Pati-Salam model in curved space-time and sheaf quantization postprint

Authors: Desheng Li, Desheng Li

Date: 2025-02-03T00:00:00+00:00

Abstract

This note aims to provide a self-contained and detailed explanation of the U(k′)×U(k) Pati-Salam model in curved spacetime, derived from a (1+n)-dimensional square root Lorentz manifold via the self-parallel transportation principle. The conceptual foundations of manifolds from the perspectives of category theory, fiber bundle theory, and sheaf theory are reviewed. The model incorporates additional U(k′)×U(k)-principal bundle and U(k)-associated bundle structures beyond the (1+n)-dimensional Lorentz manifold. Conserved currents on the square root Lorentz manifold are discussed preliminarily. A detailed proof of the relationship between sheaf quantization and path integral quantization is presented.

Full Text

Preamble

Pati-Salam Model in Curved Space-Time and Sheaf Quantization

De-Sheng Li[1]*
[1]College of Physics, Mechanical and Electrical Engineering, Jishou University, Jishou 416000, P.R. China

Abstract

This note provides a self-contained and detailed explanation of the $U(k') \times U(k)$ Pati-Salam model in curved space-time derived from a $(1{+}n)$-dimensional square root Lorentz manifold via the self-parallel transportation principle. We review the foundational concepts of manifolds from the perspective of category theory, fiber bundle theory, and sheaf theory. The square root Lorentz manifold possesses additional $U(k') \times U(k)$-principal bundles and $U(k)$-associated bundles beyond the standard $(1 + n)$-dimensional Lorentz manifold. Conservative currents on the square root Lorentz manifold are discussed preliminarily, and a detailed proof of the relationship between sheaf quantization and path integral quantization is presented.

Keywords: Sheaf quantization; Pati-Salam model; Yang-Mills theory; quantum field theory in curved space-time
lideshengjy@126.com

Introduction

A square root metric manifold carries additional $U(k') \times U(k)$ principal bundles and $U(k)$-associated bundles compared to ordinary Lorentz manifolds. These extra bundles provide an opportunity to construct Yang-Mills theory in curved space-time [1], particularly Pati-Salam type Yang-Mills theory [2, 3] in curved space-time. Sheaves, as natural mathematical structures discovered by mathematicians such as Jean Leray long ago, have deep connections with fiber bundle theory [5] (Yang-Mills theory [6, 7]) and the superposition principle.

Sheaves can be derived from contravariant functors in category theory, with sheaf cohomology and spectral sequences being both fascinating and useful. The microlocal language of sheaf theory [4] developed by Mikio Sato may become popular among future mathematical physicists. As a fundamental language of topos theory from Grothendieck [8], "we cannot even define a scheme without using sheaves" [9]. Sheaf quantization might provide a method to quantize quantum field theory in curved space-time while avoiding the problem of infinities [1, 10, 11, 12, 13, 14, 15, 16, 17].

The sheaf space is a linear space consistent with the superposition principle, even when the base manifold is curved. The sheaf quantization method is consistent with the path integral quantization method.

This paper is organized as follows: Section 2 provides a preliminary introduction to categories, functors, and the concepts of topological spaces, sheaves, manifolds, and bundles from the categorical viewpoint. Section 3 discusses the Einstein-Cartan geometry of Lorentz and Riemann manifolds. Section 4 briefly introduces generators of Clifford algebras. Section 5 describes the geometric framework of square root Lorentz manifolds, constructing the Pati-Salam model in curved space-time and Einstein-Cartan gravity based on square root Lorentz manifolds. Section 6 discusses the formulation of sheaf quantization and its relation to path integral quantization.

2 Category, Functor, Topological Space, Sheaf, Manifold and Fiber Bundle

2.1 Category

A category $\mathcal{C}$ consists of: - A class $\mathrm{ob}(\mathcal{C})$ of objects, for example $a, b, c, d \in \mathrm{ob}(\mathcal{C})$ - A class $\hom(\mathcal{C})$ of morphisms (or arrows, or maps), where $\hom_{\mathcal{C}}(a, b)$ represents all morphisms from a to b in category $\mathcal{C}$. For example, $f, g \in \hom(a, b)$, $h \in$

$\hom(b,c)$, $i \in \hom(a,c)$ - Composition of morphisms: for objects $a,b,c \in \mathrm{ob}(\mathcal{C})$, $\hom_{\mathcal{C}}(a,b) \times \hom_{\mathcal{C}}(b,c) \to \hom_{\mathcal{C}}(a,c)$

The morphisms in $\hom(\mathcal{C})$ satisfy the axioms of associativity and identity: - **(Associativity axiom)** If $f : a \to b$, $g : b \to c$, $h : c \to d$, then $h(gf) = (hg)f$ - **(Identity axiom)** For every object $x, y \in \mathrm{ob}(\mathcal{C})$, there exists a morphism $1_x : x \to x$ such that for every morphism $f : x \to y$, we have $1_x f = f = f 1_y$

2.2 Functor

Functors are structure-preserving maps between categories. A covariant functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is written as $F : \mathcal{C} \to \mathcal{D}$, where "structure-preserving" means: - For object $x \in \mathrm{ob}(\mathcal{C})$, $F(x) \in \mathrm{ob}(\mathcal{D})$ - For morphism $f \in \hom(\mathcal{C})$, $f : x \to y$, we have $F(f) : F(x) \to F(y)$, $f \in \hom(\mathcal{C})$, $F(f) \in \hom(\mathcal{D})$ - For every object $x \in \mathrm{ob}(\mathcal{C})$, $F(1_x) = 1_{F(x)}$ - For objects $x, y, z \in \mathrm{ob}(\mathcal{C})$ and all morphisms $f : x \to y$, $g : y \to z$ in $\mathcal{C}$, the functor preserves composition: $F(gf) = F(g)F(f)$

A contravariant functor is like a structure-preserving covariant functor from categories $\mathcal{C}$ to $\mathcal{D}$, but for morphisms $f, g \in \hom(\mathcal{C})$ with $f : x \to y$, we have $F(f) : F(y) \to F(x)$ and $F(gf) = F(f)F(g)$.

2.3 Topological Space

For a point $x_0 \in X$ and its neighborhood (an open covering), we can glue them to form a topological space X (more precisely, a Hausdorff space X):

$$U_{x_0} = \{x \mid x \to x_0\}$$

where $X = \cup U_x$, $x \to x_0$ is the direct limit from x to x_0. For any point $x_0 \in X$, there is an open covering partial ordered set on topological space X:

$$U_{x_0} \subset U_{x_0}^1 \subset U_{x_0}^2 \subset \cdots \subset X$$

2.4 Category View of Topological Space

- The topological space X is a category with objects $U_{x_0} \in \mathrm{ob}(X)$, $x_0 \in X$ and morphisms $\subset, \cup, \cap \in \hom(X)$
- The category **Top** has objects $X \in \mathrm{ob}(\mathbf{Top})$ and morphisms as continuous maps $\in \hom(\mathbf{Top})$

2.5 Presheaf and Sheaf

$F(U_{x_0})$ is the presheaf on U_{x_0} which is isomorphic to an Abelian group A ($F(U_{x_0})$ means all possible functions on neighborhood U_{x_0}):

$$F : U_{x_0} \to F(U_{x_0})$$

F is a functor from neighborhood U_{x_0} to presheaf $F(U_{x_0})$. To construct a sheaf $F(X)$ from presheaf $F(U_{x_0})$, we must satisfy the locality axiom and gluing axiom:

- **(Locality axiom)** If U_{x_0} is an open covering of an open set X, and if sections $s_x, t_x \in F(X)$ satisfy $s|_{U_{x_0}} = t|_{U_{x_0}}$ for any $x_0 \in X$, then $s_x = t_x$, where $s|_{U_{x_0}}$ is the section restricted to the neighborhood of x_0
- **(Gluing axiom)** If $x_0, x_1 \in X$, U_{x_0} and U_{x_1} are open coverings of an open set X, and for sections $s_{x_0} \in F(U_{x_0})$, $s_{x_1} \in F(U_{x_1})$, the sections agree on the overlap $s_{x_0}|_{U_{x_0} \cap U_{x_1}} = s_{x_1}|_{U_{x_1} \cap U_{x_0}}$, then there exists a global section $s_x \in F(X)$, $x \in X$, such that $s_{x_0} = s_x|_{U_{x_0}}$

The stalk of x_0 is the sheaf space restricted to x_0:

$$\mathcal{F}_{x_0} = F(X)|_{U_{x_0}} = F(U_{x_0})/\sim$$

where $\sim$ is an equivalence relation from restriction.

2.6 Manifold

For a manifold, in the neighborhood U_{x_0} of x_0, there is a coordinate $dx^\mu|_{x \to x_0}$. As an example of a presheaf, the collection of all possible coordinates in the neighborhood U_{x_0} of x_0 is a presheaf:

$$d(U_{x_0}) = \{dx^\mu|_{x \to x_0}\}$$

The presheaf $d(U_{x_0})$ can be glued to a sheaf $d(X)$ because:

$$d(U_{x_0}) \cap d(U_{x_1}) \to d(U_{x_0} \cap U_{x_1})$$

d is a functor from topological space X to the differential structure $d(X)$ of the topological space X:

$$d : X \to d(X)$$

where the differential structure $d(X)$ is the collection of all possible global coordinates on topological space X, d is one kind of functor F, and $d(X)$ is one kind of sheaf $F(X)$. Then we can see the manifold M is a topological space X with differential structure $d(X)$ (the global coordinates on $(1+n)$-dimensional manifold M might not be parameterized by $\mathbb{R}^{1+n}$):

$$M = (X, d(X))$$

We point out that the definition $M = (X, d(X))$ is equivalent to the definition in standard textbooks with axioms of local flatness and atlas compatibility:

- **(Local flatness axiom)** For a point x_0 in $(1+n)$-dimensional manifold, the neighborhood U_{x_0} is isomorphic to $\mathbb{R}^{1+n}$
- **(Atlas compatibility axiom)** For points x_0 and x_1 in $(1+n)$-dimensional manifold with neighborhoods U_{x_0} and U_{x_1} and parametrizations $\{x_0^\mu, \mu = 0, 1, 2, \cdots, n\}$ and $\{x_1^\mu, \mu = 0, 1, 2, \cdots, n\}$, there are

coordinates $\{dx_0^\mu, \mu = 0, 1, 2, \cdots, n\}$. For the overlap of the two neighborhoods $U_{x_0} \cap U_{x_1}$, there is a coordinate transformation:

$$dx_0^\mu = \Lambda_\nu^\mu(x_0) dx_1^\nu = \Lambda_\nu^\mu(x_1) dx^\nu$$

where $\Lambda_\nu^\mu(x_0), \Lambda_\nu^\mu(x_1) \in GL(1+n, \mathbb{R})$, and $\Lambda_\nu^\mu(x_0) = \Lambda_\nu^\mu(x)|_{x \to x_0}$

For any number of neighborhoods, we have:

$$\Lambda_{\nu_q}^\mu(x_q) \cdots \Lambda_{\nu_1}^{\nu_2}(x_1) \Lambda_{\nu}^{\nu_1}(x_0) = \mathrm{Hom}(\Lambda_\nu^\mu(x_0), \Lambda_\nu^\mu(x_q)) dx_q^\nu$$

where the element in $\mathrm{Hom}(\Lambda_\nu^\mu(x_q))$ is path-dependent and cotangent principal bundle section-dependent element of the linear transformation group $GL(n, \mathbb{R})$-valued. This means the parameters in x_0 and x_q rely on the continuous path $C : \tau \to M$, $C(\tau_k) = x_k$, $k = 0, 1, 2, \cdots, n$, linear transformation $\Lambda_\nu^\mu(x_0), \Lambda_\nu^\mu(x_q)$ and the edge function $C_q - C_0$. The solutions of equation (2.44) are just the sheaf space $d(X)$ restricted on curve $C(\tau)$:

$$d(X)|_{C(\tau)}$$

From equation (2.45) we can see that the global coordinates in $(1+n)$-dimensional manifold might not be parameterized by $\mathbb{R}^{1+n}$.

2.7 Category View of Manifold

- The manifold M is a category with objects $U_{x_0}, d(U_{x_0}) \in \mathrm{ob}(M)$ and morphisms $\subset, \cup, \cap, d \in \mathrm{hom}(M)$
- The category **Man** has objects $M \in \mathrm{ob}(\mathbf{Man})$ and morphisms as continuous differentiable maps $\in \mathrm{hom}(\mathbf{Man})$

2.8 Principal Bundle

The fiber $E(U_{x_0})$ of the cotangent principal bundle $E(M)$ on manifold M is isomorphic to the freedom $G = GL(1+n, \mathbb{R})$ of coordinates that can make transformations (left action) locally:

$$E(U_{x_0}) = \{\Lambda_\nu^\mu(x)|_{x \to x_0} \mid dx^{\mu'}|_{x' \to x_0} = \Lambda_\nu^\mu(x) dx^\nu|_{x \to x_0}, \Lambda_\nu^\mu(x)|_{x \to x_0} \in GL(1+n, \mathbb{R})\}$$

The cotangent principal G-bundle $E(M)$ on manifold M is:

$$E(M) = \cup E(U_x), \quad G = GL(1+n, \mathbb{R})$$

The cotangent principal bundle is a map π from total space E to base manifold M:

$$\pi : E \to M$$

The inverse mapping of π is a section of the sheaf $d(M)$ and bundle $E(M)$:

$$\pi^{-1} \in d(M), \quad \pi^{-1} \subset E(M)$$

Here for the inverse mapping of cotangent principal bundle, the meaning of π^{-1} is one global coordinate of the manifold M. The contravariant functor $\hat{\pi}^{-1}$ of π is the differential structure sheaf of the manifold M (all possible global coordinates). Because we have the commutative diagram:

$$d(U_{x_0}) \cup d(U_x) \to d(U_{x_0} \cup U_x) \xrightarrow{\hat{\pi}^{-1}} d(M)$$

The tangent principal bundle $E^*(M)$ is the dual bundle of cotangent principal bundle:

$$\hat{\pi}^{-1} = d$$

The section π^{*-1} in the neighborhood of U_{x_0} has the formula:

$$\pi^* : E^* \to M$$

and is dual with coordinates:

$$\langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle |_{x \to x_0}$$

The sheaf $\hat{\pi}^{*-1}$ is dual to $\hat{\pi}^{-1}$. The right action of element of $GL(1+n, \mathbb{R})$ on tangent principal bundle is consistent with the definition of left action transformation on cotangent bundle:

$$\left(\frac{\partial}{\partial x^\nu} \right)' = \Lambda^\nu_\mu(x) \frac{\partial}{\partial x^\nu}$$

The definition of dual basis (2.62) gives us:

$$\langle dx^\mu, \Lambda^\rho_\nu(x) \frac{\partial}{\partial x^\rho} \rangle |_{x \to x_0} = \Lambda^\rho_\nu(x) \langle dx^\mu, \frac{\partial}{\partial x^\rho} \rangle |_{x \to x_0}$$

2.9 Principal Bundle Connection

For a section π^{-1} of the cotangent principal bundle fiber $E(U_{x_0})$ on manifold M, the linear connection operator ∇_ρ is:

$$\nabla_\rho dx^\mu = \frac{dx^\mu - \Lambda^\mu_\nu(x_0) dx^\nu}{x^\rho - x^\rho_0} |_{x \to x_0} = -\Gamma^\mu_{\nu\rho}(x) dx^\nu$$

Then the linear connection operator ∇_ρ is a functor connecting fiber $E(U_x)$ to $E(U_{x_0})$:

$$\nabla_\rho : E(U_x) \to E(U_{x_0}), \quad x \to x_0$$

We write the connection 1-form as:

$$\Gamma^\mu_\nu(x) = \Gamma^\mu_{\nu\rho}(x) dx^\rho$$

and the linear connection 1-form operator:

$$\nabla = \nabla_\rho dx^\rho, \quad \nabla dx^\mu = -\Gamma^\mu_\nu(x)dx^\nu$$

The section of the fiber $E^*(U_{x_0})$ of tangent bundle has the connection:

$$\nabla_\rho \frac{\partial}{\partial x^\mu} = \tilde{\Gamma}^\nu_{\mu\rho}(x)\frac{\partial}{\partial x^\nu}$$

The dual relation (2.62) of bases gives us:

$$\nabla_\rho \langle dx^\mu, \frac{\partial}{\partial x^\nu}\rangle = 0 \Rightarrow \tilde{\Gamma}^\mu_{\nu\rho}(x) = -\Gamma^\mu_{\nu\rho}(x)$$

We assume that the linear connection operator ∇_ρ can be defined globally on the manifold. Under coordinate transformation in the neighborhood U_x, the transformation rule of the principal bundle connection is derived:

$$\begin{aligned}\nabla_\rho dx'^\mu &= \nabla_\rho(\Lambda^\mu_\nu(x)dx^\nu)\\ \Rightarrow -\Gamma'^\mu_{\nu\rho}(x')dx'^\nu &= \frac{\partial\Lambda^\mu_\nu(x)}{\partial x^\rho}dx^\nu - \Lambda^\mu_\nu(x)\Gamma^\nu_{\sigma\rho}(x)dx^\sigma\\ \Rightarrow \Gamma'^\mu_{\nu\rho}(x')\Lambda^\nu_\sigma(x) &= \frac{\partial\Lambda^\mu_\sigma(x)}{\partial x^\rho} - \Lambda^\mu_\nu(x)\Gamma^\nu_{\sigma\rho}(x)\\ \Rightarrow \Gamma'^\mu_{\nu\rho}(x') &= (\Lambda^\mu_\sigma(x)\Gamma^\sigma_{\nu\rho}(x) - d\Lambda^\mu_\nu(x))\Lambda^\nu_\sigma(x)\end{aligned}$$

Such that the cotangent principal bundle $E(M)$ has the structure of connection-preserving left action $G = GL(1+n,\mathbb{R})$ torsors:

$$E(M) = G \times E(M)$$

2.10 Tangent and Cotangent Associated Bundle

The tangent associated bundle $TE^*(M)$ on manifold M is glued with tangent space on neighborhood of x:

$$p^* : TE^* \to M, \quad TE^*(M) = \cup TE^*(U_x)$$

The section p^{*-1} of the bundle is a vector field of manifold M:

$$V(x) = V^\mu(x)\frac{\partial}{\partial x^\mu}, \quad (V^0(x), V^1(x), \cdots, V^n(x))|_{x\to x_0} \in \mathbb{R}^{1+n}$$

Then the fiber $TE^*(U_{x_0})$ of the bundle $TE^*(M)$ is isomorphic to $\mathbb{R}^{1+n}$. For a definite section of the tangent bundle, there is $GL(1+n,\mathbb{R})$ freedom to choose the bases of vectors in the neighborhood of x_0:

$$V(x)|_{x\to x_0} = V^{\mu'}(x)\frac{\partial}{\partial x^\nu}\Lambda^\nu_\mu(x)|_{x\to x_0} = V^\nu(x)|_{x\to x_0}, \quad \Lambda^\nu_\mu(x)|_{x\to x_0} \in GL(1+n,\mathbb{R})$$

With the help of (2.72) and (2.77), the tangent associated bundle $TE^*(M)$ has the structure of connection-preserving right action $G = GL(1+n,\mathbb{R})$ torsors:

$$TE^*(M) = TE^*(M) \times G$$

The right action structure group G of tangent associated bundle is free and transitive. The contravariant functor $\hat{p}^{*-1}$ of tangent associated bundle $TE^*(M)$ is a sheaf on manifold:

$$\hat{p}^{*-1} : M \to TE^*$$

where the sheaf $\hat{p}^{*-1}$ is the collection of all tangent vector fields on manifold M. The sheaf $\hat{p}^{*-1}$ has the structure of connection-preserving right action $G = GL(1+n,\mathbb{R})$ torsors:

$$\hat{p}^{*-1} = \hat{p}^{*-1} \times G$$

Similarly, the cotangent associated bundle is:

$$TE(M) = \cup TE(U_x)$$

and has the structure of connection-preserving left action $G = GL(1+n,\mathbb{R})$ torsors:

$$TE(M) = G \times TE(M)$$

The section p^{-1} of the cotangent associated bundle is a cotangent vector field (1-form) on manifold M:

$$\alpha(x) = \alpha_\mu(x)dx^\mu, \quad (\alpha_0(x), \alpha_1(x), \cdots, \alpha_n(x))|_{x\to x_0} \in \mathbb{R}^{1+n}$$

The contravariant functor $\hat{p}^{-1}$ is the sheaf of all cotangent vector fields on manifold M:

$$\hat{p}^{-1} : M \to TE$$

The sheaf $\hat{p}^{-1}$ has the structure of connection-preserving left action $G = GL(1+n,\mathbb{R})$ torsors:

$$\hat{p}^{-1} = G \times \hat{p}^{-1}$$

3 Lorentz Manifold, Riemann Geometry and Cartan Geometry

3.1 Metric

Pseudo-Riemannian geometry $pR = (M, g)$ is one of the most successful geometric systems. The pseudo-Riemannian geometry pR is a differentiable manifold M with smooth metric tensor g:

$$g(x) = -g_{\mu\nu}(x)dx^\mu \otimes dx^\nu$$

The metric is a symmetric rank-2 tensor field on manifold M such that the components of metric tensor satisfy $g_{\mu\nu}(x) = g_{\nu\mu}(x)$. The metric field is non-degenerate, meaning the determinants of metric tensor components at any point x_0 in manifold M are non-zero:

$$g_v|_{x\to x_0} = \det(g_{\mu\nu}(x))|_{x\to x_0} \neq 0$$

The pseudo-Riemannian manifold pR has a corresponding inverse metric $g^{-1}(x)$ where the dual basis of coordinate $dx^\mu|_{x\to x_0}$ in the neighborhood of x_0 satisfies the inner product relation:

$$\langle \partial_\mu, dx^\nu \rangle|_{x\to x_0} = \delta^\nu_\mu$$

The components of inverse metric $g^{\mu\nu}(x)$ are the inverse matrix of metric components $g_{\mu\nu}(x)$ at any point x_0. The metric is compatible with linear connection when:

$$\nabla g(x) = 0 \Rightarrow \frac{\partial g_{\mu\nu}(x)}{\partial x^\rho} - g_{\mu\sigma}(x)\Gamma^\sigma_{\nu\rho}(x) - g_{\sigma\nu}(x)\Gamma^\sigma_{\mu\rho}(x) = 0$$

We discuss the $(1+n)$-dimensional pseudo-Riemannian manifold pR with signature $(-,+,+,\cdots)$ (Lorentz manifold L) and signature $(-,-,-,\cdots)$ (Riemann manifold R):

$$L, R \subset pR$$

Then $x = (x^\mu) = (x^0, x^q) = (t, \vec{x})$, $(q = 1, 2, \cdots, n)$ parameterizes the $(1+n)$-dimensional manifolds L and R, and $dx^\mu|_{x\to x_0}(\mu = 0, 1, 2, \cdots, n)$ is a coordinate in the neighborhood of x_0.

3.2 Curve on Lorentz Manifold and Riemann Manifold

The curve $C(\tau)$ on manifolds L and R is defined as:

$$C : \tau \to L, R, \quad \tau \in \mathbb{R}$$

The curve $C(\tau)$ on manifolds L and R is an entity satisfying reparameterization symmetry $f(\tau)$:

$$C(\tau) = C'(f(\tau)), \quad \tau, f(\tau) \in \mathbb{R}$$

The metric $g(x)$ on manifolds L and R defines a line element of the curve $C(\tau)$:

$$ds = \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau$$

The length of any path $C(\tau)$ from point x_0 to point x_q on manifolds L and R is defined as:

$$s = \int_{x_0}^{x_q} ds = \int_{x_0}^{x_q} \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau$$

The variation of the length s from point x_0 to x_q screens out the geodesic curve. The definition of geodesic curve derives from $\delta s = 0$:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(\tau)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

where $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol defined by metric components:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\rho} + \frac{\partial g_{\sigma\rho}}{\partial x^\nu} - \frac{\partial g_{\nu\rho}}{\partial x^\sigma}\right)$$

The $d\tau$ is the basis of cotangent vector on curve $C(\tau)$, and the dual basis $\frac{d}{d\tau}$ is defined as:

$$\frac{d}{d\tau}|_{\tau=\tau_0} = \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}|_{\tau=\tau_0}$$

The restriction of tangent principal bundle E^* from manifolds L and R to curve $C(\tau)$ is:

$$E^*(L), E^*(R) \xrightarrow{\text{restriction}} E^*(U_\tau) \xrightarrow{\text{restriction}} U_\tau$$

The objects in $E^*(U_\tau)$ are tangent vectors on the curve $C(\tau)$. When the linear connection operator ∇_ρ acting on tangent vector $\frac{d}{d\tau}$ of curve $C(\tau)$ equals zero, the curve $C(\tau)$ is self-parallel transported:

$$\nabla_\rho \frac{d}{d\tau} = 0$$

The definition of self-parallel transport derives:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(\tau)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

3.3 Principal Bundle on Lorentz Manifold and Riemann Manifold

The freedom to choose $dx^\mu|_{x\to x_0}$ is isomorphic to the fiber $E(U_{x_0})$ of the cotangent principal bundle $E(L)$ and $E(R)$ of the $(1+n)$-dimensional Lorentz manifold L and Riemann manifold R. There is freedom to choose coordinates in the neighborhood of x_0:

$$E(U_{x_0}) = \{\Lambda^\mu_\nu(x)dx^\nu|_{x\to x_0} \mid \Lambda^\mu_\nu(x)|_{x\to x_0} \in GL(1+n,\mathbb{R})\}$$

such that the cotangent principal bundles:

$$E(L) = \cup E(U_x), \quad x \in L$$

$$E(R) = \cup E(U_x), \quad x \in R$$

have the structure of connection-preserving left action $G = GL(1+n, \mathbb{R})$ torsors:

$$E(L) = G \times E(L), \quad E(R) = G \times E(R)$$

For a definite metric $g(x)$ of manifolds L and R, there is $GL(1+n, \mathbb{R})$ freedom to choose locally to describe the same metric $g(x)$ in the neighborhood of the coordinate $dx^\mu|_{x\to x_0}$:

$$g(x)|_{x\to x_0} = -g'_{\mu\nu}(x)dx^{\mu'}dx^{\nu'}|_{x\to x_0} = -g'_{\mu\nu}(x)\Lambda^\mu_\rho(x)dx^\rho\Lambda^\nu_\sigma(x)dx^\sigma|_{x\to x_0} = -g_{\rho\sigma}(x)dx^\rho dx^\sigma|_{x\to x_0}$$

where:

$$g'_{\mu\nu}(x)\Lambda^\mu_\rho(x)\Lambda^\nu_\sigma(x)|_{x\to x_0} = g_{\rho\sigma}(x)|_{x\to x_0}$$

For the inverse metric, the analysis for tangent principal bundles $E^*(L)$ and $E^*(R)$ are similar, and the tangent principal bundle on $(1+n)$-dimensional manifolds L and R has structure of connection-preserving right $G = GL(1+n, \mathbb{R})$ action torsors:

$$E^*(L) = E^*(L) \times G, \quad E^*(R) = E^*(R) \times G$$

3.4 Orthonormal Principal Frame Bundle and Cartan Geometry

The inverse metric $g^{-1}(x)$ in Lorentz manifold L is described by orthonormal frame formalism $(a, b = 0, 1, 2, \cdots, n)$:

$$g^{-1}(x) = -\eta_{ab}\theta^a(x)\theta^b(x), \quad \eta_{ab} = \mathrm{diag}(1, -1, -1, \cdots, -1)$$

where $\theta^a(x) = \theta^a_\mu(x)\frac{\partial}{\partial x_\mu}$ are orthonormal frames describing the gravitational field.

The Riemann manifold R is described by inverse metric $\bar{g}^{-1}(x)$ in orthonormal frame formalism as:

$$\bar{g}^{-1}(x) = -I_{ab}\theta^a(x)\theta^b(x), \quad I_{ab} = \mathrm{diag}(1, 1, 1, \cdots, 1)$$

For a definite inverse metric $g^{-1}(x)$, there is $O(1, n)$ freedom to choose the orthonormal frame $\theta^a(x)|_{x\to x_0}$ to describe the same metric in the neighborhood of x_0:

$$\theta^{a'}(x)|_{x\to x_0} = \theta^b(x)\Lambda^a_b(x)|_{x\to x_0}, \quad \Lambda^a_b(x)|_{x\to x_0} \in O(1, n)$$

$$g^{-1}(x)|_{x\to x_0} = -\eta'_{ab}\theta^{a'}(x)\theta^{b'}(x)|_{x\to x_0} = -\eta'_{ab}\theta^c(x)\Lambda^a_c(x)\theta^d(x)\Lambda^b_d(x)|_{x\to x_0} = -\eta_{ab}\theta^a(x)\theta^b(x)|_{x\to x_0}$$

where:

$$\eta_{cd}\Lambda^c_a(x)\Lambda^d_b(x)|_{x\to x_0} = \eta_{ab}|_{x\to x_0} \Rightarrow \Lambda^a_b(x)|_{x\to x_0} \in O(1, n)$$

This means the fiber $OE^*(U_{x_0})$ of orthonormal principal frame bundle $OE^*(L)$ is isomorphic to the orthonormal frame transformation freedom $G = O(1,n)$ (right action) locally:

$$OE^*(U_{x_0}) = \{\theta^a(x')|_{x'\to x_0} = \theta^b(x)\Lambda_b^a(x)|_{x\to x_0} \mid \Lambda_b^a(x)|_{x\to x_0} \in O(1,n)\}$$

The orthonormal frame principal bundle is:

$$OE^*(L) = \cup OE^*(U_x), \quad x \in L$$

The fiber $OE^*(U_{x_0})$ of orthonormal principal frame bundle $OE^*(R)$ of Riemann manifold R is isomorphic to the orthonormal frame transformation freedom $G = O(1+n)$ (right action) locally:

$$OE^*(U_{x_0}) = \{\theta^a(x')|_{x'\to x_0} = \theta^b(x)\Lambda_b^a(x)|_{x\to x_0} \mid \Lambda_b^a(x)|_{x\to x_0} \in O(1+n)\}$$

The orthonormal frame principal bundle is:

$$OE^*(R) = \cup OE^*(U_x), \quad x \in R$$

The metric $g(x)$ and $\bar{g}(x)$ can be described by cotangent orthonormal frame (orthonormal co-frame) formalism as:

$$g(x) = -\eta_{ab}\theta^a(x)\theta^b(x), \quad \bar{g}(x) = -I_{ab}\theta^a(x)\theta^b(x)$$

where $\eta_{ab} = \mathrm{diag}(1,-1,-1,\cdots,-1)$, $I_{ab} = \mathrm{diag}(1,1,1,\cdots,1)$, and:

$$\theta^a(x) = \theta_\mu^a(x)dx^\mu$$

are cotangent orthonormal frames. It follows from (3.6) and (3.7) that the cotangent orthonormal frame is dual to the tangent orthonormal frame:

$$\langle\theta^a(x), \theta_b(x)\rangle|_{x\to x_0} = \theta_\mu^a(x)\theta_b^\mu(x)|_{x\to x_0} = \delta_b^a$$

From equation (3.45) we have:

$$\theta^c(x)\Lambda_c^a(x)\Lambda_c^b(x)|_{x\to x_0}$$

The structure groups of orthonormal co-frame bundles $OE(L)$ and $OE(R)$ are also $O(1,n)$ and $O(1+n)$.

The orthonormal frame connection coefficients are defined as:

$$\nabla_\rho\theta^a(x) = \nabla_\rho\left(\theta_\mu^a(x)\frac{\partial}{\partial x_\mu}\right) = \frac{\partial\theta_\mu^a(x)}{\partial x^\rho} + \theta_a^\sigma(x)\Gamma_{\sigma\rho}^\mu(x)$$

Eliminating the edge term, we can write:

$$\omega^a_{b\rho}(x) = \theta^a_\sigma(x)\Gamma^\sigma_{\mu\rho}(x)\theta^\mu_b(x) - d\theta^a_b(x)$$

The compatible connection condition (3.8) for orthonormal frame connection coefficients:

$$\eta_{ac}\Gamma^c_{b\rho}(x) + \eta_{cb}\Gamma^c_{a\rho}(x) = 0 \Rightarrow \Gamma_{ba\rho}(x) = -\Gamma_{ab\rho}(x)$$

The connection 1-form on orthonormal frame is defined:

$$\Gamma^a_b(x) = \Gamma^a_{b\rho}(x)dx^\rho$$

Then we have:

$$\nabla\theta^a(x) = \Gamma^a_b(x)\theta^b(x), \quad \nabla\theta_a(x) = -\Gamma^b_a(x)\theta_b(x), \quad \Gamma_{ab}(x) = -\Gamma_{ba}(x)$$

The structure of connection-preserving right action $G = O(1, n)$ and $\bar{G} = O(1+n)$ torsors of orthonormal frame principal bundles:

$$OE^*(L) = OE^*(L) \times G, \quad OE^*(R) = OE^*(R) \times \bar{G}$$

derives that:

$$\Gamma'^a_{b\rho}(x') = \Lambda^a_c(x)\Gamma^c_{d\rho}(x)\Lambda^d_b(x) - d\Lambda^a_c(x)\Lambda^c_b(x)$$

The curvature 2-form is defined:

$$\Omega^a_b(x) = d\Gamma^a_b(x) + \Gamma^a_c(x) \wedge \Gamma^c_b(x)$$

and equation (3.63) derives that the curvature 2-form satisfies the tensor transformation:

$$\Omega'^a_b(x')\Lambda^b_d(x) = \Lambda^a_c(x)\Omega^c_d(x)$$

The relation between curvature 2-form $\Omega^a_b(x)$ and curvature tensor $R^a_{b\mu\nu}(x)$ is:

$$\Omega^a_b(x) = \frac{1}{2}R^a_{b\mu\nu}(x)dx^\mu \wedge dx^\nu$$

where:

$$R^a_{b\mu\nu}(x) = \partial_\mu\Gamma^a_{b\nu}(x) - \partial_\nu\Gamma^a_{b\mu}(x) + \Gamma^a_{c\mu}(x)\Gamma^c_{b\nu}(x) - \Gamma^a_{c\nu}(x)\Gamma^c_{b\mu}(x)$$

Equation (3.44) gives us:

$$dx^\mu = \theta^\mu_a(x)\theta^a(x)$$

After the exterior derivative d acts on equation (3.68), the Cartan structure equation is derived:

$$0 = d(\theta_a^\mu(x)) \wedge \theta^a(x) + \theta_b^\mu(x) \wedge d\theta^b(x) + \Gamma^c(x) d(\theta_a(x))$$

$$\Rightarrow d\theta^c(x) = -\Gamma^c_{\mu\nu}(x) dx^\nu \wedge dx^\mu$$

The torsion 2-form is defined:

$$T^c(x) = \frac{1}{2} T^c_{\mu\nu}(x) dx^\mu \wedge dx^\nu = -\Gamma^c_{\mu\nu}(x) dx^\nu \wedge dx^\mu$$

Then the components of torsion are:

$$T^c_{\mu\nu}(x) = 2\Gamma^c_{[\mu\nu]}(x) = \Gamma^c_{\mu\nu}(x) - \Gamma^c_{\nu\mu}(x)$$

and the Cartan structure equation is rewritten as:

$$d\theta^c(x) + \Gamma^c_b(x) \wedge \theta^b(x) + T^c(x) = 0$$

It is easy to prove the torsion satisfies the tensor transformation rule. The exterior derivative d acting on equation (3.72) gives the Ricci identity:

$$d\Gamma^c_b(x) \wedge \theta^b(x) - \Gamma^c_b(x) \wedge d\theta^b(x) + dT^c(x) = 0$$

$$\Rightarrow \Omega^c_b(x) \wedge \theta^b(x) - \Gamma^c_b(x) \wedge T^b(x) + dT^c(x) = 0$$

Equation (3.74) is the Ricci identity in Cartan geometry with torsion, and the component formulation is:

$$\partial_{[\rho} T^a_{\mu\nu]}(x) + \Gamma^a_{\sigma[\rho}(x) T^\sigma_{\mu\nu]}(x) + R^a_{[\rho\mu\nu]}(x) = 0$$

The exterior derivative d acting on Ricci identity (3.74) derives the Bianchi identity:

$$d\Omega^c_d(x) - \Omega^c_b(x) \wedge \Gamma^b_d(x) + \Gamma^c_d(x) \wedge \Omega^d_b(x) = 0$$

The component formulation of the Bianchi identity is:

$$\partial_{[\mu} R^c_{|d|\nu\rho]}(x) - R^c_{b[\mu\nu}(x) \Gamma^b_{|d|\rho]}(x) + \Gamma^c_{d[\mu}(x) R^d_{|b|\nu\rho]}(x) = 0$$

The determinants of metric components in the neighborhood of x_0 give us the coordinate-free volume element $\theta_v(x)$:

$$g_v(x)|_{x\to x_0} = \det[g_{\mu\nu}(x)]|_{x\to x_0} = \det[\eta_{ab}\theta^a_\mu(x)\theta^b_\nu(x)]|_{x\to x_0} = \det[\theta^a_\mu(x)]^2|_{x\to x_0} = -\overset{2}{\det}[\theta^a_\mu(x)]|_{x\to x_0}$$

$$\Rightarrow \theta_v(x)|_{x\to x_0} = \sqrt{-g_v(x)}|_{x\to x_0}$$

3.5 Category View of Principal Bundle on Lorentz Manifold and Riemann Manifold

The cotangent principal bundles $E^*(L)$ and $E^*(R)$ are dual to tangent bundles $E(L)$ and $E(R)$. The orthonormal functor O acting on associated bundles gives orthonormal frame bundles and co-frame bundles:

$$O : E \to OE, \quad O : E^* \to OE^*$$

and the commutative diagram of these four kinds of associated bundles on Lorentz manifold L and Riemann manifold R follows.

4.1 $Cl_{1,n}(\mathbb{R})$ Clifford Algebra and Dirac Matrices

The $Cl_{1,n}(\mathbb{R})$ Clifford algebra has $1+n$ generators $\gamma_a (a = 0, 1, 2, \cdots, n)$. The Clifford algebra is spanned by the bases:

$$Cl_{1,n}(\mathbb{R}) = \text{span}\left\{1, \gamma_{a_1}, \gamma_{a_1}\gamma_{a_2}, \gamma_{a_1}\gamma_{a_2}\gamma_{a_3}, \cdots, \gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{1+n}}\right\}$$

The Clifford algebra $Cl_{1,n}(\mathbb{R})$ is a 2^{1+n}-dimensional linear space:

$$Cl_{1,n}(\mathbb{R}) = \{\alpha I + \alpha_{a_1}\gamma_{a_1} + \alpha_{a_1 a_2}\gamma_{a_1}\gamma_{a_2} + \cdots + \alpha_{a_1 a_2 \cdots a_{1+n}}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{1+n}}\}$$

where the coefficients before the bases are real-valued: $\alpha, \alpha_{a_1}, \alpha_{a_1 a_2}, \cdots, \alpha_{a_1 a_2 \cdots a_{1+n}} \in \mathbb{R}$.

The matrix representation of generators of Clifford algebra satisfies the restriction:

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab} I_k$$

In physics, the Hermiticity conditions for generators of Clifford algebra can always be chosen as:

$$\gamma_a\gamma_b^\dagger + \gamma_b^\dagger\gamma_a = 2I_{ab} I_k$$

The minimal faithful matrix representation for $Cl_{1,n}(\mathbb{R})$ gives the relation:

$$2^{1+n} = k \times k \Rightarrow k = 2^{(1+n)/2}$$

which means, for any matrix representation of generators of Clifford algebra, there is freedom of $U(k)$ to rotate the matrix representation:

$$\gamma_a' = \psi^\dagger\gamma_a\psi, \quad \psi \in U(k)$$

such that the γ_a' remain generators of Clifford algebra $Cl_{1,n}(\mathbb{R})$. The Dirac matrices can be represented by component formula:

$$\gamma_a = \gamma_a^{ij} \otimes e_i = \psi_i^\dagger\gamma_a\psi_j e_j^\dagger \otimes e_i, \quad \psi \in U(k)$$

where $e_i (i = 1, 2, \cdots, k)$ are orthogonal bases expanding $(1+n)$-dimensional complex space $\mathbb{C}^k$ and $\text{tr}(e_j^\dagger \otimes e_i) = e_i e_j^\dagger = \delta_{ij}$.

4.2 $Cl_{1,3}(\mathbb{R})$ Clifford Algebra and Dirac Matrices

Particularly, the solutions with:

$$n = 3 \text{ and } 1, \quad k = 4 \text{ and } 2$$

are particularly important for physics. The corresponding Clifford algebras are $Cl_{1,3}(\mathbb{R})$ and $Cl_{1,1}(\mathbb{R})$. The generators of Clifford algebra $Cl_{1,3}(\mathbb{R})$ are the well-known Dirac matrices and the bases:

$$Cl_{1,3}(\mathbb{R}) = \text{span}\{I, \gamma_a, \gamma_a\gamma_b, \gamma_a\gamma_b\gamma_c, \gamma_a\gamma_b\gamma_c\gamma_d\}$$

The Weyl representation $(q = 1, 2, 3)$ of Dirac matrices is:

$$\gamma_0 = \begin{pmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{pmatrix}, \quad \gamma_q = \begin{pmatrix} 0 & \sigma_q \\ -\sigma_q & 0 \end{pmatrix}$$

where σ_q are Pauli matrices.

The component formulation of γ'_a is:

$$\gamma'_a = \psi^\dagger_{ij}\psi_{jk}\gamma_a^{kl}e_k \otimes e_l^\dagger = \psi_i^\dagger\gamma_a\psi_j e_j \otimes e_i^\dagger, \quad \psi \in U(4)$$

with $i, j = 1, 2, 3, 4$, where ψ_1, ψ_2, ψ_3 and ψ_4 are four kinds of Dirac spinors. The element of $U(4)$ group can be presented as:

$$\psi = e^{-iV_\alpha T^\alpha}, \quad V_\alpha \in \mathbb{R}, \alpha = 0, 1, 2, \cdots, 15$$

where T^α are generators of $U(4)$ group and $T^\alpha = T^{\alpha\dagger}$.

4.3 $Cl_{1,1}(\mathbb{R})$ Clifford Algebra

The generators of Clifford algebra $Cl_{1,1}(\mathbb{R})$ can be represented by Pauli matrices:

$$Cl_{1,1}(\mathbb{R}) = \text{span}\{I, \gamma_0 = \sigma_1, \gamma_1 = i\sigma_2, -\sigma_3\}$$

4.4 Isomorphism Between Bases of $Cl_{1,3}(\mathbb{R})$ and Generators of $U(4)$ Group

An isomorphism between the bases of $Cl_{1,3}(\mathbb{R})$ and the generators of $U(4)$ group can be constructed as follows. The modified Dirac matrices could be:

$$T^{1,0} = \tilde{\gamma}_0 = \gamma_0, \quad T^{1,q} = \tilde{\gamma}_q = i\gamma_q$$

For modified Dirac matrices:

$$\tilde{\gamma}_a\tilde{\gamma}_b + \tilde{\gamma}_b\tilde{\gamma}_c = I_{ab}I_4, \quad \tilde{\gamma}_a^\dagger = \gamma_a$$

where $I_{ab} = \mathrm{diag}(1,1,1,1)$. Then, the isomorphism between the bases of $Cl_{1,3}(\mathbb{R})$ and the generators of $U(4)$ group is:

$$T^{1,a} = \tilde{\gamma}_a, \quad T^{2,ab} = i\tilde{\gamma}_a\tilde{\gamma}_b, \quad T^{3,abc} = i\tilde{\gamma}_a\tilde{\gamma}_b\tilde{\gamma}_c, \quad T^{4,abcd} = \tilde{\gamma}_a\tilde{\gamma}_b\tilde{\gamma}_c\tilde{\gamma}_d, \quad T^0 = I_4$$

It is easy to see that the constructed T^α satisfy:

$$T^{\alpha\dagger} = T^\alpha, \quad \mathrm{Tr}(T^\alpha T^\beta) = \delta^{\alpha\beta}$$

The commutative and anti-commutative relations of constructed T are:

$$[T^{1,a}, T^{1,b}] = -2iT^{2,ab}, \quad \{T^{1,a}, T^{1,b}\} = 0$$
$$[T^{1,a}, T^{2,bc}] = 0, \quad \{T^{1,a}, T^{2,bc}\} = 2T^{3,abc}$$
$$[T^{1,a}, T^{2,ab}] = 2iT^{1,b}, \quad \{T^{1,a}, T^{2,ab}\} = 0$$
$$[T^{1,a}, T^{3,bcd}] = 2iT^{4,abcd}, \quad \{T^{1,a}, T^{3,bcd}\} = 0$$
$$[T^{1,a}, T^{3,abc}] = 0, \quad \{T^{1,a}, T^{3,abc}\} = 2T^{2,bc}$$
$$[T^{1,a}, T^{4,abcd}] = -2iT^{3,bcd}, \quad \{T^{1,a}, T^{4,abcd}\} = 0$$
$$[T^{2,ab}, T^{2,cd}] = 0, \quad \{T^{2,ab}, T^{2,cd}\} = -2T^{4,abcd}$$
$$[T^{2,ab}, T^{2,bc}] = 2iT^{2,ac}, \quad \{T^{2,ab}, T^{2,bc}\} = 0$$
$$[T^{2,ab}, T^{3,bcd}] = 2iT^{3,acd}, \quad \{T^{2,ab}, T^{3,bcd}\} = 0$$
$$[T^{2,ab}, T^{3,abc}] = 0, \quad \{T^{2,ab}, T^{3,abc}\} = 2T^{1,c}$$
$$[T^{2,ab}, T^{4,abcd}] = 0, \quad \{T^{2,ab}, T^{4,abcd}\} = -2T^{2,cd}$$
$$[T^{3,abc}, T^{4,abcd}] = -2iT^{1,d}, \quad \{T^{3,abc}, T^{4,abcd}\} = 0$$

Explicitly, the constructed generators of $U(4)$ are represented by Dirac matrices:

$$T^1 = \gamma_0, \quad T^2 = i\gamma_1, \quad T^3 = i\gamma_2, \quad T^4 = i\gamma_3$$
$$T^5 = -\gamma_0\gamma_1, \quad T^6 = -\gamma_0\gamma_2, \quad T^7 = -\gamma_0\gamma_3, \quad T^8 = -i\gamma_1\gamma_2$$
$$T^9 = -i\gamma_1\gamma_3, \quad T^{10} = -i\gamma_2\gamma_3, \quad T^{11} = -i\gamma_0\gamma_1\gamma_2, \quad T^{12} = -i\gamma_0\gamma_1\gamma_3$$
$$T^{13} = -i\gamma_0\gamma_2\gamma_3, \quad T^{14} = \gamma_1\gamma_2\gamma_3, \quad T^{15} = -i\gamma_0\gamma_1\gamma_2\gamma_3, \quad T^0 = I_4$$

and we have:

$$[T^\alpha, T^\beta] = f^{\alpha\beta}_{\gamma} T^\gamma$$

5.1 Pair of Entities

We define a pair of entities:

$$l(x) = i\gamma_0(x)\gamma_a(x)\theta^a(x), \quad \tilde{l}(x) = i\gamma_a(x)\gamma_0(x)\theta^a(x)$$

which we call the square root metric of $(1+n)$-dimensional L and R. This pair of entities describes the square root Lorentz manifold rL. Direct calculations show that the definitions (5.1) and (5.2) satisfy:

$$l^\dagger(x) = -l(x), \quad \tilde{l}^\dagger(x) = -\tilde{l}(x)$$

The Dirac matrices on rL have the potential to be written as:

$$\gamma_{a'}(x) = \gamma_{a'ij}(x)e_j'^\dagger(x) \otimes e_i'(x) = u_i^\dagger(x)\gamma_{a'}(x)u_j(x)e_j^\dagger(x) \otimes e_i(x)$$

For any point $x_0 \in L$ and R:

$$\mathrm{tr}(e_j^\dagger(x) \otimes e_i(x))|_{x\to x_0} = e_i(x)e_j^\dagger(x)|_{x\to x_0} = \delta_{ij}$$

One simple choice of $e_i(x)(i = 1, 2, \cdots, k)$ on manifold is:

$$e_1(x) = (e^{i\theta_1(x)}, 0, 0, \cdots, 0), \quad e_2(x) = (0, e^{i\theta_2(x)}, 0, \cdots, 0), \quad \cdots, \quad e_k(x) = (0, 0, 0, \cdots, e^{i\theta_k(x)})$$

The bases $e_i'^\dagger(x)|_{x\to x_0}$, $(i = 1, 2, \cdots, k)$ on rL have $U(k)$ freedom to choose locally:

$$e_i'^\dagger(x)|_{x\to x_0} = u_{ij}(x)e_j^\dagger(x)|_{x\to x_0}, \quad u(x)|_{x\to x_0} \in U(k)$$

Similarly, there is another local freedom to choose representation of components of Dirac matrices:

$$\gamma_{a'}(x) = \gamma_{a'ij}(x)e_j'^\dagger(x) \otimes e_i'(x) = u_i^\dagger(x)\gamma_a(x)u_j(x)e_j'^\dagger(x) \otimes e_i'(x)$$

$$\gamma_{a'ij}(x)|_{x\to x_0} = u_{ik}^\dagger(x)\gamma_a^{kl}(x)u_{lj}(x)|_{x\to x_0} = u_i^\dagger(x)\gamma_a(x)u_j(x)|_{x\to x_0}, \quad u(x)|_{x\to x_0} \in U(k')$$

Then, there is an extra $U(k') \times U(k)$ principal bundle on $(1+n)$-dimensional square root Lorentz manifold rL compared to Lorentz manifold L, with $k' = k = 2^{1+n}$.

Under local $U(k') \times U(k)$ bases rotation equivalence relation, there still remains $U(k)$ physical freedom:

$$\gamma_a(x) = \gamma_{aj}(x) \otimes e_i(x) = \gamma_{aij}(x)e_j^\dagger \otimes e_i = \psi_i^\dagger(x)\gamma_a\psi_j(x)e_j^\dagger \otimes e_i$$

where \$ (x)|_{x →x_0} \$ $U(k)$ is isomorphic to the extra fiber space of associated bundle $UE^*_{1,2}(rL)$. In the language of mathematics, there are two extra $U(k)$ associated bundles $UE^*_{1,2}(rL)$ on $(1+n)$-dimensional square root metric rL compared to Lorentz manifold L, with structure of left $U(k')$ and right $U(k)$ action torsors:

$$UE^*_{1,2}(rL) = U(k') \times UE^*_{1,2}(rL) \times U(k)$$

where $l(x)$ and $\tilde{l}(x)$ are sections of $UE^*_1(rL)$ and $UE^*_2(rL)$ bundles, respectively. A pair of square root metric can be written as:

$$l(x) = i\gamma_0\gamma_a\psi_i^\dagger(x)\psi_j(x)e_j^\dagger \otimes e_i\theta^a(x) = i\psi_i^\dagger(x)\gamma_0\gamma_a\psi_j(x)e_j^\dagger \otimes e_i\theta^a(x)$$

$$\tilde{l}(x) = i\gamma_a\gamma_0\psi_i^\dagger(x)\psi_j(x)e_j^\dagger \otimes e_i\theta^a(x) = i\psi_i^\dagger(x)\gamma_a\gamma_0\psi_j(x)e_j^\dagger \otimes e_i\theta^a(x)$$

The total structure group of principal bundle $E^*(rL)$ on $(1+n)$-dimensional rL is:

$$G = U(k') \times U(k) \times GL(1+n,\mathbb{R}), \quad k' = k = 2^{1+n}$$

The fiber space of associated bundles $UE^*_{1,2}(rL)$ is isomorphic to:

$$UE^*_{1,2}(U_{x_0}) = U(k) \times GL(1+n,\mathbb{R})$$

and has structure of G-torsors:

$$UE^*_{1,2}(rL) = U(k') \times UE^*_{1,2}(rL) \times U(k) \times GL(1+n,\mathbb{R})$$

There are two kinds of inverse metric for the pair of entities:

$$\bar{g}^{-1}(x) = \mathrm{tr}[l(x)l(x)] = \mathrm{tr}[l(x)\tilde{l}(x)] = \mathrm{tr}[\tilde{l}(x)\tilde{l}(x)] = -I_{ab}\theta^a(x)\theta^b(x)$$

$$g^{-1}(x) = \mathrm{tr}[\tilde{l}(x)l(x)] = -\eta_{ab}\theta^a(x)\theta^b(x)$$

after using $\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0$, where $\bar{g}^{-1}(x)$ and $g^{-1}(x)$ are inverse metrics of Riemann manifold R and Lorentz manifold L, respectively.

A pair of square root metrics for metrics of R and L are:

$$\bar{l}(x) = i\gamma_0(x)\gamma_a(x)\theta^a_\mu(x)dx^\mu, \quad \bar{\tilde{l}}(x) = i\gamma_a(x)\gamma_0(x)\theta^a_\mu(x)dx^\mu$$

Direct calculation gives us that the definitions (5.20) and (5.21) satisfy:

$$\bar{l}^\dagger(x) = -\bar{l}(x), \quad \bar{\tilde{l}}^\dagger(x) = -\bar{\tilde{l}}(x)$$

The corresponding metrics for R and L are:

$$\bar{g}(x) = \mathrm{tr}[\bar{l}(x)\bar{l}(x)] = \mathrm{tr}[\bar{l}(x)\bar{\tilde{l}}(x)] = \mathrm{tr}[\bar{\tilde{l}}(x)\bar{\tilde{l}}(x)] = -I_{ab}\theta^a(x)\theta^b(x)$$

$$g(x) = \mathrm{tr}[\bar{\bar{l}}(x)\bar{l}(x)] = -\eta_{ab}\theta^a(x)\theta^b(x)$$

The entities pair (5.20) and (5.21) corresponding to principal bundle $E(rL)$ has total structure group:

$$\bar{G} = GL(1+n, \mathbb{R}) \times U(k') \times U(k), \quad k' = k = 2^{1+n}$$

The fiber space of associated bundle $UE(rL)$ is isomorphic to:

$$UE_{1,2}(U_{x_0}) = U(k) \times GL(1+n, \mathbb{R})$$

and has structure of $\bar{G}$-torsors:

$$UE_{1,2}(rL) = GL(1+n, \mathbb{R}) \times U(k') \times UE_{1,2}(rL) \times U(k)$$

where $\bar{l}(x)$ and $\bar{\bar{l}}(x)$ are sections of $UE_1(rL)$ and $UE_2(rL)$ bundles, respectively.

5.2 Connection of Extra Bundles and Gauge Field

The principal bundle connection $W_{\mu ij}(x)$, the flavor interaction gauge field, is defined as:

$$\nabla_\mu e_i^\dagger(x) = \frac{e_i^\dagger(x) - e_i^\dagger(x_0)}{x^\mu - x_0^\mu}|_{x\to x_0} = \frac{(\delta_{ij} - u_{ij}^*(x))e_j^\dagger(x)}{x^\mu - x_0^\mu}|_{x\to x_0} = iW_{\mu ij}(x)e_j^\dagger(x)|_{x\to x_0}$$

The conjugate transpose of definition (5.28) gives us:

$$\nabla_\mu e_i(x) = -iW_{\mu ij}^*(x)e_j(x), \quad W_{\mu ij}(x) = W_{\mu ji}^*(x)$$

The covariant derivative ∇_μ acting on (5.4) leads to the flavor interaction gauge field $W_{\mu ij}(x)$ which can be expanded by generators of weak interaction gauge group $U(k)$:

$$W_{\mu ij}(x) = W_\mu^\alpha(x)T_{ij}^\alpha, \quad \alpha = 0, 1, 2, \cdots, k^2 - 1$$

In summary, the flavor interaction gauge field is defined by:

$$\nabla_\mu e_i(x) = -ie_j(x)W_{\mu ji}(x), \quad \nabla_\mu e_i^\dagger(x) = iW_{\mu ij}(x)e_j^\dagger(x)$$

And the gauge fields $W_\mu^\alpha(x)$ are real-valued:

$$W_\mu^\alpha(x) = W_\mu^{\alpha *}(x)$$

In Cartan geometry and homology theory, differential forms are useful. Then, using the definition of coordinate-free covariant derivative, it is easy to see:

$$\nabla = \nabla_\mu dx^\mu, \quad W_{ij}(x) = W_{\mu ij}(x)dx^\mu$$

where $W_{ij}(x)$ is the flavor interaction gauge field connection 1-form.

Similarly, the principal bundle connection $V_\mu(x)$, the color interaction gauge field, is defined as:

$$\begin{aligned}
\nabla_\mu[\gamma_a(x)]|_{x\to x_0} &= \frac{\gamma_a(x)-\gamma_a(x_0)}{x^\mu - x_0^\mu}|_{x\to x_0} = \frac{\gamma_a(x)-u^\dagger(x)\gamma_a(x)u(x)}{x^\mu - x_0^\mu}|_{x\to x_0} \\
&= \frac{[\gamma_a(x)-u^\dagger(x)\gamma_a(x)]+[u^\dagger(x)\gamma_a(x)-u^\dagger(x)\gamma_a(x)u(x)]}{x^\mu - x_0^\mu}|_{x\to x_0} \\
&= \frac{[(I_k-u^\dagger(x))\gamma_a(x)]+[u^\dagger(x)\gamma_a(x)(I_k-u(x))]}{x^\mu - x_0^\mu}|_{x\to x_0} \\
&= i[V_\mu(x)\gamma_a(x)-\gamma_a(x)\bar{V}_\mu(x)]|_{x\to x_0}
\end{aligned}$$

We can see that:

$$\nabla_\mu[\gamma_a(x)]|_{x\to x_0} = i[V_\mu(x)\gamma_a(x)-\gamma_a(x)V_\mu^\dagger(x)]|_{x\to x_0}, \quad \bar{V}_\mu(x)=V_\mu^\dagger(x)$$

The conjugate transpose of (5.39) is:

$$\nabla_\mu[\gamma_a^\dagger(x)]|_{x\to x_0} = i[V_\mu(x)\gamma_a^\dagger(x)-\gamma_a^\dagger(x)V_\mu^\dagger(x)]|_{x\to x_0}$$

As we have the Hermiticity condition on square root Lorentz manifold rL:

$$\gamma_a^\dagger(x)\gamma_b(x)+\gamma_b^\dagger(x)\gamma_a(x)|_{x\to x_0} = I_{ab}I_k$$

we act covariant derivative ∇_μ on (5.41). After using $\gamma_a^\dagger=\gamma_0\gamma_a\gamma_0$, it is easy to find that:

$$V_\mu = V_\mu^\dagger$$

The V_μ is a $k\times k$ matrix-valued field and can be expanded by generators of $U(k)$ group:

$$V_\mu(x)=V_\mu^\alpha(x)T^\alpha, \quad \alpha=0,1,2,\cdots,k^2-1$$

In summary, the color interaction gauge field $V_\mu(x)$ is defined by:

$$\nabla_\mu(\gamma_a(x)) = i[V_\mu(x)\gamma_a(x)-\gamma_a(x)V_\mu(x)]$$

The conjugate transpose of equation (5.44) is:

$$\nabla_\mu(\gamma_a^\dagger(x)) = i[V_\mu(x)\gamma_a^\dagger(x)-\gamma_a^\dagger(x)V_\mu(x)]$$

The connections preserving G and $\bar{G}$-torsors on principal bundles $E^*(rL)$ and $E(rL)$ lead to the transformation rules of connections $W_{\mu ij}(x)$ and $V_\mu(x)$:

$$W'_{\mu ij}(x') = u_{ki}^*(x)W_{\mu kl}(x)u_{lj}(x)+u_{ki}^*(x)\partial_\mu u_{kj}(x)$$

$$V'_\mu(x') = u(x)V_\mu(x)u^\dagger(x) - (\partial_\mu u(x))u^\dagger(x)$$

where $u(x)|_{x\to x_0} \in U(k)$, $u(x)|_{x\to x_0} \in U(k')$, and:

$$u_{ji}(x)u_{jk}(x)|_{x\to x_0} = \delta_{ik}, \quad u(x)u^\dagger(x)|_{x\to x_0} = I_k$$

The gauge field strength tensors are defined as [18]:

$$F_{\mu\nu ij}(x) = \partial_\mu W_{\nu ij}(x) - \partial_\nu W_{\mu ij}(x) - iW_{\mu ik}(x)W_{\nu kj}(x) + iW_{\nu ik}(x)W_{\mu kj}(x)$$

$$H_{\mu\nu}(x) = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x) - iV_\mu(x)V_\nu(x) + iV_\nu(x)V_\mu(x)$$

and the transformation rules satisfy:

$$F'_{\mu\nu ij}(x') = u^*_{ki}(x)F_{\mu\nu kl}(x)u_{lj}(x)$$

$$H'_{\mu\nu}(x') = u(x)H_{\mu\nu}(x)u^\dagger(x)$$

From the Hermiticity condition of gauge fields $W_{\mu ij}$ and V_μ, the Hermiticity condition of gauge field strengths are:

$$F^*_{\mu\nu ij}(x) = F_{\mu\nu ji}(x), \quad H^\dagger_{\mu\nu}(x) = H_{\mu\nu}(x)$$

The gauge field strength tensors can be written as strength 2-forms:

$$H(x) = H_{\mu\nu}(x)dx^\mu \wedge dx^\nu, \quad F_{ij}(x) = F_{\mu\nu ij}(x)dx^\mu \wedge dx^\nu$$

where:

$$F_{ij}(x) = dW_{ij}(x) - iW_{ik}(x) \wedge W_{kj}(x)$$

$$H(x) = dV(x) - iV(x) \wedge V(x), \quad V(x) = V_\mu(x)dx^\mu$$

The exterior derivative acting on (5.52) and (5.53) gives the Bianchi identity of strength 2-forms:

$$dH(x) - iH(x) \wedge V(x) + iV(x) \wedge H(x) = 0$$

$$dF_{ij}(x) - iF_{ik}(x) \wedge W_{kj}(x) + iW_{ik}(x) \wedge F_{kj}(x) = 0$$

The tensor formulation of Bianchi identity in this geometry structure is:

$$\partial_{[\mu}H_{\nu\rho]}(x) = H_{[\mu\nu}(x)V_{\rho]}(x) - V_{[\mu}(x)H_{\nu\rho]}(x)$$

$$\partial_{[\mu}F_{\nu\rho]ij}(x) = F_{[\mu\nu|ik|}(x)W_{\rho]kj}(x) - W_{[\mu|ik|}(x)F_{\nu\rho]kj}(x)$$

5.3 Lagrangian Submanifold and Yang-Mills Theory in Curved Space-Time

A pair of equations satisfying $U(k') \times U(k)$ gauge invariance, local Lorentz invariance, and general covariance principles are constructed in $(1+n)$-dimensional square root Lorentz manifold rL:

$$\mathrm{tr}\nabla[l(x)] = 0, \quad \mathrm{tr}\nabla[\tilde{l}(x)] = 0$$

These equations represent the generalized self-parallel transportation principle. Eliminating index x, the explicit formulas of equations (5.59) and (5.60) are:

$$\bar{\psi}_i\gamma_a(i\partial_\mu\psi_i + V_\mu\psi_i - \psi_j W_{\mu ji})\theta^\mu_a + \bar{\psi}_i\gamma_b\psi_i\Gamma^b_{a\mu}\theta^\mu_a = 0$$

$$\psi^\dagger_i\gamma_a(i\partial_\mu\bar{\psi}^\dagger_i + V_\mu\bar{\psi}^\dagger_i - \bar{\psi}^\dagger_j W_{\mu ji})\theta^\mu_a + i\psi^\dagger_i\gamma_b\bar{\psi}^\dagger_i\Gamma^b_{a\mu}\theta^\mu_a = 0$$

where $\tilde{V}_\mu = \gamma_0 V_\mu \gamma_0$, $\bar{\psi}^\dagger = \gamma_0\psi$.

The Lagrangians corresponding to equations (5.59) and (5.60) are:

$$\mathcal{L} = \bar{\psi}_i\gamma_a(i\partial_\mu\psi_i + V_\mu\psi_i - \psi_j W_{\mu ji})\theta^\mu_a + i\bar{\psi}_i\gamma_b\psi_i\Gamma^b_{a\mu}\theta^\mu_a$$

$$\tilde{\mathcal{L}} = \psi^\dagger_i\gamma_a(i\partial_\mu\bar{\psi}^\dagger_i + V_\mu\bar{\psi}^\dagger_i - \bar{\psi}^\dagger_j W_{\mu ji})\theta^\mu_a + i\psi^\dagger_i\gamma_b\bar{\psi}^\dagger_i\Gamma^b_{a\mu}\theta^\mu_a$$

The last term in Lagrangian (5.62) is the Yukawa coupling term $\bar{\psi}_i\phi\psi_i$ and the scalar (Higgs) field is Dirac matrix-valued and originated from the gravitational field. Then, the Lagrangian (5.62) describes $U(k') \times U(k)$ Yang-Mills theory in curved space-time.

The Lagrangians (5.62) and (5.63) have the relation with (5.59) and (5.60):

$$\mathrm{tr}\nabla l(x) = \mathcal{L} - \mathcal{L}^\dagger, \quad \mathrm{tr}\nabla\tilde{l}(x) = \tilde{\mathcal{L}} - \tilde{\mathcal{L}}^\dagger$$

Thus, we say $l(x)$ and $\tilde{l}(x)$ are Lagrangian submanifolds in $UE^*_1(rL)$ and $UE^*_2(rL)$, respectively. If equations (5.59) and (5.60) are satisfied, the Lagrangians (5.62) and (5.63) are Hermitian:

$$\mathcal{L} = \mathcal{L}^\dagger, \quad \tilde{\mathcal{L}} = \tilde{\mathcal{L}}^\dagger$$

So, the unitary principle (5.67) and (5.68) of quantum field theory is consistent with the generalized self-parallel transportation principle (5.59) and (5.60).

The equations of motion for the Lagrangians (5.62) and (5.63) are:

$$\gamma_a(i\partial_\mu\psi_i + V_\mu\psi_i - \psi_j W_{\mu ji})\theta^\mu_a + \gamma_b\psi_i\Gamma^b_{a\mu}\theta^\mu_a = 0$$

$$\gamma_a(i\partial_\mu\bar{\psi}^\dagger_i + V_\mu\bar{\psi}^\dagger_i - \bar{\psi}^\dagger_j W_{\mu ji})\theta^\mu_a + \gamma_b\bar{\psi}^\dagger_i\Gamma^b_{a\mu}\theta^\mu_a = 0$$

and their conjugate transposes. Thus, a pair of Lagrangians (5.62) and (5.63) describing the $U(k') \times U(k)$ Pati-Salam model type Yang-Mills theory in curved space-time are constructed.

The Yang-Mills Lagrangian for gauge bosons in this geometry can be constructed:

$$\mathcal{L}_g = \zeta \left(\mathrm{tr}(H^{\mu\nu} H_{\mu\nu}) - \mathrm{tr}(F^{ji}_{\mu\nu} F^{\mu\nu}_{ij}) \right)$$

where $\zeta \in \mathbb{R}$ is a constant.

In this geometric framework, the equations can be derived as:

$$\nabla_{[\mu} \nabla_{\nu]} l = \nabla_{[\mu} \nabla_{\nu]} \left(\bar{\psi}_i \gamma_a \psi_j e_j^\dagger \otimes e_i \theta^a \right) = \left(\bar{\psi}_i \gamma_a \psi_k F_{\mu\nu kj} - F^*_{\mu\nu ki} \bar{\psi}_k \gamma_a \psi_j - \bar{\psi}_i \gamma_a H_{\mu\nu} \psi_j + \bar{\psi}_i \tilde{H}_{\mu\nu} \gamma_a \psi_j + \bar{\psi}_i \gamma_b \psi_j R^b_{a\mu}$$

$$\nabla_{[\mu} \nabla_{\nu]} \tilde{l} = \nabla_{[\mu} \nabla_{\nu]} \left(\psi_i^\dagger \gamma_a^\dagger \psi_k F_{\mu\nu kj} - F^*_{\mu\nu ki} \psi_k^\dagger \gamma_a^\dagger \psi_j + \psi_i^\dagger H_{\mu\nu} \gamma_a^\dagger \psi_j - \psi_i^\dagger \gamma_a^\dagger \tilde{H}_{\mu\nu} \psi_j + \psi_i^\dagger \gamma_b^\dagger \psi_j R^b_{a\mu\nu} \right) e_j^\dagger \otimes e_i \theta^a$$

where $\tilde{H}_{\mu\nu} = \gamma_0 H_{\mu\nu} \gamma_0$.

We define $\nabla^2 = \nabla_{[\mu} \nabla_{\nu]} dx^\mu \wedge dx^\nu$. The equation of motion of this gravity theory is constructed:

$$\mathrm{tr} \nabla^2 [\tilde{l}(x) l(x)] = 0$$
$$\Rightarrow -\Gamma^\mu_{\nu\rho} \cdots$$

Note: Figure translations are in progress. See original paper for figures.

Source: ChinaXiv — Machine translation. Verify with original.