Einstein-Maxwell Gravity with Magnetic Moment Term

Authors: Li Desheng, Li Desheng

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Abstract

Einstein-Maxwell gravity theory is well-defined and familiar to many theorists. This paper presents a systematic investigation of Einstein-Maxwell gravity theory incorporating a magnetic moment term. We introduce Einstein-Maxwell gravity theory coupled to an SU(4) gauge field magnetic moment term and analyse its non-vanishing electromagnetic components.

Full Text

Preamble

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Desheng LI[1*]

[1]College of Electrical and Information Engineering, Hunan Institute of Engineering, Xiangtan 411000, P. R. China

Abstract

Einstein-Maxwell gravity theory is well-defined and familiar to many theorists. What about a coherent consideration of Einstein-Maxwell gravity theory with a magnetic moment term? This paper provides a first introduction to Einstein-Maxwell gravity theory with an SU(4) gauge field magnetic moment term and analyzes the nonzero electromagnetic components of the SU(4) gauge field magnetic moment term in conjunction with Einstein-Maxwell gravity theory.

Contents

lideshengjy@126.com

1 Introduction

Einstein-Maxwell gravity theory is a renowned gravitational theory, where variation of the Einstein-Hilbert term yields the Einstein tensor, and variation of the Maxwell term yields the Maxwell energy-momentum tensor. Is it possible to coherently consider Einstein-Maxwell gravity theory with a magnetic moment term? The Pati-Salam model [1, 2] in curved spacetime [3, 4] and Einstein-Cartan gravity [3, 5] provide possible candidates for a unified field theory (UFT). When torsion vanishes, Einstein-Cartan gravity reduces to Einstein gravity. The relationship between Einstein gravity and the SU(4) gauge field magnetic moment term is intriguing [3]. In particular, the coupling between the electromagnetic components of the SU(4) gauge field magnetic moment term and the gravitational field is both interesting and important.

As a preliminary introduction to Einstein-Maxwell gravity theory with a magnetic moment term, Section 2 analyzes the relationship between Newtonian gravity and Einstein gravity. Section 3 presents the Einstein-Maxwell gravity action and the associated Einstein gravity equations. Section 4 introduces the action for Einstein-Maxwell gravity with an SU(4) gauge field magnetic moment term and analyzes the nonzero coupling between the electromagnetic components of this magnetic moment term.

2.1 Newtonian Gravity

The Newtonian gravity equation is given by

$$\vec{F} = -\frac{Mm}{r^2}\vec{e}_r, \tag{2.1}$$

where M is the mass of a large object, m is the mass of a pointlike particle, r is the distance between the center of mass M and the pointlike particle, and $\vec{e}_r$ is the unit vector in the direction of $\vec{r}$. The Newtonian gravitational force is a central force, making it a conservative field with a corresponding Newtonian gravitational potential satisfying $\vec{F} = -\vec{\nabla}V$.

For the gravitational potential, the integral form of Gauss's theorem gives

$$\int \vec{\nabla}V \cdot d\vec{S} = 4\pi Mm = 4\pi m \int \rho \, dx \, dy \, dz, \tag{2.2}$$

where ρ is the spatial mass density of the object with mass M. For Newton's gravitational potential, the differential form of Gauss's theorem is

$$\nabla^2 V = 4\pi m\rho, \tag{2.3}$$

which is mathematically equivalent to Newton's law of gravitation. The Newtonian gravitational acceleration produced by an object of mass M on any test particle of mass m is

$$\vec{g} = -\frac{M}{r^2}\vec{e}_r. \tag{2.4}$$

We can therefore define a potential function independent of the test mass m such that $\vec{g} = -\vec{\nabla}\phi$.

Thus Newton's law of gravitation can be rewritten as

$$\nabla^2 \phi = 4\pi\rho. \tag{2.5}$$

Newton's second law gives

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{x}}{dt^2}, \tag{2.6}$$

where $\vec{x} = (x, y, z) = (x_1, x_2, x_3) = (x_i)$ with $i = 1, 2, 3$. According to Einstein's equivalence principle, the gravitational mass of an object is equivalent to its inertial mass. The Newtonian gravitational acceleration field provides acceleration for the test particle. Therefore, the equation of motion for any test particle in the potential function ϕ is given by

$$\vec{g} = \vec{a}. \tag{2.7}$$

This yields

$$\frac{d^2\vec{x}}{dt^2} + \vec{\nabla}\phi = 0. \tag{2.8}$$

In component form, this becomes

$$\frac{d_i^{2x}}{dt^2} + \frac{\partial\phi}{\partial x_i} = 0. \tag{2.9}$$

2.2 From Newtonian Gravity to Einstein Gravity

Einstein's geometric theory of gravity, general relativity, posits that the gravitational field is the curvature of spacetime. The trajectory of a test particle follows geodesics in this curved spacetime.

The metric of Riemannian geometry in general relativity is written as

$$ds^2 = -g_{\mu\nu}dx^\mu \otimes dx^\nu, \quad \mu, \nu = 0, 1, 2, 3. \tag{2.10}$$

where $x^\mu = (x^0, x^i) = (x^0, x^1, x^2, x^3)$, and c is the speed of light. The line element is the square root of the metric:

$$\sqrt{-g_{\mu\nu} dx^\mu \otimes dx^\nu}. \tag{2.11}$$

The length of any path λ_{AB} on a hypersurface with endpoints A and B is defined by the integration of the line element:

$$s = \int_{\lambda_{AB}} \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda. \tag{2.12}$$

Taking an extremal path from the infinite number of possible paths between two points, the variational principle for the path length λ_{AB} gives us the geodesic equation:

$$\delta s = 0 \implies \frac{d^{2x^\mu}}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0, \tag{2.13}$$

where $\Gamma^\mu_{\nu\rho}$ is the Christoffel connection. The parameter τ parameterizes the geodesic, and $dx^\mu/d\tau$ is the tangent vector of the geodesic. The Christoffel connection is determined by the metric:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}). \tag{2.14}$$

We choose the parameter $\tau = t + b$, allowing the equation to be rewritten as

$$\frac{d^{2x^i}}{dt^2} + \Gamma^i_{00} = 0. \tag{2.15}$$

Comparing equation (2.15) with equation (2.13), we find

$$\Gamma^i_{00} = -\frac{\partial \phi}{\partial x^i}, \tag{2.16}$$

where the gravitational acceleration is $\vec{g} = (g_x, g_y, g_z) = (g_1, g_2, g_3)$.

The curvature can be defined through the Christoffel connection:

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\tau\rho} \Gamma^\tau_{\nu\sigma} - \Gamma^\mu_{\tau\sigma} \Gamma^\tau_{\nu\rho}. \tag{2.17}$$

Equation (2.17) yields

$$R^i_{0j0} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}, \quad i, j = 1, 2, 3. \tag{2.18}$$

The Ricci tensor component is

$$R_{00} = R^i_{0i0} = \nabla^2 \phi = \frac{4\pi G \rho}{c^2}. \tag{2.19}$$

There exists the Schwarzschild solution to Einstein's gravity equations, which describes the spacetime metric outside a massive star:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.20}$$

From the Schwarzschild solution, the relationship between the gravitational potential ϕ and the metric can be found:

$$g_{00} = 1 + 2\phi. \tag{2.21}$$

3 Einstein Gravity with Maxwell Term

The action of Einstein-Maxwell theory is

$$S = \int d^{4x}\sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right), \tag{3.1}$$

where R is the Einstein-Hilbert term, $F^{\mu\nu}F_{\mu\nu}$ is the Maxwell term, and $\sqrt{-g}$ is the volume density with $g = \det(g_{\mu\nu})$.

Variation of the action (3.1) yields the Einstein gravity equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}, \tag{3.2}$$

with the energy-momentum tensor given by

$$T_{\mu\nu} = \left(g_{\mu\rho}F^{\rho\sigma}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^{\sigma\rho}F_{\sigma\rho}\right). \tag{3.3}$$

Here $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature, and the electromagnetic field strength tensor $F_{\mu\nu}$ is

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{3.4}$$

4 Einstein-Maxwell Gravity with SU(4) Gauge Field Magnetic Moment Term

The action of Einstein-Maxwell gravity with an SU(4) gauge field magnetic moment term is

$$S = \int d^{4x}\sqrt{-g}\left(R + i\,\mathrm{tr}\left(H_{ab}(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger})\right) - \frac{1}{4}\mathrm{tr}(H^{\mu\nu}H_{\mu\nu})\right), \tag{4.1}$$

where V_μ is the SU(4) gauge field $V_\mu = V_\mu^\alpha T^\alpha$ ($\alpha = 1, 2, \cdots, 15$), and $H_{\mu\nu}$ is the SU(4) gauge field strength tensor:

$$H_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu], \tag{4.2}$$

$$H_{\mu\nu} = H^{\alpha}_{\mu\nu} T^{\alpha} \quad (\alpha = 1, 2, \cdots, 15). \tag{4.3}$$

The term $\mathrm{tr}(H^{\mu\nu} H_{\mu\nu})$ is the Yang-Mills term [6] for the SU(4) gauge field, which represents the non-Abelian generalization of the Maxwell term.

According to Appendices A and B, the relationships between the 15 Hermitian matrices T^{α} constructed from Dirac matrices and the generators of the U(4) group are found. In particular, the photon field component of the gauge field (4.2) is interesting. The electromagnetic field strength tensor corresponds to T^{15}:

$$F_{\mu\nu} = H^{15}_{\mu\nu} = \partial_{\mu} V^{15}_{\nu} - \partial_{\nu} V^{15}_{\mu}, \tag{4.4}$$

with the generator T^{15} of the photon gauge field V^{15}_{μ}:

$$T^{15} = \frac{1}{2\sqrt{6}} \mathrm{diag}(1, 1, 1, -3). \tag{4.5}$$

The Lagrangian density of Einstein-Maxwell theory with electromagnetic components of the SU(4) gauge field magnetic moment term is

$$\mathcal{L} = \sqrt{-g} \left(R + i F_{ab} \mathrm{tr} \left(T^{15} (\gamma^a \gamma^b - \gamma^{b\dagger} \gamma^{a\dagger}) \right) - \frac{1}{4} (F^{\mu\nu} F_{\mu\nu}) \right). \tag{4.6}$$

The nonzero part of $\gamma^a \gamma^b - \gamma^{b\dagger} \gamma^{a\dagger}$ can be preliminarily written as

$$\gamma^a \gamma^b - \gamma^{b\dagger} \gamma^{a\dagger} = 2\gamma^{[A} \gamma^{B]}, \quad A, B = 1, 2, 3. \tag{4.7}$$

Note that the trace

$$\mathrm{tr} \left(T^{15} (\gamma^a \gamma^b - \gamma^{b\dagger} \gamma^{a\dagger}) \right) = \epsilon_{AB} \delta^{aA} \delta^{bB}, \quad A, B = 1, 2, \tag{4.8}$$

is nonzero. The Lagrangian density of Einstein-Maxwell theory with the electromagnetic moment term is therefore

$$\mathcal{L} = \sqrt{-g} \left(R + F_{AB} \epsilon^{AB} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right). \tag{4.9}$$

A Dirac Matrices and Hermitian Matrices

The Weyl representation of Dirac matrices is

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{A.1}$$

where σ^i are the Pauli matrices.

The 15 Hermitian matrices can be constructed from Dirac matrices:

$$T^1 = \gamma^0, \quad T^2 = i\gamma^1, \quad T^3 = i\gamma^2, \quad T^4 = i\gamma^3,$$

$$
\begin{gathered}
T^5 = -\gamma^0\gamma^1, \quad T^6 = -\gamma^0\gamma^2, \quad T^7 = -\gamma^0\gamma^3, \\
T^8 = -i\gamma^1\gamma^2, \quad T^9 = -i\gamma^1\gamma^3, \quad T^{10} = -i\gamma^2\gamma^3, \\
T^{11} = -i\gamma^0\gamma^1\gamma^2, \quad T^{12} = -i\gamma^0\gamma^1\gamma^3, \\
T^{13} = -i\gamma^0\gamma^2\gamma^3, \quad T^{14} = \gamma^1\gamma^2\gamma^3, \\
T^{15} = -i\gamma^0\gamma^1\gamma^2\gamma^3.
\end{gathered} \tag{A.2}
$$

These 15 Hermitian matrices can be written explicitly as:

$$
\begin{gathered}
T^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
T^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad T^4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
T^5 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad T^6 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \\
T^7 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T^8 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \\
T^9 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad T^{10} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \\
T^{11} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T^{12} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
T^{13} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad T^{14} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
T^{15} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.
\end{gathered} \tag{A.3}
$$

B Generators of SU(4) Group

The generators of SU(4) can be represented as:

$$\mathcal{T}^1=\frac{1}{2}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \mathcal{T}^2=\frac{1}{2}\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\mathcal{T}^3=\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \mathcal{T}^4=\frac{1}{2}\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\mathcal{T}^5=\frac{1}{2}\begin{pmatrix}0&0&-i&0\\0&0&0&0\\i&0&0&0\\0&0&0&0\end{pmatrix},\quad \mathcal{T}^6=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix},$$

$$\mathcal{T}^7=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{pmatrix},\quad \mathcal{T}^8=\frac{1}{2\sqrt{3}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&0\\0&0&0&0\end{pmatrix},$$

$$\mathcal{T}^9=\frac{1}{2}\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix},\quad \mathcal{T}^{10}=\frac{1}{2}\begin{pmatrix}0&0&0&-i\\0&0&0&0\\0&0&0&0\\i&0&0&0\end{pmatrix},$$

$$\mathcal{T}^{11}=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix},\quad \mathcal{T}^{12}=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&-i\\0&0&0&0\\0&i&0&0\end{pmatrix},$$

$$\mathcal{T}^{13}=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\quad \mathcal{T}^{14}=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix},$$

$$\mathcal{T}^{15}=\frac{1}{2\sqrt{6}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{pmatrix}. \tag{B.1}$$

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