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Authors: Zhipeng LIANG, Chengzhi LIU, Jinsong PING, Makram Ibrahim,
Chengzhi LIU

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Abstract

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Full Text

Preamble

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Instant Determination of Earth's Polar Motion by Lunar Laser Ranging Common View

Zhipeng LIANG¹², Chengzhi LIU¹³, Jinsong PING²⁴, Makram Ibrahim⁵

¹ Changchun Observatory of National Astronomical Observatories, Chinese Academy of Sciences, Changchun, 130117, China; lcz@cho.ac.cn

² Key Laboratory of Space Object and Debris Observation, Purple Mountain Observatory, Chinese Academy of Sciences, Nanjing, 210008, China

³ National Research Institute of Astronomy and Geophysics, Helwan, 11421, Cairo, Egypt

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Abstract

The Lunar Laser Ranging (LLR) technology has been used to determine Earth Rotation Parameters (ERP) since the 1970s. Conventional determination methods incorporate state-of-the-art models and LLR data spread over decades. The most recent studies achieve an accuracy of tens of microseconds for UT1 and several milliarcseconds (mas) for polar motion coordinates (PMC). This paper determines instant PMC using the lunar laser ranging common view method (LLRCV). Through quantitative analysis of derivatives of lunar range residual with respect to ERP variation, we formulated a linear equation to solve for instant PMC with simultaneous range measurements. The precision of the solution was evaluated from the measurement precision through analysis of the linear equation. We identified 184 LLRCV events in the LLR data during 2012–2022 to solve for PMC. To assess the solution errors, the results of our method were compared to the IERS C04 data. The results showed that 29.3% of the solution errors were less than 150 mas, and 21.7% of the solutions were more accurate than the prediction data. This method has the potential to address short-term and transient geophysical changes. Nonetheless, its dependence on long-term geodetic data means it is not a substitute for conventional techniques.

Key words: reference systems –Moon –methods: data analysis –planets and satellites: general

1 INTRODUCTION

The Lunar Laser Ranging (LLR) technique measures the round-trip time between a ground laser ranging station and a lunar retro-reflector array. Over the past decade, several LLR observation stations have produced precise data. These stations include Grasse in France (GRSM), Wettzell in Germany (WETL), and Matera in Italy (MATM) in Europe, as well as Apache Point (APOL) and McDonald Observatory (MDOL) in the United States of America in North America [?, ?].

The precision of LLR measurement has improved to the centimeter level [?, ?].

In recent years, two laser ranging stations in China reported acquisition of LLR data [?, ?].

LLR data have been used to determine components of the Earth rotation parameter (ERP) since the 1970s [?, ?]. Early research typically determined UT0 and variation of latitude from one night of lunar ranging data, achieving an accuracy of 0.7 milliseconds or 10 milliarcseconds (mas) [?]. Modern LLR data analysis, which incorporates up-to-date reference frame data and state-of-the-art models for all aspects of the Earth and Moon, is able to determine all components of ERP [?, ?, ?, ?]. Typical modern research on ERP determination selects observation nights with 10–15 normal points [?, ?, ?]. The most recent studies achieve accuracies of several mas for polar motion coordinates (PMC) [?, ?].

The ERP components fitted using traditional methods represent the long-term evolution of Earth rotation rather than the real-time status of Earth rotation. Major geodynamic events, such as earthquakes, can cause rapid changes in Earth' s rotation parameters [?, ?]. Therefore, it is of interest to determine instant ERP components to provide data for time-sensitive geodetic studies.

The concept of lunar laser ranging common view (LLRCV) dates back to as early as the 1980s [?]. Leick proposed the concept of simultaneous observation from pairs of co-observing stations and investigated the determination of ERP components using simulated lunar laser range differencing data [?]. It was claimed that the method could provide ERP components with measurement accuracy on Earth' s surface, which was about 30 cm for LLR in the early 1980s [?]. This accuracy corresponds to 10 mas for PMC, or 0.6 ms for UT1 [?].

Another form of lunar range common view was proposed by Müller et al., involving placement of an optical transponder on the lunar surface to illuminate the entire facing hemisphere of Earth, thus enabling common-view ranging at standard SLR stations and yielding a 1000-fold improvement in detected photons compared to traditional LLR [?].

The challenge in the common view approach is the poor geometry. Given a single session of LLRCV measurements, it is not possible to determine all three ERP components at that epoch because LLR range measurements are invariant to rotation about the axis from Earth' s center to the reflector. As a result, the rotation angle cannot be determined from LLR data. The LLRCV method can only determine two independent linear combinations of the three ERP components, and if the reflector is near the equatorial plane, the polar motion components become unsolvable.

To overcome this rank deficiency, more measurements are needed at different lunar hour angles, as suggested in early studies [?, ?]. However, taking more measurements at multiple epochs involves the dynamics of both Earth and Moon, which may introduce complexities in modeling. This paper focuses on instant determination with simultaneous common-view measurements at a single epoch.

In this paper, we successfully determined PMC using LLRCV data sessions, with

each session acquired simultaneously from a pair of stations. The method of determination with LLRCV is presented in Section 2, where the LLR measurement model and ERP variation model are formulated for common view scenarios, and the scheme for instant determination and assessment is described. In Section 3, we introduce the measurement data and ERP data used in this paper. The results are shown and discussed in Section 4, and a broader discussion is given in Section 5.

2 INSTANT ERP DETERMINATION

The model for lunar laser ranging has been presented in numerous works [?, ?, ?, ?, ?, ?].

The lunar ranging measurement conducted from a ground station to a lunar reflector can be modeled as:

$$\tau = s_{12}/c + s_{23}/c + \tau_{gr} + \tau_{tropo} + \Delta\tau,$$

where τ represents the time of flight measured at the station; the distances s_{12} and s_{23} represent the vacuum light path from station to reflector (outbound) and reflector to station (inbound), respectively, accounted in barycentric inertial coordinates; τ_{gr} represents the gravitational delay effect on the propagating pulse; τ_{tropo} represents the tropospheric refraction delay during light propagation; and the correction term $\Delta\tau$ accounts for the difference between TT and TDB clock rates.

These terms are quantified using either Terrestrial Time (TT), Barycentric Dynamical Time (TDB), or International Atomic Time (TAI). The round-trip time interval τ is measured in an Earth observatory with an atomic time scale, TAI. Laser flight paths s_{12} and s_{23} are calculated with TDB-compatible barycentric ephemeris, therefore the corresponding time of flight is calculated in TDB. The general relativity correction τ_{gr} is also calculated in TDB. The tropospheric delay τ_{tropo} is modeled in TT. The correction term $\Delta\tau$ is calculated in TT. During the lunar ranging round-trip time of less than 3 seconds, $\Delta\tau$ is at the 10^{-12} second level, which is negligible for this paper.

Light paths s_{12} and s_{23} are Euclidean distances. The term s_{12} is from the station at transmit time t_1 to the reflector at hit time t_2 , and s_{23} is from the reflector at t_2 to the station at receive time t_3 , as follows:

$$s_{12} = |\vec{s}_{12}| = |-\vec{r}_{ES}(t_1) - \vec{r}_{BE}(t_1) + \vec{r}_{BM}(t_2) + \vec{r}_{MA}(t_2)|$$

$$s_{23} = |\vec{s}_{23}| = |-\vec{r}_{MA}(t_2) - \vec{r}_{BM}(t_2) + \vec{r}_{BE}(t_3) + \vec{r}_{ES}(t_3)|,$$

where \vec{r}_{ES} , \vec{r}_{BE} , \vec{r}_{BM} , \vec{r}_{MA} are barycentric vectors with B for barycenter of the solar system, E for Earth center, S for ground station, M for Moon center, and A for retro-reflector array, respectively.

The barycentric vectors \vec{r}_{BE} and \vec{r}_{BM} are computed with the TDB-compatible DE430 planetary ephemeris. The selenocentric vector \vec{r}_{MA} is computed with selenocentric coordinates and lunar libration angles, both provided by DE430 ephemeris [?, ?]. The geocentric station vector \vec{r}_{ES} is calculated using the ITRF2020 coordinates and transformed into a geocentric celestial reference frame.

Algorithms for reference system conversion between celestial and terrestrial frames, terrestrial tide offsets, tropospheric refraction delay of light propagation, relativistic delay of light propagation in gravitational fields, and relativistic motion deformation of Earth and Moon are adopted from IERS Conventions 2010 [?, ?]. Time scale conversion algorithms are adopted from SOFA [?], particularly the PyMsOfa software package [?]. Other astronomical calculations are conducted with AstroPy [?].

Comparing the LLR model against historical LLR data from 2012 to 2022, we obtain range residuals, as shown in Subsection 3.3.

2.1 Determination of Earth Rotation Parameters with LLRCV

The Earth rotation parameters (ERP) represent Euler angles of Earth rotation in the geocentric celestial reference system (GCRS). Conventionally, the Earth rotation parameter contains one set of three numbers. The rotation axis is represented by terrestrial intermediate pole (TIP) coordinates, or polar motion coordinates (PMC), denoted as x_p , y_p . The Earth rotation angle (ERA) ψ_{ERA} in the celestial reference frame, concerning the equinox, is represented by UT1. The conversion formula between ERA (mod 2π) and UT1 (mod 24h) is $\psi_{ERA} = 15 \times 1.00273781191135448 \times \text{UT1}$, with UT1 in seconds and ERA in arc-seconds.

The three ERP components y_p , x_p , and ψ_{ERA} are related to frame rotations about the X, Y, and Z axes, respectively.

We assume that the vector of ERP variation solely causes the variation of LLR measurement. For simultaneously conducted range measurements, we form a range residual vector $\vec{q} = (\delta\rho_A, \delta\rho_B, \dots)^T$. At a given epoch, assuming two or more stations (A, B, ...) observe simultaneously, we form the linear equation $M\vec{p} = \vec{q}$, where the design matrix M holds partial derivatives of range measurements with respect to ERP components. The calculation of design matrix M involves only predicted ERP data (see Subsection 3.1).

In this paper, we focus on determination of the polar motion components δx_p and δy_p using two simultaneous measurements from a pair of common view stations. In such a problem, the design matrix M is shaped 2×2 , and the parameter vector $\vec{p} = (\delta y_p, \delta x_p)^T$. The x_p and y_p predictions were calculated with a linear model as described in Subsection 3.1. The UT1 data are set to

reference values. After solving for \vec{p} , we adjust the prediction with \vec{p} to form an estimate of x_p and y_p . The estimated ERP components were then compared with reference data for assessment.

2.2 Variation Analysis of Lunar Range Measurements

To investigate the variation $\delta\rho$ of lunar range measurement ρ with respect to ERP components y_p , x_p , and ψ_{ERA} , or partial derivatives, we start with the geometrical relation:

$$\delta\rho = H(\theta, \phi)\delta\vec{r}_{GCRS},$$

where $H(\theta, \phi) = (-\cos\theta\cos\phi, -\sin\theta\cos\phi, -\sin\phi)$ is the transpose of the unit vector along the line of sight from the reflector to the station. Angles θ and ϕ are the celestial right ascension and declination of the reflector as seen from the station. The variation vector $\delta\vec{r}_{GCRS}$ is the variation of the station's geocentric inertial vector caused by ERP variation.

According to the IERS Conventions 2010 [?], the transformation from an international terrestrial reference system (ITRS) to a geocentric celestial reference system (GCRS) is:

$$\vec{r}_{GCRS} = Q(t)R_3(-s')(R_3(-\psi_{ERA})R_2(x_p)R_1(y_p))\vec{r}_{ITRS},$$

where the bracketed term $R_3(-\psi_{ERA})R_2(x_p)R_1(y_p)$ is where the ERPs take effect. The term $Q(t)$ is the transformation matrix arising from the motion of the celestial pole in the celestial reference system (i.e., precession and nutation), and R_1 , R_2 , R_3 are matrices for coordinate frame rotation about axes X, Y, and Z. We adapted the equation to its current form using commutativity between the terrestrial intermediate origin (TIO) locator $R_3(-s')$ and Earth rotation $R_3(-\psi_{ERA})$.

The variations of astronomical angles θ and ϕ due to ERP changes are negligible. Applying partial derivative rules to the above equation and letting the ERP components be predicted values $(y_p^0, x_p^0, \psi_{ERA}^0)$, we have partial derivatives as:

$$\frac{\partial\rho}{\partial y_p} = M_{HQR}R_3(-\psi_{ERA}^0)R_2(x_p^0)R_1'(y_p^0) \cdot \vec{r}_{ITRS}$$

$$\frac{\partial\rho}{\partial x_p} = M_{HQR}R_3(-\psi_{ERA}^0)R_2'(x_p^0)R_1(y_p^0) \cdot \vec{r}_{ITRS}$$

$$\frac{\partial\rho}{\partial\psi_{ERA}} = M_{HQR}R_3'(-\psi_{ERA}^0)R_2(x_p^0)R_1(y_p^0) \cdot \vec{r}_{ITRS},$$

where $M_{HQR} = H(\theta, \phi)Q(t)R_3(-s')$. The matrices R'_k ($k = 1, 2, 3$) denote the derivatives of each rotation matrix with respect to the rotation angle.

2.3 Precision of Solution

The precision of the solution is used for quality assessment before comparison with reference data. It mainly depends on the stiffness of the design matrix M .

According to the perturbation theorem of linear algebra, for any linear system $Ax = b$ and its perturbed form $(A + \delta A)(x + \delta x) = b + \delta b$, we have the following inequality [?]:

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\text{cond}(A)}{1 - c} \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} \right),$$

where $\text{cond}(A) = \|A\|\|A^{-1}\|$ is the condition number of matrix A , and $c = \|A^{-1}\delta A\|$ should satisfy $c < 1$. The latter requirement means the perturbation should be minor. In this work, we assume the perturbations in the coefficient matrix are relatively small, i.e., $\|\delta A\|/\|A\| \approx 0$, letting them be zero in the formula. The inequality becomes:

$$\frac{\|\delta x\|}{\|x\|} \leq \text{cond}(A) \frac{\|\delta b\|}{\|b\|}.$$

Therefore, we neglect c and $\|\delta A\|/\|A\|$.

For the problem of PMC determination, we substitute the design matrix M , the PMC correction vector \vec{p} , and range residual vector \vec{q} into the formula. If we put the range measurement precision (NP RMS) vector $\delta\vec{q}$ in place of δb and move it to the right side, the resulting inequality:

$$\|\delta\vec{p}\| \leq \|\delta\vec{q}\| \cdot \text{cond}(M) \frac{\|\vec{p}\|}{\|\vec{q}\|}$$

puts an upper bound on the norm of the solution error $\delta\vec{p}$. We call this upper bound the precision of the solution. It represents how much the solution would be perturbed given the measurement precision as variation in the range measurement, and it also represents the resolving power of the instant determination method.

3 DATA AND METHODS

3.1 ERP Data

The reference ERP dataset was adopted from the IERS EOP 20 C04 time series, retrieved from the IERS website [?]. The precision of polar motion coordinates

in the reference data was better than 0.1 mas in the selected time span.

Predicted polar motion data were needed as initial input for the solution. The IERS (2010) mean pole model [?] was adopted to calculate predictions. After epoch 2010.0, the model is formulated as:

$$\bar{x}_p(t) = 23.513 + 7.6141 \times (t - t_0)$$

$$\bar{y}_p(t) = 358.891 - 0.6287 \times (t - t_0),$$

where t and t_0 are Besselian epochs of current time and year 2000, respectively, with \bar{x}_p and \bar{y}_p units in mas. The deviations between prediction and reference are no more than 200 mas for both components in the selected period. See Figure 1 [Figure 1: see original paper].

3.2 Terrestrial and Lunar Coordinates

The coordinates of ground observatories are mostly adopted from the International Terrestrial Reference Frame (ITRF) 2020 [?]. Since the APOL station is not included in ITRF2020, we adopted its coordinates from [?]. Small adjustments to station coordinates were made to optimize LLR range residuals. The adjusted station coordinates are listed in Table 1 .

The reflector coordinates were initially adopted from the DE430 document and adjusted using range residuals, which are listed in Table 2 .

3.3 Lunar Laser Ranging

We retrieved 10,513 LLR normal point (NP) data from the EUROLAS Data Center (EDC)³, covering the period from May 2, 2012 to December 7, 2022, provided by stations APOL, GRSM, MATM, MDOL, and WETL, as shown in Figure 2 Figure 2: see original paper and (b).

In the LLR data, the Consolidated Ranging Data (CRD) format allows recording of normal point root mean square (RMS) values, which are considered equivalent to 1-sigma. Figure 3 [Figure 3: see original paper] shows the measurement precision of the existing LLR stations during 2012-2022.

We filtered 0.8% (88 of 10,513) outlier NPs with range residuals larger than 0.4 meter. In the adopted 99.2% (10,425 of 10,513) NP data, the range measurement residuals against the model are 0.05 meters.

3.4 Common View Events

We searched the historical LLR normal point data for common view events, where two normal points from different stations have a temporal separation of less than one hour. The epoch of a common view event was set to the midpoint

of the pair of NP epochs. From the LLR data, we found 184 common view events, as listed in Table 3 . The GRSM station formed common-view events with MATM 126 times and with WETL 55 times. The APOL station formed common view events with MDOL 3 times. Due to geographical separation, the European stations did not form common view events with stations in America.

Out of all common view events, the Apollo-15 reflector was the most frequently tracked, with a total of 119 events. The other reflectors tracked included 30 events from Luna-17, 17 events from Apollo-11, 13 events from Luna-21, and 5 events from Apollo-14. The common view events are dense during certain times when the lunar phase and weather were suitable for LLR observation by adjacent stations.

4 DATA ANALYSIS

For each common view event, we solved for the corresponding polar motion components x_p and y_p . The solution represents the instant value of these parameters at the epoch of the common view event. We analyzed both the precision and accuracy of the solutions.

Comparing the polar motion solutions with reference polar motion data generated the solution errors. By fitting solution error versus solution precision, we found an empirical uncertainty trend curve $y = 0.1x$ to depict the quantitative relation between them, as shown in Figure 4 [Figure 4: see original paper]. The uncertainty trend curve allows estimation of the solution error before comparison with reference values. If the goal is to filter out errors exceeding 1.5 arcseconds, we filter those precision values exceeding 22.5 arcseconds instead.

Out of 184 solutions, we filtered out 9.2% (17 common view solutions) with solution precision larger than 22.5 arcseconds. The remaining 90.8% (167 solutions) yielded for x_p a mean of -54 mas, with a standard error of 394 mas or RMS of 398 mas, and for y_p a mean of -14 mas, with a standard error of 392 mas or RMS of 392 mas. There were 29.3% (54 solutions) inside the circle with a 150 mas radius, as shown in Figure 6 Figure 6: see original paper. There were 21.7% (40 solutions) that exhibited less error than the predicted data, i.e., improved over the prediction.

The solutions from GRSM+MATM spread wider in the x_p component, while the solutions from GRSM+WETL spread wider in the y_p component, as shown in Figure 6(a).

5 DISCUSSION AND CONCLUSION

This paper established a method to solve Earth' s PMC using instant LLRCV measurements. We compared the solutions with the IERS EOP 20 C04 time

series. The x_p and y_p solution errors had means of -54 mas and -14 mas, with RMS errors of 398 mas and 392 mas, respectively. There is no similar result in the literature, but if we compare our results with the best results of the mainstream method (WRMS values of 1.30 mas and 1.63 mas) [?], the error is about 300 and 240 times larger, respectively. In 21.7% of cases, our solutions outperformed the linear polar motion model predictions. Overall, this result indicates that instant determination of polar motion coordinates is possible.

There remain unsolved problems in this approach. The quality of solution differs between station pairs. Systematic trends exist in the solutions of both European station pairs, as shown in Figure 6(a). This may be due to the near distance between members of a station pair that can affect the precision of the solution. For the GRSM+MATM pair, station coordinate error may be a significant error source, since the MATM portion of range residuals had decimeter-level bias in this paper, as shown in Figure 2(a). The quality of solution could be improved using a better LLR model, more precise station coordinates, and more precise PMC predictions.

The rapid solution has the potential to address short-term and transient geophysical changes. However, the method relies on long-term geodetic products and therefore cannot replace conventional solutions.

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