The general propagator for S-wave threshold states (Postprint)

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Abstract

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Full Text

Preamble

The General Propagator for S-Wave Threshold States

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Abstract

We demonstrate that the propagator derived from an Effective Field Theory (EFT) that incorporates Weinberg' s compositeness theorem provides a more general formula for describing S-wave near-threshold states. By fitting the line-shape using this propagator, we can extract the Z factor for these states and elucidate their structures.

Keywords: Tetraquarks, Exotic Hadron Resonances.

Introduction

The Belle Collaboration first observed the $X(3872)$ state [?]. Since then, an increasing number of nonconventional (exotic) states have been discovered [?]. Despite renewed interest in the spectroscopy of these exotic (or XYZ) states over the past two decades [?, ?], their nature remains uncertain.

Effective field theory is widely used in hadron physics and also for studying exotic states. For example, in Ref.~[?], an effective field theory called XEFT was proposed to study $X(3872)$. In XEFT, the $X(3872)$ is assumed to be a weakly bound molecule of a charm meson pair. Recently, LHCb reported the observation of the decay mode $X(3872) \to \psi(2S)\gamma$, and measured the ratio of this partial width to that of $X(3872) \to J/\psi\gamma$ to be $R_{\psi\gamma} = \Gamma[X(3872) \to \psi(2S)\gamma]/\Gamma[X(3872) \to J/\psi\gamma] = 1.67 \pm 0.21 \pm 0.12 \pm 0.04$ [?]. The value $R_{\psi\gamma}$ is sensitive to the nature of $X(3872)$. The measured ratio makes the interpretation of $X(3872)$ as a pure $D\bar{D}^*$ molecule questionable and indicates a sizeable compact component in $X(3872)$ [?]. In view of this, it is necessary to consider the compact component in effective field theory.

In Ref.~[?], an effective field theory that incorporates Weinberg' s compositeness theorem and considers the $X(3872)$ as a compact state that couples strongly to $D\bar{D}^*$ was proposed. To distinguish such an effective field theory from XEFT, we will call the effective field theory proposed in Ref.~[?] CEFT throughout this paper. The CEFT has been further used to study the structure of the Z_b states [?] and, very recently, to study the structure of $X(3872)$ [?, ?].

The aim of this paper is to discuss some properties of CEFT, particularly the propagator function in CEFT. Typically, three amplitudes are used to describe these near-threshold states: Breit-Wigner, Flatté, and low-energy scattering amplitude. Let us consider a two-body channel, denoted as DD, with a threshold M_{th} and a near-threshold state X with mass M and width Γ. In the center-of-mass frame, the Breit-Wigner propagator is defined as $P^\mu = (E_0, \mathbf{P})$, where E_0 and M can be parameterized as $E_0 = M_{\text{th}} + E$ and $M = M_{\text{th}} - B$, respectively. Here E is the energy relative to the threshold, and B is the binding energy (we call it binding energy in the sense that it is defined relative to the threshold). We choose the convention that $B > 0$ if the state lies below threshold. Near the threshold, terms suppressed by $O(p^2)$ can be neglected. The amplitude can then be written as $f(E) = D_{\text{BW}}(E)$, where $D_{\text{BW}}(E) = E + B + i\Gamma/2$.

The Flatté formula is a general model used to parameterize the resonant structure near a hadron-hadron threshold [?]. The Flatté amplitude can be written as $f(E) = 1/D_{\text{FL}}(E)$, with $D_{\text{FL}}(E)$ defined as $D_{\text{FL}}(E) = E - E_f - \sqrt{-2\mu E - i\epsilon} + i\Gamma_f/2$. Here, E_f is the Flatté energy parameter related to the mass, μ is the reduced mass of the two-body state DD, g_1 characterizes the coupling between X and DD, and Γ_f accounts for decay modes other than DD. As we will illustrate later, the Flatté amplitude assumes the existence of a compact object.

Finally, the low-energy amplitude is written as $f(E) = 1/D_{\text{LE}}(E)$, with $D_{\text{LE}}(E)$

defined as $D_{\rm LE}(E) = -1/a + \sqrt{-2\mu E - i\epsilon}$, where a is the scattering length. We call it low-energy amplitude; one should note to distinguish it from the low-energy expansion amplitude $f = a + \frac{1}{2}rp^2 - ip$, where r is the effective range.

In this paper, we investigate the relationship between the propagator derived from CEFT and the three amplitudes commonly used. We find that the CEFT-derived propagator serves as a more general form, making it particularly useful for characterizing the lineshape of S-wave near-threshold states.

Revisiting Weinberg' s Compositeness Theorem

To proceed, we will first revisit Weinberg's method [**?**, **?**]. The total Hamiltonian H of interest can be divided into a free part H_0 and an interaction part V:

$$H = H_0 + V.$$

Specifically, H_0 can be the Hamiltonian in quark models, where only the interactions between the quarks are considered. The eigenstates of the free part H_0 include continuum states $|\alpha\rangle$ (for example, the two-body state DD) and possible discrete bare elementary states $|B\rangle$ near the DD threshold. $|B\rangle$ is a compact state, possibly a tetraquark state or a quarkonium. V describes the interaction between $|B\rangle$ and DD. We then have:

$$H_0|\alpha\rangle = E(\alpha)|\alpha\rangle, \quad \langle\beta|\alpha\rangle = \delta(\beta - \alpha), \tag{1}$$

$$H_0|B\rangle = -B_0|B\rangle, \quad \langle\alpha|B\rangle = 0, \quad \langle B|B\rangle = 1, \tag{2}$$

$$\langle\alpha|V|B\rangle = g_0, \tag{3}$$

where the energies are defined relative to the DD threshold. We adopt the convention that $B_0 > 0$ if the mass of $|B\rangle$ is below the DD threshold. We parameterize the matrix element $\langle\alpha|V|B\rangle$ as a constant g_0, since the coupling between $|B\rangle$ and $|\alpha\rangle$ is an S-wave coupling. Higher-order corrections in effective field theory can be considered by replacing g_0 with $g_0 + g_{1p}^2$, where g_1 is a constant and p is the momentum of D in the center-of-mass frame of the DD system. Near the threshold, p is a small scale, allowing us to neglect the term g_{1p}^2 in the leading-order result.

A physical bound state $|X\rangle$ is a normalized eigenstate of the total Hamiltonian H, with:

$$H|X\rangle = -B|X\rangle, \quad \langle X|X\rangle = 1.$$

Without loss of generality, the wave function of $|X\rangle$ can be written as:

$$|X\rangle = Z|B\rangle + \int d\alpha\, C_\alpha|\alpha\rangle,$$

where Z is the probability of finding $|B\rangle$ in the bound state $|X\rangle$. We refer to $|X\rangle$ as a bound state in the sense that it includes continuum states in its wave function, and its mass is below the threshold, or equivalently, $B > 0$. If $Z = 0$, then $|X\rangle$ contains only the continuum states in its wave function, and we will refer to such a state as a pure molecular state to distinguish it from a bound state $|X\rangle$ with $Z \neq 0$.

The coefficient C_α can be explored as:

$$C_\alpha = \langle\alpha|X\rangle = \frac{\langle\alpha|H - H_0|X\rangle}{E(\alpha) + B} = -\frac{\langle\alpha|V|X\rangle}{E(\alpha) + B}.$$

Using the wave function of $|X\rangle$, the matrix element $\langle\alpha|V|X\rangle$ in the above can be expressed as:

$$\langle\alpha|V|X\rangle = Z\langle\alpha|V|B\rangle + \int d\beta\, C_\beta \langle\alpha|V|\beta\rangle,$$

noting that $\langle\alpha|V|B\rangle = g_0$, as defined in Eq.~(5). If we parameterize the matrix element $\langle\alpha|V|X\rangle$ as Weinberg did, $\langle\alpha|V|X\rangle = g$, where g is a constant, and by setting $\langle\alpha|V|\beta\rangle = 0$, we immediately obtain the relation $g = Zg_0$, or equivalently $g^2 = Zg_0^2$.

Actually, the above relation is one of the matching relations given in Ref.~[**?**]. There, the relation is obtained through the matching between a non-relativistic effective field theory and Weinberg' s compositeness theorem. This relation can be reproduced by setting $\langle\alpha|V|\beta\rangle = 0$, which simply reflects the fact that the direct four-body $DD - DD$ contact interaction terms and also the $D - D$ interactions mediated by t-channel meson exchange are not explicitly included in the effective field theory of Ref.~[**?**]. This is because their contributions are of order $O(p^0)$ (where p is the momentum of D) or higher, which is smaller than the leading-order $D - D$ scattering amplitude (which is $O(p^{-1})$) in CEFT. The contribution of the matrix element $\langle\beta|V|\alpha\rangle$ can be taken into account in the next-to-leading-order results of CEFT by including the four-body $DD - DD$ contact interaction terms and also the t-channel meson exchange interactions.

With $\langle\alpha|V|X\rangle = g$, we can write Eq.~(8) as:

$$C_\alpha = -\frac{g}{E(\alpha) + B}.$$

Using the normalization condition of $|X\rangle$, i.e., $\int d\alpha\, |C_\alpha|^2 = 1 - Z$, with $\int d\alpha = \int \frac{d^3p}{(2\pi)^3} = \frac{4\pi p^2 dp}{(2\pi)^3} = \sqrt{2\mu E}\, dE$ where $E \equiv p^2/2\mu$, we then obtain:

$$\frac{g^2}{2\pi\sqrt{2\mu B}} = 1 - Z.$$

With Eq.~(13), Weinberg derived the well-known relations between Z, B and the effective range expansion parameters, i.e., the scattering length a and effective range r, as:

$$a = \frac{2(1-Z)/(2-Z)}{\sqrt{2\mu B}} + O(m_\pi^{-1}), \tag{4}$$

$$r = -\frac{Z/(1-Z)}{\sqrt{2\mu B}} + O(m_\pi^{-1}). \tag{5}$$

It is worth mentioning that in the case of $X(3872)$, Ref.~[**?**] finds that the corrections to the low-energy expansion parameters from pions are extremely small (see also Ref.~[**?**]). Thus, it is a very good approximation to neglect the correction terms $O(m_\pi^{-1})$ in Eq.~(14) and Eq.~(15). For the $X(3872)$, the charged DD channel may also need to be taken into account besides the neutral channel. We will denote the charged continuum state as $|\alpha_c\rangle$, while $|\alpha\rangle$ denotes the neutral continuum DD states. We then have:

$$H_0|\alpha_c\rangle = (E(\alpha_c)+\delta)|\alpha_c\rangle, \quad \langle\alpha|\alpha_c\rangle = 0, \quad \langle B|\alpha_c\rangle = 0, \tag{6}$$

$$\langle\beta_c|\alpha_c\rangle = \delta(\beta_c - \alpha_c), \quad \langle\alpha_c|V|B\rangle = g_{0c}, \tag{7}$$

where δ is the mass splitting between the charged channel and neutral channel. δ is included because $E(\alpha_c)$ is defined relative to the threshold of the charged channel, while the eigenvalue of H_0 is defined relative to the threshold of the neutral channel. We use a new constant g_{0c}, which may differ from g_0 defined in Eq.~(5), to parameterize the matrix element $\langle\alpha_c|V|B\rangle$, as the isospin violation is sizeable for the $X(3872)$. With the charged channel taken into account, the wave function of the $X(3872)$ can be written as:

$$|X(3872)\rangle = Z|B\rangle + \int d\alpha\, C_\alpha|\alpha\rangle + \int d\alpha_c\, C_{\alpha_c}|\alpha_c\rangle.$$

Along the same line to treat C_α, we can have:

$$C_{\alpha_c} = \langle\alpha_c|X(3872)\rangle = \frac{\langle\alpha_c|H-H_0|X(3872)\rangle}{(E(\alpha_c)+\delta)+B} = -\frac{\langle\alpha_c|V|X(3872)\rangle}{E(\alpha_c)+\delta+B},$$

where we have used the wave function of the $X(3872)$ in Eq.~(17), $\langle\alpha_c|V|B\rangle = g_{0c}$ in Eq.~(16), and $\langle\alpha_c|V|\alpha\rangle = \langle\alpha_c|V|\beta_c\rangle = 0$ in the last step. Thus, the matrix element $\langle\alpha_c|V|X(3872)\rangle$ can be expressed as:

$$\langle\alpha_c|V|X(3872)\rangle = Z\langle\alpha_c|V|B\rangle = Zg_{0c} \equiv g_c.$$

Again, the matrix elements $\langle\alpha_c|V|\alpha\rangle$ and $\langle\alpha_c|V|\beta_c\rangle$ are set to be zero, as their effects will only appear in the next-to-leading-order results. Now the wave function normalization of $X(3872)$ reads:

$$\int d\alpha\, |C_\alpha|^2 + \int d\alpha_c\, |C_{\alpha_c}|^2 = 1.$$

After performing the phase space integral, we have:

$$\frac{g^2}{2\pi\sqrt{2\mu B}} + \frac{g_c^2}{2\pi\sqrt{2\mu_c(B+\delta)}} = 1 - Z,$$

where μ is the reduced mass of neutral DD and μ_c is the reduced mass of charged DD. Eq.~(21) is the extension of Eq.~(13) to further include another continuum spectrum. Note that if we further assume $g^2 = g_c^2$, we can obtain the expression for g^2 with Eq.~(21).

We will further discuss $X(3872)$ later, and now we will turn to study the matrix element $\langle B|X\rangle$. We have:

$$\langle B|X\rangle = \langle B|\frac{H-H_0}{-B+B_0}|X\rangle = \frac{\langle B|V|X\rangle}{-B+B_0},$$

or equivalently:

$$\langle B|V|X\rangle = Z(B_0 - B).$$

On the other hand, $\langle B|V|X\rangle$ can also be estimated by using the wave function of X, i.e., Eq.~(7), as:

$$\langle B|V|X\rangle = \int d\alpha\, C_\alpha \langle B|V|\alpha\rangle.$$

Note that we set $\langle B|V|B\rangle = 0$, since at leading order V describes the interactions between $|B\rangle$ and DD. The above integral has ultraviolet divergence and can be handled in momentum space with dimensional regularization. Using the definition $\langle \alpha|V|B\rangle = g_0$ in Eq.~(5), and relations in Eq.~(10), (11) and (12), we have:

$$\begin{aligned}
\langle B|V|X\rangle &= \int d\alpha\, C_\alpha \langle B|V|\alpha\rangle && (8)\\
&= \int \frac{d^{3p}}{(2\pi)^3}\frac{-g}{p^2/(2\mu)+B} g_0 && (9)\\
&= -gg_0\nu^{4-D}\int \frac{d^{D-1}p}{(2\pi)^{D-1}}\frac{1}{p^2/(2\mu)+B} && (10)\\
&= -gg_0\nu^{4-D}\frac{(2\mu)^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}\int_0^\infty \frac{p^{D-2}dp}{p^2+2\mu B} && (11)\\
&= -gg_0\nu^{4-D}\frac{(2\mu)^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}\frac{\Gamma\left(\frac{3-D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{2(2\mu B)^{\frac{3-D}{2}}} && (12)\\
&= -\frac{gg_0}{2\pi}\sqrt{2\mu B}, && (13)
\end{aligned}$$

where ν is the additional scale introduced in dimensional regularization. Note that there is no pole at $D = 4$. After taking the $D \to 4$ limit and using Eq.~(13), we obtain:

$$\langle B|V|X\rangle = \frac{g^2}{2\pi}\sqrt{2\mu B} = 2(1-Z)B.$$

Combining Eq.~(23) and Eq.~(26), we have:

$$B_0 = \frac{2-Z}{Z}B,$$

which is exactly the other matching relation that was obtained in Ref.~[**?**]. Here we reproduced it using a different approach as a cross-check. With the new approach, we can straightforwardly extend the above relation to $X(3872)$, where the charged channel may also be included. Using the wave function for the $X(3872)$ in Eq.~(17), C_{α_c} in Eq.~(18), and the matrix element $\langle \alpha_c|V|B\rangle$ in Eq.~(19), we obtain:

$$\langle B|V|X(3872)\rangle = \int d\alpha\, C_\alpha \langle B|V|\alpha\rangle + \int d\alpha_c\, C_{\alpha_c} \langle B|V|\alpha_c\rangle \tag{14}$$

$$= \frac{g^2}{2\pi}\sqrt{2\mu B} + \frac{g_c^2}{2\pi}\sqrt{2\mu_c(B+\delta)}. \tag{15}$$

The integrals in the above are treated in the same way as in Eq.~(25). Combining Eq.~(28) and Eq.~(23) (note that Eq.~(23) now becomes $\langle B|V|X(3872)\rangle = Z(B_0 - B)$), we can then generalize Eq.~(27) to:

$$B_0 = \frac{1}{Z}\left[\frac{g^2}{2\pi}\sqrt{2\mu B} + \frac{g_c^2}{2\pi}\sqrt{2\mu_c(B+\delta)} + ZB\right].$$

We will utilize this relation later to obtain the propagator for the $X(3872)$.

Before ending this section, we would like to discuss the interpretation of Z in Weinberg' s compositeness theorem. As previously illustrated, Z represents the probability of finding $|B\rangle$ in a physical bound state $|X\rangle$. However, $|B\rangle$ is not a physical state but rather an eigenstate of H_0, with the interaction V turned off. When the interaction V turns on, $|B\rangle$ will couple with the continuum state DD, resulting in $0 < Z < 1$. Consequently, a large Z, for example close to one, indicates that the compact state couples weakly with the continuum state. Conversely, a small but non-vanishing Z indicates that the compact state couples strongly with the continuum state. On the other hand, $Z = 0$ implies that the physical state $|X\rangle$ is dynamically generated by the DD interaction without a compact state. It is worth noting that this point has also been addressed in Ref.~[**?**].

The General Propagator for S-Wave Near-Threshold States

Weinberg' s compositeness theorem has been incorporated into a non-relativistic effective field theory (CEFT) in Ref.~[**?**]. There, the propagator for the S-wave near-threshold state is written as:

$$G_X(E) = \frac{1}{D_{\rm EFT}(E)}, \tag{16}$$

$$D_{\rm EFT}(E) = E + B + \tilde{\Sigma}'(E) + i\Gamma/2, \tag{17}$$

$$\tilde{\Sigma}' = -g^2\left[\sqrt{-2\mu E - i\epsilon} + (E-B)\right]. \tag{18}$$

The term $i\Gamma/2$ was directly inserted into the denominator of the above propagator, thereby rendering the physical interpretation of Γ ambiguous [**?**]. Now, we will re-derive the propagator and find that Γ can be naturally incorporated in field theory. This methodology will elucidate that Γ has a distinct physical interpretation, namely, it is the renormalized width resulting from the non-DD decays of X.

Consider a bare state $|B\rangle$ with bare mass $-B_0$, width Γ_0, and coupling g_0 to the two-particle state DD. If $|B\rangle$ is near the two-particle threshold, the propagator of this bare state can be written as $1/(E+B_0+i\Gamma_0/2)$, as shown in Eq.~(1). The full propagator can be obtained by summing the Feynman diagrams in Fig.~1 [Figure 1: see original paper]. Near threshold, the momenta of the particles are non-relativistic. The loop integral in Fig.~1 can be calculated straightforwardly with the minimal subtraction (MS) scheme; the result can be written as [**?**]:

$$I_{\text{MS}} \equiv \nu^{4-D}\int \frac{d^D\ell}{(2\pi)^D}\frac{i}{[\ell^0-\vec{\ell}^2/(2m_1)+i\epsilon][E-\ell^0-\vec{\ell}^2/(2m_2)+i\epsilon]} \tag{19}$$

$$= i\nu^{4-D}\int \frac{d^{D-1}\ell}{(2\pi)^{D-1}}\frac{1}{E-\vec{\ell}^2/(2\mu)+i\epsilon} \tag{20}$$

$$= -i\nu^{4-D}\frac{\Gamma\left(\frac{3-D}{2}\right)}{(4\pi)^{\frac{D-1}{2}}}\sqrt{-2\mu E-i\epsilon}. \tag{21}$$

Here m_1 and m_2 are the masses of the DD states. Note that the above loop integral is linearly divergent and has a pole at $D=3$, but it has no pole at $D=4$. In the minimal subtraction scheme, counterterms are added to subtract the pole at $D=4$. Since the result has no pole at $D=4$, no counterterm is needed in the MS scheme. It is easy to consider the widths of DD states by replacing the propagator in Eq.~(33) with $i/[\ell^0-\vec{\ell}^2/(2m)+i\Gamma/2]$. The result can be obtained by simply replacing the ϵ in the final expression of Eq.~(33) with $\mu(\Gamma_1+\Gamma_2)$, where Γ_1 and Γ_2 are the decay widths of the DD states (a similar result is also given in Ref.~[**?**] when X is a pure molecular state).

It is worth mentioning that due to the absence of log divergences, the additional renormalization scale does not appear in the final result of the loop integral. This feature makes it possible to incorporate the relation Eq.~(13), which is obtained from quantum mechanics, into the EFT (as in Ref.~[**?**]). Because if the result of the loop integral depends on the renormalization scale, the coupling g^2 will also depend on the renormalization scale through the renormalization group equation. A similar argument can be applied to the field renormalization constant Z, thus Z does not depend on the renormalization scale and can have a physical interpretation, i.e., the probability.

After summing the contributions from all the diagrams, the full propagator reads:

$$G_X(E) = \frac{1}{E+B_0-g_0^2\sqrt{-2\mu E-i\epsilon}+i\Gamma_0/2}.$$

Actually, the above amplitude is just the Flatté amplitude. The separation of the self-energy into two terms $i\Gamma_0/2$ and $-g_0^2\sqrt{-2\mu E - i\epsilon}$ may not make too much sense if the mass of the bare state is well above the threshold, because in this case $-g_0^2\sqrt{-2\mu E - i\epsilon}$ is pure imaginary as the $i\Gamma_0/2$ term in the concerned energy region. But such separation is necessary if the mass of the bare state is below the threshold, as $-g_0^2\sqrt{-2\mu E - i\epsilon}$ is real for $E < 0$ [**?**].

To incorporate Weinberg' s compositeness theorem, one should use the matching relations between the bare parameters (g_0, B_0) and the renormalized parameters (g, B) as given in Ref.~[**?**] (and they are also reproduced in the previous section):

$$g_0^2 = g^2/Z, \tag{22}$$

$$B_0 = \frac{2-Z}{Z}B, \tag{23}$$

where g^2 is defined in Eq.~(13) and B is the binding energy. Note that g_0, B_0 become infinite in the limit $Z \to 0$. The matching relation Eq.~(35) is obtained in the limit $\Gamma_0 \to 0$ in Ref.~[**?**]. We assume that this relation still holds for a non-vanishing Γ_0, as the long-distance physics (the coupling between X and DD channel) should not be sensitive to the short-distance physics (the coupling between X and non-DD channels). Notice that B and B_0 have the same sign for $0 \leq Z \leq 1$, so the bare state is below the threshold if X is below the threshold, and vice versa.

Using Eq.~(35), Eq.~(36) and Eq.~(13), the full propagator can be rewritten as:

$$G_X(E) = \frac{Z}{ZE + (2-Z)B - g^2\sqrt{-2\mu E - i\epsilon} + iZ\Gamma_0/2} \tag{24}$$

$$= \frac{Z}{E + B - g^2\sqrt{-2\mu E - i\epsilon} - (1-Z)(E-B) + iZ\Gamma_0/2} \tag{25}$$

$$= \frac{Z}{E + B - \frac{g^2}{2\pi\sqrt{2\mu B}}\sqrt{-2\mu E - i\epsilon} - (1-Z)(E-B) + iZ\Gamma_0/2} \tag{26}$$

$$= \frac{Z}{E + B - \frac{g^2}{4\pi B}\sqrt{-2\mu E - i\epsilon} - (1-Z)(E-B) + iZ\Gamma_0/2}, \tag{27}$$

which is precisely the form of $G_X(E)$ as defined in Eq.~(30). By comparing Eq.~(30) and Eq.~(37), we can find that Γ satisfies:

$$\Gamma = Z\Gamma_0.$$

Thus, we have demonstrated that the width Γ can be naturally incorporated into CEFT. This width can be interpreted as the renormalized width, which comes from the non-DD decay of the elementary state. In this manner, one can naturally extend Weinberg' s compositeness theorem to resonances. Actually, what we have addressed above is the coupled-channel effect between the DD decay channel and the non-DD channels. The parameters B_0 and Γ_0 may have physical interpretation; in other words, they may be calculated from some specific tetraquark model. For $0 < Z < 1$, the coupled-channel effect will shift the mass of the state near to the threshold, i.e., $B < B_0$ in Eq.~(36).

We now come to discuss the relation between the propagator $G_X(E)$ and the Breit-Wigner, Flatté, and low-energy amplitudes. Firstly, for $Z = 1$ ($g^2 = 0$), one can find that $D_{\mathrm{EFT}}(E)$ is just $D_{\mathrm{BW}}(E)$ defined in the near-threshold form of Breit-Wigner, i.e., Eq.~(1). This can be easily understood as by using Breit-Wigner for the propagator of the bare state in Fig.~1, we assume the interactions V have been turned off. If the interactions have been turned on ($g^2 \neq 0$), the propagator of the compact state should be replaced with the bubble sum shown in Fig.~1. In other words, the Breit-Wigner amplitude assumes the below-threshold compact state does not couple with the continuum state. Note that this conclusion cannot be extended to a compact state with mass above the threshold, because for a state above the threshold the relation $g^2 = 2\pi\sqrt{2\mu B}(1-Z)$ does not hold anymore, and the coupling between the compact state and the continuum states can be absorbed into the width term Γ of the Breit-Wigner. That means, if one uses $G_X(E)$ to fit some near-threshold structure, and the best fit gives $B < 0$ (above the threshold), then the meaningful fitting result should have $Z \to 1$ at the same time. In such a case one can claim that the near-threshold structure is a compact state with mass above the threshold. Therefore $G_X(E)$ can also be used to identify the structure of threshold states with mass above the threshold.

Secondly, for $Z = 0$, by using the relation in Eq.~(14), $D_{\mathrm{EFT}}(E)$ is equal to $-1/D_{\mathrm{LE}}(E)$, with $D_{\mathrm{LE}}(E)$ defined in the low-energy amplitude, i.e., Eq.~(3) (in other words, the low-energy amplitude can only be used for a pure molecule). This point has already been mentioned in Ref.~[**?**]; we address it again for the completion of our discussions.

Finally, for $Z \neq 1$ and $Z \neq 0$, B_0 and g_0 are well-defined. The form of the denominator of Eq.~(34), i.e., the propagator in its bare form, is precisely $D_{\mathrm{FL}}(E)$ as defined in Eq.~(2). The matching between them yields the relations:

$$E_f = -B_0, \quad g_1 = g_0^2\mu/\pi, \quad \Gamma_f = \Gamma_0.$$

By the way, the first two have been given in Ref.~[**?**] and the last one is new. It can be observed that as Z approaches 0, the Flatté parameters (E_f, g_1) become infinite, due to (B_0, g_0) becoming infinite. Conversely, as Z approaches 1, g_1 becomes 0, coinciding with the vanishing of g^2. Therefore, the Flatté parameterization can only be successfully applied in the case $0 < Z < 1$. Hence the Flatté amplitude assumes the existence of a compact object.

Thus far, we have demonstrated that the Breit-Wigner amplitude assumes a compact state lies below the threshold and weakly couples to the continuum state ($Z \to 1$) or a compact state lies above threshold without knowing the coupling strength between this state and the continuum state. The low-energy amplitude proposed in Ref.~[**?**] assumes the below-threshold state to be a pure molecule. The Flatté amplitude assumes a below-threshold compact state with its coupling strength with the continuum state determined by the value of Z. Therefore, all three amplitudes have assumptions. In contrast, the propagator $G_X(E)$ derived from CEFT includes the factor Z explicitly, thus it makes no assumptions on the structure of the near-threshold states and provides a more general formula to describe S-wave threshold states.

We now come to further discuss the $X(3872)$, which may also consider the charged DD channel. With Fig.~1, the full propagator, which includes the charged DD channel, can be written as:

$$G_{X(3872)} = \frac{1}{E + B_0 - g_0^2\sqrt{-2\mu E - i\epsilon} - g_{0c}^2\sqrt{-2\mu_c(E-\delta) - i\epsilon} + i\Gamma_0/2}.$$

Using the matching relations $g_0^2 = g^2/Z$ (i.e., Eq.~(10)), $g_{0c}^2 = g_c^2/Z$ (i.e., Eq.~(19)), and Eq.~(29), we can have:

$$G_{X(3872)} = \frac{Z}{E + B + \tilde{\Sigma}'_{X(3872)} + i\Gamma/2}, \tag{28}$$

where
$$\tilde{\Sigma}'_{X(3872)} = \frac{g^2}{2\pi\sqrt{2\mu B}}\left(\sqrt{2\mu B} - \sqrt{-2\mu E - i\epsilon}\right) + \frac{g_c^2}{2\pi\sqrt{2\mu_c(B+\delta)}}\left(\sqrt{2\mu_c(B+\delta)} - \sqrt{-2\mu_c(E-\delta) - i\epsilon}\right) \tag{29}$$

$$= -\frac{g^2}{4\pi B}\left(\sqrt{-2\mu E - i\epsilon} + (E-B)\right) - \frac{g_c^2}{4\pi(B+\delta)}\left(\sqrt{-2\mu_c(E-\delta) - i\epsilon} + (E-B-2\delta)\right) \tag{30}$$

$$= -g^2\left[\frac{\sqrt{-2\mu E - i\epsilon} + (E-B)}{4\pi B} + \frac{\sqrt{-2\mu_c(E-\delta) - i\epsilon} + (E-B-2\delta)}{4\pi(B+\delta)}\right], \tag{31}$$

and $\Gamma = Z\Gamma_0$. Note that g^2 is determined by Eq.~(21) instead of Eq.~(13), and we replace the factor $(1-Z)$ using Eq.~(21) in the last step of Eq.~(40). One can also reobtain Eq.~(29) by noting that $E = -B$ is the pole of Eq.~(38) in the limit $\Gamma_0 \to 0$.

It is worth mentioning that the Flatté amplitude used by LHCb [**?**] for the $X(3872)$ (which was first proposed in Ref.~[**?**]) is essentially the same as Eq.~(38). As previously emphasized, using the Flatté amplitude involves an assumption, as the Flatté parameters B_0, g_0, g_{0c} become infinite in the limit $Z \to 0$. However, with the renormalized form, i.e., Eq.~(39), to parameterize the propagator of the $X(3872)$, no assumption is made about the structure of $X(3872)$ since all quantities are finite for $0 \leq Z \leq 1$. Another important point is that by using the coupled-channel Flatté amplitude to extract Z, LHCb further uses the compositeness relations Eq.~(14) and Eq.~(15) (or equivalently, the relation between Z and the asymmetry of the pole location in momentum space [**?**]). However, these compositeness relations are only valid in the single-channel scattering. Therefore, such analysis relies on extrapolation to the single-channel case, and it is unclear whether this extrapolation is valid. In contrast, one can directly determine the value of Z by using Eq.~(39) to fit the lineshape without any extrapolation, since Eq.~(39) is derived from a coupled-channel analysis.

Finally, we would like to discuss implications of our results in phenomenological studies. Considering a scattering process: initial states $\rightarrow X + \text{others} \rightarrow$ final states, the scattering amplitude can be generally written as $i\mathcal{M} = A_{PX} G_{XB_{DX}}$, where A_{PX} is the production vertex of X, G_X is the propagator of X defined in Eq.~(30), and B_{DX} is the decay vertex of X. The production vertex of X can be further separated into a short-distance part A^S_{PX} and a long-distance part A^L_{PX}, i.e., $A_{PX} = A^S_{PX} + A^L_{PX}$. In the short-distance part A^S_{PX}, X is produced directly, while in the long-distance part A^L_{PX}, X is produced through DD rescattering. Specifically, if X is a tetraquark (for $X(3872)$, it can also be $c\bar{c}$), A^S_{PX} can be estimated by the tetraquark model, i.e., no additional factors of Z are needed to be included in A^S_{PX}. This can be immediately realized by replacing the propagator of X in the Feynman diagram of the scattering process with the bubble sum in Fig.~1 and noticing that in the production vertex a bare state $|B\rangle$ was first produced and then renormalized by the interaction. Similarly, if we consider the non-DD decay modes of X, B_{DX} can be estimated by the tetraquark model. Note that in this way, we assume the non-DD decay modes come from the direct decay of the tetraquark, instead of through the final state interaction of DD rescattering. This assumption is justified for the radiative decay of $X(3872)$ [**?**]. For the $X(3872)$, it is also found that $A^S_{PX} \gg A^L_{PX}$ [**?**, **?**], so the scattering amplitude is dominated by $i\mathcal{M} = A^S_{PX} G_X B_{DX}$. Therefore, if we consider the non-DD decay modes, A^S_{PX} and B_{DX} can be estimated by the tetraquark models and the factor Z only manifests itself in G_X of the scattering amplitude. Otherwise, if we consider the DD decay modes, the dominated scattering amplitude is written as $i\mathcal{M} = A^S_{PX} G_X i g_0$. It is then interesting to constrain the parameters of the tetraquark models by fitting the cross section data with the above amplitudes. A non-vanishing Z ensures that A^S_{PX} and the non-DD decay vertex B_{DX} can be estimated with the tetraquark model. This feature ensures that the large production cross section of $X(3872)$ at CDF [**?**] and the large $R_{\psi\gamma}$ measured by LHCb [**?**] and also the strong coupling of $X(3872)$ with $D\bar{D}^*$ can be well described in a compact state model for $X(3872)$. This picture is consistent with the result of Ref.~[**?**], which found that while the nucleon-nucleon interaction can form a pure molecular state, the meson-meson interaction cannot form such a pure molecular state in large N_c QCD. This discrepancy arises because the nucleon mass and the meson mass have different N_c scaling.

Summary

We have re-derived the propagator for near-threshold states in CEFT. In this way, Γ can be naturally included in the propagator and can be interpreted as the renormalized non-DD decay width of the compact state. We further discuss the relations between the propagator in CEFT and the Breit-Wigner, Flatté, and low-energy amplitudes. We find that the propagator in CEFT is the more general formula to describe S-wave near-threshold states. By using this propagator to fit the lineshape, one can extract Z for these states and eventually clarify their structure. A study in this direction is presented in Ref.~[**?**]. Finally, we discuss the implication of our results in phenomenological studies.

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Note: Figure translations are in progress. See original paper for figures.