thermal evolution of holographic models in the LTB inhomogeneous case by examining the generalized second law of thermodynamics, horizon entropy evolution, and its maximization in near-future times.

4.1. Entropy

The second law of thermodynamics requires that entropy of any closed system always increase, a principle that should hold for cosmological systems and the entire universe. For systems with evolving boundaries, boundary entropy must be added to the content entropy. The Hubble horizon serves as the thermodynamic boundary of the observable universe (Davis et al. 2003; John et al. 2023), so the total entropy is the sum of horizon entropy S_H and matter-energy content entropy S_m. The generalized second law can be written as:

$$\frac{dS_{total}}{dt} = \frac{dS_H}{dt} + \frac{dS_m}{dt} \geq 0.$$

Since horizon entropy exceeds matter entropy by several orders of magnitude, total entropy can be approximated by horizon entropy alone (Egan & Lineweaver 2010). The cosmological horizon entropy follows the Bekenstein law (Bekenstein 1973):

$$S_H = \frac{k_B A}{4l_P^2},$$

where A is the horizon surface area, k_B is Boltzmann’ s constant, and l_P is the Planck length. For the locally flat observable universe, the horizon radius is $r_H = c/H$ and the horizon entropy becomes:

$$S_H = \frac{\pi c^3 k_B}{\hbar G H^2}.$$

In natural units, this simplifies to $S_H = 1/H^2$. Substituting the Hubble parameter from Equation (20) allows us to study the evolution of horizon entropy (and thus total entropy) with redshift for both holographic models.

Figures 6 [Figure 6: see original paper] and 7 [Figure 7: see original paper] plot the total entropy evolution. For the RHDE model, entropy increases with time, satisfying the generalized second law, but reaches a maximum at $z \approx -1$ then decreases, violating thermodynamics. For the THDE model, entropy increases with cosmic evolution, reaching a maximum near $z = -1$, but then increases again at later times, suggesting greater thermodynamic stability in the future.

To further understand thermodynamic stability, we investigate the first and second derivatives of entropy with respect to the scale factor. The first derivative reveals stationary points where $dS/da = 0$. The second derivative determines whether entropy has a convex maximization point satisfying $S'' < 0$. Natural systems require this convexity over long evolution times to satisfy the second law and achieve thermal stability. For example, the ΛCDM model satisfies this condition at the end of its de Sitter phase (Krishna & Mathew 2017).

Figures 8 [Figure 8: see original paper] and 9 [Figure 9: see original paper] show the entropy derivatives. For the RHDE model, stationary points occur at $z = -1$ for both $\xi = -0.209$ and $\xi = 0.290$, corresponding to entropy maxima in Figure 7. In the far future ($z < -1$), the first derivative increases then decreases to zero, indicating entropy decreases after reaching its maximum. This behavior appears in the second derivative evolution (Figures 8c and 8d), which becomes positive for $z < -2$, showing no entropy convexity in the far future. This violates the generalized second law and demonstrates thermal instability of the RHDE model in future LTB universe times.

Figure 9 [Figure 9: see original paper] shows THDE entropy stationary points and second derivative evolution. The entropy fails to satisfy the convexity condition in the long-run evolution, violating the second law of thermodynamics. Consequently, both HDE models lack bounded entropy and ultimate thermal stability in the future, consistent with the phantom-like behavior discussed previously. These thermal instabilities may cause dynamic instabilities in the future evolution of the inhomogeneous LTB universe (John et al. 2023), analyzed in detail in the next section.

5. Dynamical Stability and Phase Space Analysis

The thermal analysis reveals that both interacting holographic models contradict conventional thermodynamics in the long run. To investigate the implications for the LTB universe’ s dynamical behavior, we examine the evolution of relevant dynamical variables to understand the asymptotic limits.

We adopt the convenient dynamical variables (Mathew et al. 2022):

$$u \equiv \Omega_D, \quad v \equiv \Omega_m,$$

which satisfy $u + v = 1$ from the Hubble parameter equation. The conservation equations for interacting HDE models can be written as:

$$\dot{u} = -3Hu(1-u)(1+\omega_{eff}) + \frac{Q}{\rho_{crit}},$$
$$\dot{v} = -3Hv(1-v) - \frac{Q}{\rho_{crit}}.$$

Using these with the definition of Q yields the coupled differential equations:

$$\frac{du}{d\ln a} = -3u(1-u)(1+\omega_D) + 3\xi u(1-u),$$
$$\frac{dv}{d\ln a} = -3v(1-v) - 3\xi u(1-u).$$

These equations hold for any interacting HDE model. The equilibrium (critical) points reveal dynamical properties at different cosmic epochs and are key elements for phase space analysis. Critical points are obtained by setting $du/d\ln a = 0$ and $dv/d\ln a = 0$, yielding the set $(u_c, v_c) = \{(0,0), (1,0), (u_*, v_*)\}$, where (u_*, v_*) depends on ξ and ω_D.

The point $(0, 0)$ represents an empty universe, $(1, 0)$ represents a matter-dominated early universe state, and (u_*, v_*) is most relevant for understanding late-time evolution (Usman & Jawad 2023). For non-interacting models ($\xi = 0$), this becomes $(0, 1)$, the de Sitter state of DE domination. Figures 10 [Figure 10: see original paper] and 11 [Figure 11: see original paper] plot the evolution of ω_D for both models.

To study stability at the nontrivial critical points, we apply linear perturbation theory. Dynamical variables are expressed around their critical values as $u = u_c + \delta u$ and $v = v_c + \delta v$, where δu and δv are infinitesimal deviations. Linearizing the coupled differential equations yields a matrix equation for the perturbations, with the Jacobian matrix:

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix},$$

where f and g represent the right-hand sides of the evolution equations for u and v. Diagonalizing this matrix gives eigenvalues (λ_1, λ_2) for each critical point. Stability is determined by these eigenvalues: both negative indicates a stable sink, both positive indicates an unstable source, and opposite signs indicate a saddle point requiring further analysis.

5.1. Dynamical Stability of RHDE Model

For the first nontrivial critical point $(1, 0)$, the eigenvalues are $(3, 3(\xi - \omega_D))$. In the RHDE model, ω_D is always negative in the redshift range of interest (Figure 10). For $\xi = -0.209$, the eigenvalues are always positive, and for $\xi = 0.290$, they are also positive for all ω_D values. Thus, $(1, 0)$ represents a dynamically unstable source point in phase space, shown as the black dot in Figure 12 [Figure 12: see original paper] from which all trajectories diverge, consistent with its representation of the early matter-dominated state.

The second critical point (u_*, v_*) has eigenvalues that can be analyzed using mathematical software. In future times when ω_D converges to -1.3, the critical point degenerates into two possibilities depending on ξ. For $\xi = -0.209$, the point $(0.16, 0.84)$ has eigenvalues $(-0.79, 4.01)$, representing a saddle point. For $\xi = 0.290$, the point $(-0.22, 1.22)$ has eigenvalues $(-0.57, 3.571)$, also a saddle point. These saddle points appear as red dots in Figure 12. While stable points should attract trajectories, here trajectories converge to a line passing through these points rather than the points themselves. However, in an infinitesimally small region around them, phase space trajectories appear to converge, suggesting local stability.

5.2. Dynamical Stability of THDE Model

Repeating the analysis for the THDE model, Figure 11 [Figure 11: see original paper] shows the evolution of ω_D. The parameter passes through -1 (the de

Sitter asymptotic state of non-interacting models and ΛCDM), allowing investigation of dynamical stability around this value.

The first critical point $(1, 0)$ again indicates an unstable matter-dominated state, appearing as the source point (black dot) in Figure 13 [Figure 13: see original paper] from which all trajectories diverge. The second critical point (u_*, v_*) shows different behavior: for $\xi = -0.209$, ω_D saturates at -1 in the future $(z > -2)$, giving the point $(0.26, 0.74)$ with eigenvalues $(-0.43, 3.43)$—a saddle point. In Figure 13(a), phase space trajectories locally diverge around this red dot, indicating at least local dynamical instability.

For $\xi = 0.290$, ω_D converges to -1 at $z \approx -4$, giving the critical point $(0.29, 1.29)$ with complex eigenvalues $(1.5 + 1.06i, 1.5 - 1.06i)$. Since these complex conjugates have positive real parts, the phase portrait shows a growing counterclockwise spiral (Figure 13b). The universe passes through $\omega_D = -1$ but rapidly evolves to more negative values, indicating dynamical instability.

The explicit dynamical instability of the LTB universe with THDE is consistent with previous results showing unbounded entropy and thermodynamic instability, as well as phantom-like behavior violating all energy conditions. In contrast, the RHDE scenario shows at least local dynamical stability at one critical point despite being thermodynamically unstable. To resolve this paradox, we investigate the sound speed squared parameter in the next section.

6. Sound Speed Squared Parameter

Cosmological model stability can be investigated through the evolutionary behavior of the sound speed squared parameter V_s^2. Real sound speed corresponds to regular density perturbation propagation modes and system stability, while imaginary sound speed indicates irregular modes and classical instability. Physical V_s^2 values must lie between 0 and 1; values outside this range indicate tachyonic or superluminal propagation instabilities (Vagnozzi et al. 2020).

The sound speed squared parameter is:

$$V_s^2 = \frac{dP_D}{d\rho_D} = \omega_D + \rho_D \frac{d\omega_D}{d\rho_D},$$

where $P_D = \rho_D \omega_D$ is the dark energy pressure. Using ρ_D from Equations (5) and (14), we plot V_s^2 for both holographic models in Figures 14 [Figure 14: see original paper] and 15 [Figure 15: see original paper].

These figures show that V_s^2 has only negative values in recent and far-future times, indicating instabilities at the perturbation level with irregular superluminal propagation modes where density perturbations grow exponentially (Myung 2007; Kim et al. 2008; Sharma & Dubey 2022). This result is expected for THDE, given its instability across all analysis methods. However, for RHDE, despite local dynamical stability at the critical point, the sound speed analysis confirms that this model is also unstable in the LTB inhomogeneous universe scenario.

7. Conclusion

This work investigates interacting Rényi and Tsallis holographic models—statistical generalizations of black hole entropy introduced by Bekenstein and Hawking—in the generalized LTB inhomogeneous universe (Abd Elrashied et al. 2019; Aly et al. 2020; Grande & Perivolaropoulos 2011). Our primary objective is analyzing stability conditions of the inhomogeneous LTB universe where dark energy is represented by interacting RHDE and THDE models, focusing on thermodynamic evolution and dynamical behavior in recent and future times.

We first examined gravitational energy conditions, finding that both holographic models violate nearly all conditions. SEC violation (Figure 5) is expected for dark energy systems with negative pressure. NEC violation (Figure 4) suggests the dark energy density evolves unstably. WEC violation indicates phantom-like behavior (Pasqua et al. 2014), as both models have equation of state parameters $\omega_D < -1$ (Figures 10 and 11).

Thermodynamic stability analysis shows both models are unstable in the far future, though they appear temporarily stable in the near future. THDE entropy always increases with time (Figure 7), while RHDE entropy increases then rapidly decreases at $z \approx -1$ (Figure 6). Neither model satisfies the convexity condition $S'' < 0$ in the future for either coupling constant ξ, though both show temporary stability with $S'' > 0$ near $z \approx -1.5$ (Figures 8 and 9), eventually losing stability as S'' becomes positive.

Dynamical stability analysis through phase space and critical points (Figures 12 and 13) confirms THDE is explicitly unstable at all critical points, while RHDE shows local stability at one critical point. However, investigation of the sound speed squared parameter reveals negative V_s^2 values for both models (Figures 14 and 15), confirming phantom-like behavior and instability in recent and near-future times.

Consequently, LTB universe scenarios with interacting RHDE and THDE models are not promising candidates for explaining recent universe behavior or providing reasonable future predictions, due to their explicit phantom-like behavior and thermodynamic instability. We conclude that these holographic dark energy models in the inhomogeneous LTB framework fail to provide a stable, viable alternative to the standard ΛCDM cosmology.

References

Abd Elrashied, M., Aly, A. A., & Selim, M. M. 2019, JApA, 40, 15
Ade, P., Balbi, A., Bock, J., et al. 2000, ApJ, 545, L1
Aditya, Y., Mandal, S., Sahoo, P., & Reddy, D. 2019, EPJC, 79, 1020
Aly, A. A., Elrashied, M. A., & Selim, M. M. 2020, IJMPD, 29, 2050023

Bekenstein, J. D. 1973, PhRvD, 7, 2333
Benisty, D., Pan, S., Staicova, D., Di Valentino, E., & Nunes, R. C. 2024, A&A, 688, A156
Chunlen, S., & Rangdee, P. 2020, arXiv:2008.13730
Davis, T. M., Davies, P., & Lineweaver, C. H. 2003, CQGra, 20, 2753
Egan, C. A., & Lineweaver, C. H. 2010, ApJ, 710, 1825
Enqvist, K. 2008, GReGr, 40, 451
Grande, J., & Perivolaropoulos, L. 2011, PhRvD, 84, 023514
Hawking, S. W., & Ellis, G. F. 2023, The Large Scale Structure Of Space-time (Cambridge: Cambridge Univ. Press)
Hooft, G. 1993, arXiv:gr-qc/9310026
Huang, Z.-Y., Wang, B., Abdalla, E., & Su, R.-K. 2006, JCAP, 2006, 013
Jahromi, A. S., Moosavi, S., Moradpour, H., et al. 2018, PhLB, 780, 21
Jawad, A., Bamba, K., Younas, M., Qummer, S., & Rani, S. 2018, Symm, 10, 635
John, R., Sarath, N., & Mathew, T. K. 2023, EPJC, 83, 697
Kim, K. Y., Lee, H. W., & Myung, Y. S. 2008, PhLB, 660, 118
Komatsu, N. 2017, EPJC, 77, 229
Krishna, P., & Mathew, T. K. 2017, PhRvD, 96, 063513
Landim, R. G. 2022, PhRvD, 106, 043527
Li, X., Wang, S., Huang, Q., Zhang, X., & Li, M. 2012, SCPMA, 55, 1330
Mathew, T. K., et al. 2022, arXiv:2204.12097
Moradpour, H., Bonilla, A., Abreu, E. M., & Neto, J. A. 2017, PhRvD, 96, 123504
Moradpour, H., Moosavi, S., Lobo, I., et al. 2018, EPJC, 78, 829
Myung, Y. S. 2007, PhLB, 652, 223
Nayak, B. 2020, GrCo, 26, 273
Pasqua, A., Chattopadhyay, S., Khurshudyan, M., & Aly, A. A. 2014, IJTP, 53, 2988
Perlmutter, S., Aldering, G., Goldhaber, G., et al. 1999, ApJ, 517, 565
Ribeiro, M. B. 2008, arXiv:0807.0866
Riess, A. G., Filippenko, A. V., Challis, P., et al. 1998, AJ, 116, 1009
Rodriguez-Benites, C., Gonzalez-Espinoza, M., Otalora, G., & Alva-Morales, M. 2024, EPJC, 84, 276
Saha, A., Chanda, A., Dey, S., Ghose, S., & Paul, B. 2023, MPLA, 38, 2350100
Sharma, U. K., & Dubey, V. C. 2022, IJGMM, 19, 2250010
Shekh, S. 2023, InJPh, 97, 983
Sheykhi, A. 2011, PhRvD, 84, 107302
Som, S., & Sil, A. 2014, Ap&SS, 352, 867
Susskind, L. 1995, JMP, 36, 6377
Thorn, C. B. 1994, arXiv:hep-th/9405069
Usman, M., & Jawad, A. 2023, IJMPD, 32, 2350100
Vagnozzi, S., Visinelli, L., Mena, O., & Mota, D. F. 2020, MNRAS, 493, 1139
Wang, B., Abdalla, E., & Osada, T. 2000, PhRvL, 85, 5507

Note: Figure translations are in progress. See original paper for figures.