Contribution of Wiggler to Radiation Integral in an Electron Storage Ring

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Abstract

With the development of accelerator light sources, the application of wiggler could become more and more important, for example to speed up damping or generate synchrotron radiation. The quantum excitation contribution of such a wiggler to electron beam emittance in a storage ring should be carefully evaluated when small emittance is desired. What found in literature is an approximate formula. Here we present a more exact result.

Full Text

Preamble

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Abstract

With the advancement of accelerator light sources, wigglers have become increasingly important for applications such as accelerating radiation damping or generating synchrotron radiation. When small emittance is desired, the quantum excitation contribution of such wigglers to the electron beam emittance in a storage ring must be carefully evaluated. Existing literature provides only an approximate formula for this contribution. Here we present a more exact result that will be valuable for future light source development. This paper aims to provide an accurate formula for evaluating the quantum excitation of a wiggler on beam emittance in an electron storage ring. Our motivation stems from the fact that wigglers may become essential components in future storage ring-based

light sources, for instance, to combat intra-beam scattering by significantly increasing the natural radiation damping rate, where ultrasmall transverse or longitudinal emittance and high beam current are simultaneously desired. Along with radiation damping comes quantum excitation from the wiggler, which may not be negligible in determining the equilibrium beam emittance, especially when the wiggler field is strong and its length is long. While an approximate radiation integral formula can be found in the literature to describe the wiggler's quantum excitation contribution, the demand for ever-smaller emittance necessitates an accurate formula that gives the exact dependence of the radiation integral on various optical functions at the wiggler. This paper fills that gap. In the following discussion, we focus on the case of a planar uncoupled ring, which represents the typical setup for present synchrotron radiation sources. Note, however, that the presented method applies to a general coupled lattice [1]. Similarly, although we use the horizontal plane for our example analysis, the method also applies to the vertical and longitudinal planes. Furthermore, the method can be used to evaluate the quantum excitation contribution of other bending-related elements.

Main Content

We now present the detailed derivation. Typically, the central part of a wiggler exhibits a sinusoidal field strength pattern along the longitudinal axis s. We set $s = 0$ to be the location of the peak magnetic field closest to the wiggler center. The vertical magnetic field of a horizontal planar wiggler is then given by $B_y = B_{0w}\cosh(k_x x)\cosh(k_y y)\cos(k_w s)$, where B_{0w} is the peak magnetic field and $k_w = 2\pi/\lambda_w$ is the wiggler wavenumber. In this paper, we use the particle state vector $\mathbf{X} \equiv (x, x', y, y', z, \delta)^T$, whose components represent horizontal position, horizontal angle, vertical position, vertical angle, longitudinal position, and relative energy deviation with respect to the reference particle, respectively. The superscript T denotes transpose.

The linear transfer matrix of $\mathbf{X}$ from $s = 0$ to $s \in [-L_w/2, L_w/2]$, where L_w is the wiggler length, is then given by [2]:

$$\mathbf{W}(s|0) = \begin{pmatrix} \cos(k_y s) & \frac{1}{k_y}\sin(k_y s) & 0 & 0 & 0 & -\frac{K}{k_y^2}[1-\cos(k_w s)] \\ -\frac{k_y}{\gamma}\sin(k_y s) & \cos(k_y s) & 0 & 0 & 0 & -\frac{K}{\gamma k_y}\sin(k_w s) \\ 0 & 0 & \cos(k_y s) & \frac{1}{k_y}\sin(k_y s) & 0 & 0 \\ 0 & 0 & -\frac{k_y}{\gamma}\sin(k_y s) & \cos(k_y s) & 0 & 0 \\ W_{51} & W_{52} & 0 & 0 & 1 & W_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $W_{51} = -W_{26}$, $W_{52} = -sW_{26} + W_{16}$, and $W_{56} = \frac{2\lambda_0}{\lambda_w}$ with $\lambda_0 = \frac{1+K^2/2}{2\gamma^2}\lambda_w$ being the fundamental on-axis resonant wavelength. Here γ is the Lorentz factor, which for the relativistic cases of interest is much larger than 1. The dimensionless undulator parameter of the wiggler is $K = \frac{eB_{0w}}{m_e c k_w}$, where e is the elementary charge, m_e is the electron mass, and c is the speed of light in free space.

In a planar uncoupled ring, the normalized eigenvector of the storage ring one-turn map corresponding to the horizontal eigenmode at $s = 0$ can be expressed as:

$$\mathbf{E}_I(0) = \left(-\alpha_{x0} + i,\ -(\alpha_{x0} D_{x0} + \beta_{x0} D'_{x0}) + i D_{x0},\ 0,\ 0,\ \frac{E_{I5}(0)}{E_{I1}(0)},\ 1\right)^T e^{i\Phi_{I0}}$$

where α_{x0}, β_{x0} are the Courant-Snyder functions and D_{x0}, D'_{x0} are the dispersion and dispersion angle corresponding to the horizontal plane at $s = 0$, and i is the imaginary unit. The chromatic function H_{x0} at $s = 0$ is defined as $H_{x0} \equiv 2|E_{I5}(0)|^2 = \gamma_{x0} D_{x0}^2 + 2\alpha_{x0} D_{x0} D'_{x0} + \beta_{x0} {D'}^2_{x0}$, where $\gamma_{x0} = \frac{1+\alpha_{x0}^2}{\beta_{x0}}$.

The evolution of H_x from $s = 0$ to $s \in [-L_w/2, L_w/2]$ is given by $H_x(s) \equiv 2|E_{I5}(s)|^2 = 2|(\mathbf{W}(s|0)\mathbf{E}_I(0))_5|^2$, which expands to:

$$H_x(s) = \frac{(D_{x0} + W_{16} - W_{26} s)^2 + [\alpha_{x0}(D_{x0} + W_{16} - W_{26} s) + \beta_{x0}(D'_{x0} + W_{26})]^2}{\beta_{x0}}$$

where W_{ij} denotes the matrix element in the i-th row and j-th column of $\mathbf{W}(s|0)$. Substituting the explicit expressions for the wiggler matrix elements yields:

$$H_x(s) = \frac{\left[D_{x0}\rho_w k_w^2 + \beta_{x0} k_w \left(\rho_w k_w D'_{x0} - \sin(k_w s)\right) + \alpha_{x0}\left(D_{x0}\rho_w k_w^2 + \cos(k_w s) - 1\right)\right]^2 + \left[\alpha_{x0}\left(D_{x0}\rho_w k_w^2 + \frac{\sin(k}{k_w}\right.\right.}{\beta_{x0}}$$

where $\rho_w = \frac{\gamma m_e \beta c}{eB_{0w}}$ corresponds to the bending radius at the location of peak magnetic field, with $\beta = \sqrt{1 - 1/\gamma^2}$.

Assuming that the quantum excitation contribution from the entrance and exit regions of the wiggler, where the field strength in reality deviates from the ideal sinusoidal pattern, is much smaller than that from the central sinusoidal field region, the quantum excitation of a wiggler to the horizontal beam emittance can be evaluated by the integral [3]:

$$I_{5w} = \int_{-L_w/2}^{L_w/2} \frac{H_x(s)}{|\rho(s)|^3} ds = \int_{-L_w/2}^{L_w/2} \frac{H_x(s)}{|\rho_w \cos(k_w s)|^3} ds$$

Substituting the expression for $H_x(s)$ yields:

$$I_{5w} = \left[\beta_{x0} + \gamma_{x0}\frac{L_w^2}{12} + 5\rho_w^2 H_{x0} - \left(\frac{8}{\pi}\right)\rho_w(D_{x0} + \alpha_{x0}D_{x0} + \beta_{x0}D'_{x0})\right]\frac{L_w}{5\rho_w^2}R$$

where the integral R is defined as:

$$R = \int_{-N_w\pi}^{N_w\pi}\left[\frac{\sin(x)}{x} + \cos(x) - 1\right]^2 |\cos(x)|^3 dx$$

with $N_w = L_w/\lambda_w$ being the number of wiggler periods, which is assumed to be an integer. Equation (8) above represents the exact formula for the radiation integral I_5 of a wiggler in an electron storage ring. For a specific N_w, the value of R can be obtained straightforwardly by evaluating the integral in Eq. (9).

When $N_w \gg 1$, we have $R \approx 1/12$, and the expression simplifies to:

$$I_{5w} \approx \left[\langle\beta_x\rangle_w + 5\rho_w^2 H_{x0} - \left(\frac{8}{\pi}\right)\rho_w(D_{x0} + \alpha_{x0}D_{x0} + \beta_{x0}D'_{x0})\right]\frac{L_w}{5\rho_w^2}$$

where $\langle\beta_x\rangle_w$ is the average β_x along the wiggler: $\langle\beta_x\rangle_w = \int_{-L_w/2}^{L_w/2}\beta_x(s)ds = \beta_{x0} + \gamma_{x0}L_w^2/12$.

Let us denote $\chi_{x0} = \arg\left(\frac{E_{I5}(0)}{E_{I1}(0)}\right)$, where $E_{I5}(0)$ and $E_{I1}(0)$ represent the fifth and first components of the eigenvector in Eq. (3) and $\arg(\cdot)$ denotes the phase angle of a complex number. Then Eq. (10) can be rewritten as:

$$I_{5w} \approx \left[\langle\beta_x\rangle_w + 5\rho_w^2 H_{x0} - \left(\frac{8}{\pi}\right)\rho_w\sqrt{H_{x0}\beta_{x0}}\cos\left(\chi_{x0} - \frac{\pi}{4}\right)\right]\frac{L_w}{5\rho_w^2}$$

The first term in the bracket corresponds to the approximate formula found in the literature: $I_{5w,\text{intrinsic}} \approx L_w\langle\beta_x\rangle_w/(5\rho_w^2)$. This can be viewed as the intrinsic contribution of a wiggler to the radiation integral I_5, since intrinsic dispersion and dispersion angle are generated inside the wiggler even when $D_{x0} = 0$ and $D'_{x0} = 0$, as evident from the matrix terms W_{16} and W_{26}.

The second and third terms in the bracket arise from a nonzero H_{x0}. When D'_{x0} or D_{x0} is of order K/k_w, which can easily occur in a real lattice, the contribution from this nonzero H_{x0} can be comparable to or even larger than the first term and cannot be neglected. In such cases, the more accurate formula derived here should be used to calculate the wiggler's quantum excitation of beam emittance. When the third term is much smaller than the second term—that is, roughly when $H_{x0}\beta_{x0} \gg (8/5\pi)^2$—Eq. (12) can be further approximated as:

$$I_{5w} \approx \frac{L_w\langle\beta_x\rangle_w}{5\rho_w^2} + \frac{L_w H_{x0}}{\langle\beta_x\rangle_w}$$

where the first term accounts for the intrinsic contribution and the second term accounts for the nonzero H_{x0}.

From the above analysis, we see that the minimum I_{5w} is achieved when $\alpha_{x0} = 0$, $\beta_{x0} = \sqrt{12}\rho_w/K$, $D_{x0} = 0$, and $D'_{x0} = 0$ (i.e., $H_{x0} = 0$), yielding the minimal value:

$$I_{5w,\min} \approx \frac{L_w\sqrt{12}\rho_w}{5K\rho_w^2} = \frac{L_w\sqrt{12}}{5K\rho_w}$$

We believe that the calculation method and analysis presented here will be useful for optimizing wigglers in future accelerator light sources to achieve ultrasmall beam emittance.

References

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Note: Figure translations are in progress. See original paper for figures.