Postprint: Application of Unified Conic Orbital Elements to Orbit Prediction of Cislunar Space Targets

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Abstract

When using traditional elliptical orbital elements to describe the motion of a spacecraft, it becomes difficult to continue the calculation if its orbit changes from elliptical to hyperbolic. To solve this problem, improvements are made based on classical orbital elements such as elliptical orbital elements, and a set of orbital elements applicable to any conic section is used to integrate the equations of motion. This set of elements is applicable for any eccentricity e 0 and inclination 0 i<180{ }, with a singularity occurring only when i=180{ }. The basic conversion formulas and perturbation equations of motion are presented. Subsequently, orbit prediction is performed for cislunar space targets, and compared with calculation results using position and velocity. The results show that this element set has sufficient accuracy, and offers computational efficiency advantages when the element variations are not particularly severe. For issues that may arise in practical applications of the elements, such as hyperbolic orbits with large e or the i=180{ } singularity, solutions including conditional state variable switching and fixed integration step size are proposed, and their applicability is evaluated.

Full Text

Preamble

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The Application of Unified Orbital Elements for Conic Orbits in Cislunar Orbit Prediction

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Abstract

When describing spacecraft motion using traditional elliptical orbital elements, calculations become difficult if the orbit transitions from elliptical to hyperbolic. To address this issue, we improve upon classical orbital elements by employing a unified set applicable to any conic section for integrating the equations of motion. This element set works for any eccentricity $e > 0$ and inclination $0 \leq i < 180^\circ$, with singularities occurring only at $i = 180^\circ$. We present the fundamental conversion formulas and perturbation equations. Subsequently, we perform orbit predictions for cislunar space objects and compare the results with those obtained using position-velocity coordinates. The findings demonstrate that the unified elements achieve sufficient accuracy and offer computational efficiency advantages when the element variations are not excessively dramatic. For practical applications involving hyperbolic orbits with large eccentricity or the $i = 180^\circ$ singularity, we propose solutions such as conditional state variable switching and fixed integration step sizes, and evaluate their applicability.

Keywords: celestial mechanics, Earth, Moon, orbit prediction, methods: numerical

1. Introduction

With advancing space technology, human exploration and utilization of cislunar space have deepened, drawing attention to the safety implications of spacecraft and small celestial bodies operating in this region. Keplerian elements $(a, e, i, \Omega, \omega, M)$ are the most commonly used orbital elements for describing such orbits, representing semi-major axis, eccentricity, inclination, longitude of ascending node, argument of perigee, and mean anomaly, respectively. However, Keplerian elements have limitations: they only apply to elliptical motion and exhibit singularities at eccentricity $e = 0$ and inclinations $i = 0^\circ, 180^\circ$.

To resolve these issues, various alternative element sets have been proposed. Reference [1] introduces nonsingular elements, with the second-type nonsingular elements expressed as:

$$a, \quad h = \sin i \cos \Omega, \quad k = \sin i \sin \Omega, \quad \xi = e \cos(\omega + \Omega), \quad \eta = e \sin(\omega + \Omega), \quad \lambda = M + \omega + \Omega.$$

These elements apply to eccentricities $0 \leq e < 1$ and inclinations $0 \leq i < 180^\circ$, though singularities appear at $i = 180^\circ$. Some formulations replace $\sin i$ with $\tan i$ in the expressions for h and k, introducing a $1/\cos i$ factor in related problems (such as the partial derivative matrix of elements [2]), which adds a singularity at $i = 90^\circ$.

References [3-4] present element sets that eliminate the $i = 180^\circ$ singularity:

For direct orbits:

$$h = e\sin(\Omega{+}\omega), \quad k = e\cos(\Omega{+}\omega), \quad \phi = \tan\frac{i}{2}\cos\Omega, \quad \psi = \tan\frac{i}{2}\sin\Omega, \quad \lambda = M{+}\Omega{+}\omega.$$

For retrograde orbits:

$$h_r = e\sin(\Omega{-}\omega), \quad k_r = e\cos(\Omega{-}\omega), \quad \phi_r = \cot\frac{i}{2}\cos\Omega, \quad \psi_r = \cot\frac{i}{2}\sin\Omega, \quad \lambda_r = M{-}\Omega{+}\omega.$$

These elements apply to $0 \le e < 1$ and $0 \le i \le 180^\circ$. Although they distinguish between direct and retrograde orbits, the boundary need not be strictly at $i = 90^\circ$; it suffices to avoid $i = 180^\circ$ for direct elements or $i = 0^\circ$ for retrograde elements during numerical integration.

For eccentricities $e > 1$, References [5-6] introduce elements that use the semi-latus rectum p instead of semi-major axis a, making them applicable to all eccentricities:

$$p, \quad h = \tan\frac{i}{2}\cos\Omega, \quad k = \tan\frac{i}{2}\sin\Omega, \quad \xi = e\cos(\omega{+}\Omega), \quad \eta = e\sin(\omega{+}\Omega), \quad L = \frac{h^*}{\mu}\sqrt{\frac{p}{\mu}}f,$$

where f is the true anomaly. These elements exhibit singularities only at $i = 180^\circ$. Combining these approaches—replacing a with p and λ with L in equations (2) and (3)—eliminates the $i = 180^\circ$ singularity while remaining applicable to all eccentricities. This element set has found widespread application in spacecraft trajectory design [7], space situational awareness [8], and space debris removal [9].

Additionally, some scholars have used vector combinations [10]:

$$\vec{a} = e\vec{P}, \quad \vec{b} = \frac{e}{p}\vec{Q}, \quad \vec{c} = p\vec{R}, \quad \vec{g} = \frac{e}{p}\vec{Q},$$

where $\vec{P}$ and $\vec{R}$ are unit vectors in the direction of periapsis and orbit normal, respectively, and $\vec{Q} = \vec{R} \times \vec{P}$. For instance, Reference [11] used $\vec{a}$ and $\vec{b}$ in studying asteroid Icarus (1566), while Reference [12] introduced Milankovich elements using the eccentricity vector $\vec{e}$ and angular momentum vector $\vec{H}$ (equivalent to $\vec{a}$ and $\vec{c}$ above). Selecting two such vector combinations yields only five independent variables, requiring an additional variable such as mean anomaly, true longitude, or time of periapsis passage.

Many current cislunar spacecraft have complex orbits. In the Earth-centered mean equator frame, Earth-Moon libration point orbits appear as large elliptical orbits [13-14], while some transfer orbits or deep space probes follow hyperbolic trajectories. In March 2022, rocket debris temporarily designated WE0913A by the Minor Planet Center (MPC) impacted the lunar surface. This object operated in cislunar space on a geocentric highly elliptical orbit for an extended period, transitioning to a geocentric hyperbolic orbit in its final orbital phase. This paper employs a unified set of conic orbit elements to analyze their applicability in describing the orbital motion of such cislunar objects with complex and significantly varying orbit types. Furthermore, with cislunar space development entering a new "boom period"(e.g., China's Chang'e Program[1], NASA's Artemis Program[2], and the Sino-Russian International Lunar Research Station[3]), cislunar objects will increase substantially. These will join other near-Earth objects as targets for cataloging and early warning systems. However, cislunar objects operate in relatively complex dynamical environments, including highly elliptical orbits with eccentricities exceeding 0.9, hyperbolic orbits, and objects like WE0913A that transition from elliptical to hyperbolic orbits during operation. These complex orbits limit the application of elliptical elements. While Cartesian coordinates can handle all orbit types, their computational efficiency is inferior to orbital elements for the vast majority of near-circular Earth orbits that dominate cataloged objects [15]. Therefore, this paper explores the applicability of unified conic orbit elements for cislunar objects.

The following sections present the basic conversion formulas and perturbation equations for these elements. Through extrapolation of various cislunar orbit types and comparison with Cartesian coordinate calculations, we conduct numerical verification and analysis of accuracy and computational efficiency.

3. Conversion Between Unified Elements and Position-Velocity and Perturbation Equations

To calculate spacecraft position, we present the conversion method between unified elements (p, ξ, η, h, k, u) and position-velocity vectors $(\vec{r}, \dot{\vec{r}})$ in the epoch mean equator Earth-centered frame. In spacecraft motion studies, a spacecraft-centered coordinate system is typically established with x, y, z axes representing the radial direction relative to the central body, the transverse direction in the instantaneous orbital plane, and the normal to the orbital plane, respectively. The components of perturbing acceleration along these three directions are denoted as S, T, W (shown in Figure 1 [Figure 1: see original paper]).

where h^* is the orbital angular momentum, $\mu = Gm_e$ is Earth' s gravitational parameter (G is the gravitational constant, m_e is Earth' s mass), and f is the true anomaly. The relationship between p and semi-major axis a varies by convention: Reference [1] adopts $a > 0$, where for ellipses $p = a(1 - e^2)$ and for hyperbolas $p = a(e^2 - 1)$; Reference [16] uses $p = a(1 - e^2)$ for both, with $a > 0$ for ellipses and $a < 0$ for hyperbolas.

The rationale for using this element set includes: First, compared to various elliptical element sets, it uses the semi-latus rectum p instead of semi-major axis a and employs true anomaly f rather than mean anomaly M, enabling unified definition for hyperbolic orbits and avoiding singularities of a in parabolic orbits while preserving the advantages of orbital elements over position-velocity coordinates [15]. Second, the definitions of ξ, η, h, and k match those of the second-type nonsingular elements, maintaining maximum consistency with commonly used elliptical nonsingular elements [1-2], allowing existing formulas and programs for elliptical orbits to be largely retained. Using $\sin i$ instead of $\sin i$ eliminates the $i = 90°$ singularity mentioned earlier. This element set has a singularity at $i = 180°$, though this generally does not occur for satellite-type spacecraft [1]. If unavoidable, possible solutions are discussed in Section 4.3.

3.1. Conversion from Elements to Position-Velocity

The radial and transverse unit vectors $\hat{r}$ and $\hat{t}$ can be obtained from:

$$\hat{r} = \cos u \hat{P}^* + \sin u \hat{Q}^*, \quad \hat{t} = -\sin u \hat{P}^* + \cos u \hat{Q}^*,$$

where $\hat{P}^*$ and $\hat{Q}^*$ are calculated as:

$$\hat{P}^* = \frac{1 - 2k^2}{1 - h^2 - k^2}, \quad \hat{Q}^* = \frac{-2k}{1 - h^2 - k^2}.$$

From conic orbit integration, the position magnitude r is easily obtained, and the position and velocity vectors are calculated as:

$$\vec{r} = r\hat{r} = \frac{p}{1 + \xi \cos u + \eta \sin u}\hat{r},$$

$$\dot{\vec{r}} = \frac{\sqrt{\mu/p}}{(1 + \xi \cos u + \eta \sin u)} \left[(\xi \cos u - \eta \sin u)\hat{r} + \hat{t} \right].$$

3.2. Conversion from Position-Velocity to Elements

First, compute the angular momentum vector $\vec{H}$ and its magnitude h^*:

$$\vec{H} = \vec{r} \times \dot{\vec{r}} = h^* \hat{w}, \quad h^* = \sqrt{\mu p}.$$

Then p is immediately obtained from $p = h^{*2}/\mu$. The components of $\vec{H}$ in equation (11) yield h and k. Substituting the expressions for $\hat{P}^*$ and $\hat{Q}^*$ from (7) into (6) for $\hat{r}$ allows solving for $\cos u$ and $\sin u$ from the components, thus determining u.

On the other hand, rearranging equation (10) gives:

$$\dot{\vec{r}} \cdot \hat{t} = \frac{\sqrt{\mu/p}}{(1 + \xi \cos u + \eta \sin u)} e \sin f.$$

Any component of equation (12) yields $e\sin f$. From conic orbit integration we have $e\cos f = \frac{p}{r} - 1$, so both $e\sin f$ and $e\cos f$ are known. Substituting into:

$$e\cos f = \xi\cos u + \eta\sin u, \quad e\sin f = \xi\cos u - \eta\sin u,$$

solves for ξ and η. All six unified elements (p, ξ, η, h, k, u) are now determined.

3.3. Perturbation Equations

The Gauss-type perturbation equations for these unified elements can be derived from those in Reference [1] using the chain rule:

$$\frac{dp}{dt} = \frac{2p}{\sqrt{\mu p}}\left[\frac{p}{r}T\right],$$

$$\frac{d\xi}{dt} = \frac{\sqrt{\mu p}}{h^*}\left\{\sin uS + \left[\cos u + \frac{\xi + \cos u}{1 + \xi\cos u + \eta\sin u}\right]T - \frac{h\sin u - k\cos u}{\cos(i/2)}W\right\},$$

$$\frac{d\eta}{dt} = \frac{\sqrt{\mu p}}{h^*}\left\{-\cos uS + \left[\sin u + \frac{\eta + \sin u}{1 + \xi\cos u + \eta\sin u}\right]T + \frac{h\sin u - k\cos u}{\cos(i/2)}W\right\},$$

$$\frac{dh}{dt} = \frac{r}{\sqrt{\mu p}}\frac{\sin\Omega}{\cos(i/2)}W, \quad \frac{dk}{dt} = \frac{r}{\sqrt{\mu p}}\frac{\cos\Omega}{\cos(i/2)}W,$$

$$\frac{du}{dt} = \frac{h^*}{r^2} + \frac{1}{\sqrt{\mu p}}\frac{h\sin u - k\cos u}{\cos(i/2)}W.$$

Let the perturbing acceleration be $\vec{F}_p$. The three components S, T, W are obtained from:

$$S = \vec{F}_p \cdot \hat{r}, \quad T = \vec{F}_p \cdot \hat{t}, \quad W = \vec{F}_p \cdot \hat{w},$$

where $\hat{r}$, $\hat{t}$, $\hat{w}$ are the radial, transverse, and normal unit vectors, with $\hat{r}$ and $\hat{t}$ from (6) and $\hat{w} = \hat{r} \times \hat{t}$.

Clearly, equations (14) and (15) have no singularities for any eccentricity $e > 0$ and inclination i $(0 \le i < 180^\circ)$. When i approaches 180°, singularities appear; solutions are discussed in Section 4.3.

4. Numerical Verification and Analysis

For convenience, normalized units are used in the following calculations. The length unit $[L]$, mass unit $[M]$, and time unit $[T]$ are defined as:

$$[L] = a_e, \quad [M] = m_e, \quad [T] = \sqrt{\frac{a_e^3}{Gm_e}},$$

where a_e is Earth' s reference ellipsoid radius and m_e is Earth' s mass.

We select actual or simulated targets with various orbit types in cislunar space, perform orbit predictions, and compare with Cartesian coordinate calculations to analyze the computational efficiency and accuracy of the unified elements.

4.1. Earth-Orbiting Elliptical Orbit Example

We first examine actual Earth-orbiting targets. The efficiency advantage of orbital elements for near-Earth object prediction is well-established [15]. To demonstrate the applicability of our unified elements to cislunar targets, we select the three targets with the largest semi-major axes from TLE (Two-Line Element) data released by space-track in early October 2023. The initial Keplerian elements are shown in Table 1 . While many cislunar targets (including the libration point orbit in the next section) are in geocentric elliptical orbits, we refer to these three TLE-cataloged targets as "Earth-orbiting elliptical orbits" for convenience.

Numerical calculations using both unified elements and position-velocity vectors are performed to compare their ephemerides. Perturbations include 10$×$10 Earth and lunar non-spherical gravity, solar point-mass gravity, and solar radiation pressure. A fixed-step RKF7(8) integrator (Runge-Kutta-Fehlberg) with 1-hour step size integrates for 100 days. The trajectories are shown in Figure 2 [Figure 2: see original paper], Figure 3 [Figure 3: see original paper] shows selenocentric and geocentric distances, and Figure 4 [Figure 4: see original paper] shows the position deviation between the two methods during integration. The maximum difference reaches millimeter-level over 100 days, demonstrating high accuracy of the unified elements.

For computational efficiency, a variable-step RKF7(8) integrator with error tolerance 1×10^{-14} (normalized units) performs 100-day predictions. Figure 5 [Figure 5: see original paper] shows the cumulative right-hand-side function evaluations. In these cases, both methods show steady growth, with the element method maintaining an advantage. The speed gain is less pronounced for Chang' e-5 in Figure 5(a) because its elements vary more dramatically, resulting in larger accelerations in the equations of motion and smaller step sizes. Overall, even for distant, high-eccentricity Earth-orbiting targets, orbital elements maintain efficiency advantages over Cartesian coordinates, similar to conventional Earth satellites.

Since unified elements use true anomaly f to define position, integration speed for highly elliptical orbits is limited by step size near perigee. Compared to classical elliptical elements (Keplerian or nonsingular elements based on mean anomaly M), Chang' e-5 ($e_0 \approx 0.37$) shows higher efficiency with unified elements, while TESS ($e_0 \approx 0.49$) and SPEKTR-R ($e_0 \approx 0.60$) are more efficient with classical elliptical elements.

4.2. Earth-Moon Libration Point Target Example

This example uses initial position and velocity ($\vec{r}_0 = (x_0, y_0, z_0)^T, \dot{\vec{r}}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0)^T$) in GCRS (Geocentric Celestial Reference System) as shown in Table 2 . The epoch is 0:00:00.0 UTC on January 1, 2028. The target is on a Lissajous orbit near the Earth-Moon Lagrange point L1, remaining near L1 for approximately 20 days before gradually departing the equilibrium position.

Numerical calculations compare ephemerides from unified elements and position-velocity vectors. Perturbations include 10$×$10 Earth and lunar gravity, solar gravity, and solar radiation pressure. A fixed-step RKF7(8) integrator with 15-second step integrates for 50 days. The trajectory is shown in Figure 6 Figure 6: see original paper, while Figures 6(b) and 7 [Figure 7: see original paper] show selenocentric distance and variations in eccentricity and inclination. The ephemeris difference is shown in Figure 8 [Figure 8: see original paper], with position deviations at the 10^{-3} km level over 50 days. For the first 20+ days, Earth' s two-body gravity dominates. After approximately 21 days, without station-keeping, the target leaves L1 and moves extensively in cislunar space where lunar gravity exceeds Earth' s. The ephemeris deviation variations in Figure 8 reflect changes in equation-of-motion accelerations during integration, related to the changing dominant forces in cislunar space and the inherent variation characteristics of different state variables.

Efficiency comparison uses a variable-step RKF7(8) integrator with error tolerance 1×10^{-12} for 50 days. Figure 9 [Figure 9: see original paper] shows cumulative right-hand-side function evaluations. The element method maintains an advantage for the first ~20 days, but after departing the equilibrium position, element calculations experience several sharp increases and fall behind Cartesian coordinates. Analysis of Figure 7 reveals the cause: at ~22 and ~44 days, eccentricity rapidly increases from <1 to >3. From the relationship in Section 2, with small semi-major axis a variation, p increases by about an order of magnitude. Since equation (14) contains a $\sqrt{p}$ factor in the denominator, larger p increases acceleration, reducing step size and increasing function evaluations. At ~30 and ~35 days, orbital inclination suddenly increases, bringing the target near retrograde motion. The $1/\cos(i/2)$ factor in equation (14) makes the denominator small, again increasing acceleration and reducing efficiency.

In summary, the element method maintains efficiency advantages over Cartesian coordinates for elliptical orbits, consistent with Section 4.1. However, efficiency decreases and may fall behind for hyperbolic orbits, which are generally distant from Earth with dramatically varying elements that reduce step sizes.

4.3. Lunar Impact Target Example

Target WE0913A' s Keplerian elements at UTC 23:58:50.816 on January 20, 2022 are shown in Table 3 [4]. After ~40 days of flight, it impacted the Moon. Using Cartesian integration results as reference, the trajectory, selenocentric distance, inclination, and eccentricity are shown in Figures 10 [Figure 10: see original paper] and 11 [Figure 11: see original paper] (with a magnified inset in Figure 11(a)). Eccentricity increases from <1 to >1, indicating an initial elliptical orbit that becomes hyperbolic near the Moon, making elliptical elements unsuitable. Additionally, near impact (the integration end), elements undergo dramatic changes, with inclination i approaching 180°—the singularity of unified elements. The small denominator near the singularity causes abrupt right-hand-side variations that numerical integration cannot cross, manifested as step sizes shrinking to zero. Relaxing local truncation error is one option to cross the singularity, but at the cost of increased integration error and still faces significant step size reduction. Singularity handling is discussed in the next section. We now calculate the ephemeris from initial time to impact using unified elements and compare with position-velocity results.

4.3.1. Ephemeris Comparison Figure 12 [Figure 12: see original paper] shows position errors between unified elements and position-velocity calculations. Perturbations include 10$×$10 Earth and lunar gravity, solar gravity, and solar radiation pressure. For comparison convenience, ephemeris output intervals are 30 minutes when selenocentric distance > 10,000 km, and 1 second when < 10,000 km. The lunar surface shape is not strictly considered; impact is defined as selenocentric distance < lunar radius (1,737.1 km). Truncation error tolerance is 1×10^{-10} (normalized units). Both methods yield impact time as UTC 12:29:31.816 on March 4, 2022.

The spikes in Figure 12 correspond to close lunar proximity where lunar perturbations are strong. After ~42.5 days of prediction, maximum errors between unified elements and position-velocity coordinates are at meter-level, showing consistent results.

4.3.2. Computational Efficiency For efficiency comparison, a variable-step RKF7(8) integrator integrates to specific times with tolerance 1×10^{-10}. Table 4 and Figure 13 [Figure 13: see original paper] show right-hand-side function evaluations for different integration durations.

The element method maintains efficiency advantages for most of the early integration period, but efficiency drops dramatically when integrating to impact. As shown in Figure 11(a), inclination i approaches 180° near the end, making $\cos(i/2)$ a small denominator in the perturbation equations and causing large acceleration values. To satisfy error tolerance, step sizes must become extremely small, reducing efficiency. Table 5 shows that most function evaluations occur when i is near the singularity, confirming it as the primary cause of efficiency loss.

To improve variable-step integration efficiency near singularities without significantly affecting accuracy, we propose monitoring step size changes and applying strategies when thresholds are reached: (1) switch to fixed-step integration; (2) limit minimum step size; (3) switch to Cartesian coordinates. Table 6 presents function evaluation counts, impact time differences, and impact point variations for various improved methods (using highest-precision Cartesian integration (1) as reference. TOL denotes variable-step integration error tolerance in normalized units):

(1) Full Cartesian variable-step integration (TOL = 1×10^{-14})

(2) Full Cartesian variable-step integration (TOL = 1×10^{-10})

(3) Full unified element variable-step integration (TOL = 1×10^{-10})

(4) Same as (3), but switch to 0.01 s fixed-step when step $<$ 0.01 s

(5) Same as (3), but switch to 0.05 s fixed-step when step $<$ 0.05 s

(6) Same as (3), but switch to 0.07 s fixed-step when step $<$ 0.07 s

(7) Same as (3), but limit minimum step to 0.01 s (use actual step if $>$ 0.01 s)

(8) Same as (3), but limit minimum step to 0.05 s

(9) Same as (3), but limit minimum step to 0.07 s

(10) Same as (3), but switch to Cartesian variable-step when step $<$ 0.01 s

(11) Same as (3), but switch to Cartesian variable-step when step $<$ 0.05 s

Strategies (4)–(11) all provide substantial efficiency improvements over full element integration (3). Except for (6) and (9), impact time differences from high-precision Cartesian results (1) are at 10^{-2} s level, and impact point differences are at 10^{-1} km level, maintaining high accuracy. However, in (6) and (9) where step size is relaxed to 0.07 s, impact time differences reach ~1 s and impact point differences reach 5 km, showing significant accuracy loss. Therefore, while these methods improve efficiency, step size thresholds cannot be excessively enlarged to avoid substantial precision degradation. Normally, controlling integration step size through local truncation error remains the most stable and reliable approach.

In conclusion, the $i = 180°$ singularity imposes objective and non-negligible limitations on unified elements, but noticeable efficiency gains can be achieved with modest precision relaxation. These improvements make unified elements applicable even when encountering the $i = 180°$ singularity, rendering them suitable for all orbit types in cislunar space with clear computational advantages over Cartesian coordinates in most scenarios (e.g., $e < 3$ and $i \neq 180°$).

References

[1] Liu L, Hu S J, Wang X. *Introduction to Spaceflight Dynamics*. Nanjing: Nanjing University Press, 2006
[2] Liu L. *Spacecraft Orbit Theory*. Beijing: National Defense Industry Press, 2000: 439
[3] Cefola P J. *Equinoctial Orbit Elements–Application to Artificial Satellite Orbits*, AIAA 1972-937. Astrodynamics Conference. Palo Alto, CA, U.S.A., September 11-12, 1972
[4] McClain W D. NASA-CR-156782. *A Recursively Formulated First-order Semianalytic Artificial Satellite Theory Based on the Generalized Method of Averaging. Volume 1: The Generalized Method of Averaging Applied to the Artificial Satellite Problem*, 1977: A-1
[5] Walker M J H, Ireland B, Owens J. *CeMec*, 1985, 36: 409
[6] Walker M J H. *CeMec*, 1986, 38: 391
[7] Peloni A, Ceriotti M, Dachwald B. *JGCD*, 2016, 39: 2712
[8] Horwood J T, Poore A B. *SIAM/ASA*, 2014, 2: 276
[9] Qi R, Misra A K, Zuo Z. *JGCD*, 2017, 40: 722
[10] Roy A E, Moran P E. *CeMec*, 1973, 7: 236
[11] Herrick S. *AJ*, 1953, 58: 156
[12] Rosengren A J, Scheeres D J. *CeMDA*, 2014, 118: 197
[13] Liu L, Tang J, Hou X. *ChA&A*, 2019, 43: 278
[14] Liu L, Tang J, Hou X. *Acta Astronomica Sinica*, 2018, 59: 29
[15] Wang H H. *Comprehensive Research and Application of Precise Orbit Determination for Artificial Earth Satellites*. Nanjing: Nanjing University, 2007: 15
[16] Vallado D A. *Fundamentals of Astrodynamics and Applications*. Fourth Edition. Hawthorne, CA: Microcosm Press, 2013

Note: Figure translations are in progress. See original paper for figures.