On the Relationship Between Constant Product (Sum) of Slopes and Fixed-Point Chords in Conic Sections

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Abstract

Objective: To investigate the relationship between the product (or sum) of slopes of lines PA and PB being constant and whether line AB passes through a fixed point, where A, B, and P are three distinct points on a conic section C. Methods: Through strategic assumption without direct computation, appropriate transformations, and slope homogenization to simplify calculations, we explore and discover underlying relationships. Conclusion: Under specific conditions, the sum (or product) of slopes of lines PA and PB being constant is equivalent to line AB passing through a fixed point, which enables the formulation of several specific problems.

Full Text

An Investigation into the Relationship Between Constant Product/Sum of Slopes and Fixed-Point Chords in Conic Sections

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Abstract

Objective: Let P, A, and B be three distinct points on a conic section C. This paper investigates the relationship between the product (or sum) of the slopes of lines PA and PB being constant and whether line AB passes through a fixed point. **Methods:** The approach employs a "set without solving" strategy, appropriate transformations, and slope homogenization to simplify calculations and enable discovery. **Conclusions:** Under specific conditions, the sum (or product) of the slopes of PA and PB being constant is equivalent to line AB passing through a fixed point. Based on this finding, several concrete problems can be formulated.

Keywords: sum of slopes; product of slopes; constant value; passing through; fixed point
Classification: O123.5

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1. Central Conic Sections (Circle, Ellipse, Hyperbola)

1.1 Problem Statement

Let $P(x_0, y_0)$ be a fixed point on a central conic section C (circle, ellipse, or hyperbola), and let A and B be two other points on C distinct from P. If the sum (or product) of the slopes of lines PA and PB is constant, does line AB pass through a fixed point? Conversely, if line AB passes through a certain fixed point, is the product (or sum) of the slopes of PA and PB equal to a constant?

1.2 Analysis and Derivation

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be points on C, and assume the slopes of PA and PB exist, denoted as k_1 and k_2. Through substitution and transformation, we can establish a unified framework. The equation of the conic section C can be expressed in a form that facilitates homogenization. Since $P(x_0, y_0)$ lies on C, we can rewrite the equation of C accordingly.

Let the equation of line AB be $mx + ny = 1$ (which does not pass through point P). Substituting this into the conic equation yields a homogeneous quadratic equation. Since A and B are the intersection points of line AB with conic C, the slopes k_1 and k_2 of lines PA and PB are precisely the two real roots of this quadratic equation. By Vieta's theorem, we obtain relationships between the sum and product of the slopes and the coefficients of the equation.

When the sum of slopes $k_1 + k_2 = \lambda$ is constant, we derive the equation of line AB. Analysis of this equation reveals that: - If certain coefficient conditions are met, the system of equations has no solution, meaning line AB does not pass through a fixed point. - In special degenerate cases where the discriminant vanishes, line AB becomes tangent to C at P, which violates the problem conditions. - Under specific parameter relationships, line AB represents a vertical line $x = x_0$.

When the product of slopes $k_1 k_2 = \mu$ is constant, similar analysis shows that the behavior depends on whether μ equals particular values related to the conic's parameters. When μ matches the slope of the tangent at P, line AB is parallel to this tangent and does not pass through a fixed point.

Conclusion 1

Given a fixed point $P(x_0, y_0)$ on a central conic section C and moving points A, B on C, when the sum of the slopes of PA and PB equals zero, line AB is parallel to the tangent of C at P and does not pass through a fixed point.

Conclusion 2

Given a fixed point $P(x_0, y_0)$ on a central conic section C and moving points A, B on C, when the sum of the slopes of PA and PB equals a non-zero constant λ, line AB passes through a fixed point. The coordinates of this fixed point can be determined from the coefficients of the conic equation and the value of λ.

Conclusion 3

Given a fixed point $P(x_0, y_0)$ on a central conic section C and moving points A, B on C, when the product of the slopes of PA and PB equals a non-zero constant μ, the behavior of line AB depends on the relationship between μ and the conic's parameters. Under specific conditions, line AB passes through a fixed point; otherwise, it does not.

1.3 When Line AB Passes Through a Fixed Point, Is the Slope Sum/Product Constant?

If line AB passes through a fixed point $M(s, t)$, then as AB varies, certain parameters become variable. For the sum of slopes $k_1 + k_2$ to remain constant, specific relationships must hold between the coordinates of M and the conic's parameters. This leads to the following conclusions:

Conclusion 4

Given a fixed point $P(x_0, y_0)$ on a central conic section C and moving points A, B on C, if line AB passes through a fixed point $M(s, t)$ and satisfies $t = y_0$, then the sum of the slopes of PA and PB equals a constant. If $t \neq y_0$, the sum is not constant.

Conclusion 5

Given a fixed point $P(x_0, y_0)$ on a central conic section C and moving points A, B on C, if line AB passes through a fixed point $M(s, t)$ and satisfies certain coordinate relationships, then the product of the slopes of PA and PB equals a constant. Otherwise, the product is not constant.

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2. Parabola

2.1 Problem Statement

Let $P(x_0, y_0)$ be a fixed point on a parabola C, and let A and B be two other points on C distinct from P. If the product (or sum) of the slopes of lines PA and PB is constant, does line AB pass through a fixed point? Conversely, if line AB passes through a certain fixed point, is the product (or sum) of the slopes of PA and PB equal to a constant?

2.2 Analysis and Derivation

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be points on the parabola C, and assume the slopes of PA and PB exist. Through substitution and transformation, we can establish a unified equation for lines PA and PB. The parabola's equation can be rewritten in a form that facilitates analysis, using the fact that $P(x_0, y_0)$ lies on C.

Let the equation of line AB be $mx + ny = 1$ (not passing through P). Substituting into the parabola equation yields a quadratic equation in terms of the slopes. Since A and B are intersection points, the slopes k_1 and k_2 are the roots of this equation. By Vieta's theorem, we obtain relationships between the sum and product of the slopes and the line parameters.

When the sum of slopes $k_1 + k_2 = \lambda$ is constant: - If $\lambda = 0$, line AB is parallel to the tangent at P and does not pass through a fixed point. - If $\lambda \neq 0$, line AB passes through a fixed point whose coordinates depend on λ and the parabola's parameters.

When the product of slopes $k_1 k_2 = \mu$ is constant, line AB passes through a fixed point determined by μ and the parabola's equation.

Conclusion 6

Given a fixed point $P(x_0, y_0)$ on a parabola C and moving points A, B on C, when the sum of the slopes of PA and PB equals zero, line AB is parallel to the tangent of C at P and does not pass through a fixed point.

Conclusion 7

Given a fixed point $P(x_0, y_0)$ on a parabola C and moving points A, B on C, when the sum of the slopes of PA and PB equals a non-zero constant λ, line AB passes through a fixed point. The coordinates of this fixed point can be expressed in terms of λ and the parameters of the parabola.

Conclusion 8

Given a fixed point $P(x_0, y_0)$ on a parabola C and moving points A, B on C, when the product of the slopes of PA and PB equals a non-zero constant μ, line AB passes through a fixed point.

2.3 When Line AB Passes Through a Fixed Point, Is the Slope Sum/Product Constant?

If line AB passes through a fixed point $M(s,t)$, then for the sum of slopes k_1+k_2 to be constant, the coordinates of M must satisfy specific relationships derived from the parabola's equation. This leads to the following conclusions:

Conclusion 9

Given a fixed point $P(x_0, y_0)$ on a parabola C and moving points A, B on C, if line AB passes through a fixed point $M(s,t)$ and satisfies $t = y_0$, then the sum of the slopes of PA and PB equals a constant. If $t \neq y_0$, the sum is not constant.

Conclusion 10

Given a fixed point $P(x_0, y_0)$ on a parabola C and moving points A, B on C, if line AB passes through a fixed point $M(s,t)$ and satisfies $t = -y_0$, then the product of the slopes of PA and PB equals a constant. If this condition is not met, the product is not constant.

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3. Illustrative Examples

Example 1. Given point $P(1,2)$ and moving points A, B on the circle $x^2+y^2 = 5$ that do not coincide with P, if the sum of the slopes of lines PA and PB equals -1, does line AB pass through a fixed point?

Solution: By Conclusion 2, line AB must pass through a fixed point. We can verify this through direct computation. Let the equation of line AB be $y = kx+b$. Substituting into the circle equation and using Vieta's theorem, we find that when the slope sum condition holds, line AB passes through the fixed point $Q(5,0)$. The vertical line through Q does not intersect the circle at two distinct points, so the valid fixed point is $Q(5,0)$.

Example 2. Given point $D(0,-2)$ and moving points A, B on the hyperbola $x^2-y^2 = 2$, if the sum of the slopes of lines DA and DB equals 2, does line AB pass through a fixed point?

Solution: Applying Conclusion 2, we deduce that line AB passes through a fixed point $Q(-2,-4)$. To verify, we consider two cases: when line AB has undefined slope (vertical) and when it has defined slope. For the latter, let AB be $y = kx+m$. Substituting into the hyperbola equation and applying the slope sum condition yields $m = 2k-4$, showing that AB always passes through $Q(-2,-4)$. The vertical case does not satisfy the given conditions.

Example 3. A line l intersects parabola C at points A and B. Does there exist a fixed point D on C such that the sum of the slopes of lines AD and BD equals 2?

Solution: Since line l passes through a fixed point, Conclusion 9 indicates that such a point D exists. To find it, we parameterize the intersection and apply the slope sum condition. Solving the resulting system yields $D(2, 2)$ as the unique point on the parabola where the slope sum condition holds for all lines l through the given fixed point.

Example 4. A line l passing through $Q(0, 6)$ intersects ellipse $C : \frac{x^2}{4} + y^2 = 1$ at points A and B. Let D be a point on C such that the product of the slopes of lines DA and DB equals 1. Find the maximum area of triangle ABQ.

Solution: By Conclusion 3, the point D satisfying the slope product condition is $D(0, 2)$. To verify, let line l be $y = kx + 6$. Substituting into the ellipse equation and applying the slope product condition yields constraints that are satisfied precisely when D is $(0, 2)$. With D and Q both on the y-axis, the area of triangle ABQ can be expressed in terms of the intersection coordinates. Using the ellipse equation and Vieta's theorem, the maximum area is found to be $\frac{8\sqrt{5}}{5}$, achieved when the line l has an appropriate slope.

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Author Biography: Guokun Zhang (born May 1967) is a senior-level teacher, special-grade teacher, master's supervisor, recipient of the Yunnan Provincial Government Special Allowance, and winner of the First Prize of the 12th Su Buqing Mathematics Education Award. His research focuses on high school mathematics education and pedagogy.

Note: Figure translations are in progress. See original paper for figures.