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Abstract

In this paper, the main aim is to demonstrate the boundedness for commutatorsof sharp maximal function in the context of the p-adic variable Lebesgue spaces and Morreyspaces. where the symbols of the commutators belong to the p-adic version of Lipschitz orBMO spaces. Moreover, the boundedness of commutators of fractional maximal operatorin the p-adic Morrey space is also given, where the symbols of the commutators belong tothe p-adic version of Lipschitz space, by which some new characterizations of Lipschitz andBMO spaces are obtained.

Full Text

Boundedness for Commutators of Some Maximal Functions on the p-adic Vector Space

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Abstract: In this paper, our main aim is to demonstrate the boundedness for commutators of the sharp maximal function in the context of p-adic variable Lebesgue spaces and Morrey spaces, where the symbols of the commutators belong to the p-adic version of Lipschitz or BMO spaces. Moreover, we also establish the boundedness of commutators of the fractional maximal operator in the p-adic Morrey space, where the symbols belong to the p-adic version of Lipschitz space, which yields some new characterizations of Lipschitz and BMO spaces.

Keywords: p-adic field; sharp maximal function; fractional maximal function; commutator; variable Lebesgue space; Lipschitz space; BMO space

AMS(2020) Subject Classification: 11E95, 42B25, 42B35, 26A33

Introduction and Main Results

For a prime number p, the p-adic field consists of $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm. Let $x = p^\gamma \frac{a}{b}$, where $x \in \mathbb{Q}$ and $\gamma \in \mathbb{Z}$, with a and b integers not divisible by p. Then the p-adic norm is defined by $|x|_p = p^{-\gamma}$ and satisfies the following properties: (i) $|x|_p \geq 0$, with $|x|_p = 0$ if and only if $x = 0$; (ii) $|xy|_p = |x|_p |y|_p$; (iii) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$, with equality when $|x|_p \neq |y|_p$.

From standard p-adic analysis, any p-adic number $x \neq 0$ can be written as $x = p^\gamma(a_0 + a_{1p} + a_{2p}^2 + \cdots) = p^\gamma \sum_{j=0}^{\infty} a_{jp}^j$, where $a_j \in \{0, \dots, p-1\}$ and $a_0 \neq 0$. Naturally, the aforementioned p-adic number x converges. For any vector $x = (x_1, x_2, \dots, x_n)$ with $x_i \in \mathbb{Q}_p$ $(i = 1, \dots, n)$, the p-adic norm is defined by $|x|_p = \max_{1\leq j\leq n} |x_j|_p$. Moreover, the p-adic ball is denoted by $B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\}$, where the center $a \in \mathbb{Q}_p^n$ and radius p^γ with $\gamma \in \mathbb{Z}$. The p-adic sphere is written as $S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\} = B_\gamma(a) \quad B_{\gamma-1}(a)$. It is easy to see that $B_\gamma(a) = \bigcup_{k\leq\gamma} S_k(a)$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists a Haar measure on $\mathbb{Q}_p^n$. It is easy to verify that the unique Haar measure dx on $\mathbb{Q}_p^n$ (up to positive constant multiple) satisfies translation invariance. Normalizing the measure dx by $\int_{B_0(0)} dx = |B_0(0)|_h = 1$, we have $\int_{B_\gamma(a)} dx = |B_\gamma(a)|_h = p^{n\gamma}$ and $\int_{S_\gamma(a)} dx = |S_\gamma(a)|_h = p^{n\gamma}(1 - p^{-n}) = |B_\gamma(a)|_h - |B_{\gamma-1}(a)|_h$ for any $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$, where $|B_0(0)|_h$ denotes the Haar measure of the p-adic unit ball. For more details about p-adic analysis, see [**?**, **?**].

The study of p-adic harmonic analysis holds significant research importance and occupies a pivotal position in mathematics. It plays an indispensable role in enhancing our understanding of various branches such as number theory, algebraic geometry, and representation theory [**?**, **?**, **?**]. Additionally, p-adic harmonic analysis exhibits broad prospects for practical applications, possessing potential value in areas like cryptography, signal processing, and data analysis [**?**, **?**].

Let T be the classical singular integral operator. The Coifman-Rochberg-Weiss type commutator $[b, T]$ generated by T and a suitable function b is defined by $[b, T]f = bT(f) - T(bf)$. In [**?**, **?**], it was shown that $[b, T]$ is bounded on $L^s(\mathbb{R}^n)$ $(1 < s < \infty)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Janson [**?**] also established characterizations of the Lipschitz function space $\Lambda_\beta(\mathbb{R}^n)$ via the commutator and proved that $[b, T]$ is bounded from $L^s(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < s < n/\beta$ and $1/q = 1/s - \beta/n$ $(0 < \beta < 1)$ if and only if $b \in \Lambda_\beta(\mathbb{R}^n)$ (see also Paluszyński [**?**]).

For $0 \leq \alpha < n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$, the p-adic fractional maximal function of f is defined by $M_{\alpha,p}(f)(x) = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y)|\, dy$, where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$. For $\alpha = 0$, we write $M_p = M_{0,p}$. If $b \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$, the fractional maximal commutator generated by b with $M_{\alpha,p}$

is given by $M^b_{\alpha,p}(f)(x) = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(x) - b(y)||f(y)|\,dy$. For $\alpha = 0$, we write $M^b_{0,p}$. The commutator generated by b with $M_{\alpha,p}$ is defined by $[b, M_{\alpha,p}](f)(x) = b(x)M_{\alpha,p}(f)(x) - M_{\alpha,p}(bf)(x)$. For $\alpha = 0$, we write $[b, M_p] = [b, M_{0,p}]$.

Kim [?] introduced the p-adic sharp maximal function $M^\sharp_p(f)(x) = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|\,dy$, where $f_{B_\gamma(x)}$ is the average of f over $B_\gamma(x)$. The nonlinear commutator generated by b with $M^\sharp_p$ is defined by $[b, M^\sharp_p](f)(x) = b(x)M^\sharp_p(f)(x) - M^\sharp_p(bf)(x)$. It is worth noting that the operators $M^b_{\alpha,p}$ and $[b, M_{\alpha,p}]$ essentially differ from each other. Indeed, $M^b_{\alpha,p}$ is positive and sublinear, but $[b, M_{\alpha,p}]$ is neither positive nor sublinear. The same holds for $[b, M^\sharp_p]$.

In the Euclidean setting, many researchers have studied the operators $M^b_{\alpha,p}$, $[b, M_{\alpha,p}]$, and $[b, M^\sharp_p]$; see [?, ?] for instance. We can study analogous results on p-adic fields by borrowing similar methods.

Recently, when b belongs to Lipschitz spaces, the authors [?, ?] gave necessary and sufficient conditions for the boundedness of the commutators in variable Lebesgue spaces and Morrey spaces. When b belongs to BMO spaces, similar results were obtained in [?].

Inspired by the above literature [?], we focus on studying the boundedness for commutators of the sharp maximal function in p-adic variable Lebesgue spaces and Morrey spaces. Moreover, we also establish the boundedness of commutators of the fractional maximal operator in the p-adic Morrey space, which yields some new characterizations of Lipschitz and BMO spaces.

To introduce the following theorem, we define $b^-(x) = \begin{cases} |b(x)|, & \text{if } b(x) < 0 \\ 0, & \text{if } b(x) \geq 0 \end{cases}$ and $b^+(x) = |b(x)| - b^-(x)$. The following result gives boundedness for the commutator of the sharp maximal function on p-adic variable Lebesgue spaces and provides new characterizations of BMO spaces.

Theorem 1.1. Assume that $b \in L^1_{\mathrm{loc}}(\mathbb{Q}^n_p)$ and $b^- \in L^\infty(\mathbb{Q}^n_p)$. Then the following statements are equivalent: 1. $b \in \mathrm{BMO}(\mathbb{Q}^n_p)$; 2. $[b, M^\sharp_p]$ is bounded on $L^{q(\cdot)}(\mathbb{Q}^n_p)$ for all $q(\cdot) \in C^{\log}(\mathbb{Q}^n_p)$ with $q(\cdot) \in \mathcal{P}(\mathbb{Q}^n_p)$; 3. $[b, M^\sharp_p]$ is bounded on $L^{q(\cdot)}(\mathbb{Q}^n_p)$ for some $q(\cdot) \in C^{\log}(\mathbb{Q}^n_p)$ with $q(\cdot) \in \mathcal{P}(\mathbb{Q}^n_p)$; 4. There exists some $q(\cdot) \in C^{\log}(\mathbb{Q}^n_p)$ with $q(\cdot) \in \mathcal{P}(\mathbb{Q}^n_p)$ such that

$$\frac{\|b - \frac{p^2}{2(p-1)} M^\sharp_p(b\chi_{B_\gamma(x)})\|_{L^{q(\cdot)}(\mathbb{Q}^n_p)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}^n_p)}} \leq C; \tag{1}$$

5. For all $q(\cdot) \in C^{\log}(\mathbb{Q}^n_p)$ with $q(\cdot) \in \mathcal{P}(\mathbb{Q}^n_p)$, condition (1.5) holds.

Remark 1. If $q(\cdot)$ is a constant exponent, this result can be found in Theorem 1.4 of [**?**].

The following result gives boundedness for the commutator of the sharp maximal function on p-adic Morrey spaces and provides new characterizations of Lipschitz spaces.

Theorem 1.2. Assume that $b \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ and $0 < \beta < 1$. Then the following statements are equivalent: 1. $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$; 2. $[b, M_p^\sharp]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$ for all r, q with $1 < r < n/\beta$ and $1/q = 1/r - \beta/(n-\lambda)$; 3. $[b, M_p^\sharp]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$ for some r, q with $1 < r < n/\beta$ and $1/q = 1/r - \beta/(n-\lambda)$; 4. There exists some r, q with $1 < r < n/\beta$ and $1/q = 1/r - \beta/(n-\lambda)$ such that

$$\frac{\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} \leq C; \tag{2}$$

5. For all r, q with $1 < r < n/\beta$ and $1/q = 1/r - \beta/(n-\lambda)$, condition (1.6) holds.

Remark 2. In the Euclidean setting, see Theorem 1.3 of [**?**].

The following result gives boundedness for the commutator of the sharp maximal function on p-adic Morrey spaces and provides new characterizations of BMO spaces.

Theorem 1.3. Assume that $b \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ and $b^- \in L^\infty(\mathbb{Q}_p^n)$. Then the following statements are equivalent: 1. $b \in \text{BMO}(\mathbb{Q}_p^n)$; 2. $[b, M_p^\sharp]$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^n)$ for all q with $1 < q < \infty$; 3. $[b, M_p^\sharp]$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^n)$ for some q with $1 < q < \infty$; 4. There exists some q with $1 < q < \infty$ such that

$$\frac{\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} \leq C; \tag{3}$$

5. For all q with $1 < q < \infty$, condition (1.7) holds.

Theorem 1.4 (Spanne type result). Assume that b is a locally integrable function on $\mathbb{Q}_p^n$, and $0 \leq \alpha < \alpha + \beta < n$. Let $1 < r < n/(\alpha+\beta)$, $0 < \lambda < n - r(\alpha+\beta)$, and $1/q = 1/r - (\alpha+\beta)/n$ with $\lambda/r = \kappa/q$. Then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ if and only if $M_{\alpha,p}^b$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\kappa}(\mathbb{Q}_p^n)$.

Remark 3. When $\alpha = 0$, the above result can be found in Theorem 3 of [**?**].

Theorem 1.5 (Adams type result). Assume that b is a locally integrable function on $\mathbb{Q}_p^n$, and $0 < \alpha < \alpha + \beta < n$. Let $1 < r < n/(\alpha+\beta)$, $0 < \lambda < n - r(\alpha+\beta)$, and $1/q = 1/r - (\alpha+\beta)/(n-\lambda)$. Then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ if and only if $M_{\alpha,p}^b$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$.

Remark 4. For the case $\alpha = 0$, the above result can be obtained from Theorem 2 of [?].

Theorem 1.6 (Spanne type result). Assume that b is a locally integrable function on $\mathbb{Q}_p^n$, and $0 \leq \alpha < \alpha + \beta < n$. Let $1 < r < n/(\alpha+\beta)$, $0 < \lambda < n - r(\alpha+\beta)$, and $1/q = 1/r - (\alpha+\beta)/n$ with $\lambda/r = \kappa/q$. Then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$ if and only if $[b, M_{\alpha,p}]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\kappa}(\mathbb{Q}_p^n)$.

Remark 5. For the case $\alpha = 0$, the above result can be obtained from Theorem 7 of [?].

Theorem 1.7 (Adams type result). Assume that b is a locally integrable function on $\mathbb{Q}_p^n$, and $0 < \alpha < \alpha + \beta < n$. Let $1 < r < n/(\alpha+\beta)$, $0 < \lambda < n - r(\alpha+\beta)$, and $1/q = 1/r - (\alpha+\beta)/(n-\lambda)$. Then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$ if and only if $[b, M_{\alpha,p}]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$.

Remark 6. For the case $\alpha = 0$, the above result can be obtained from Theorem 6 of [?].

Throughout this paper, the letter C always denotes a constant independent of the primary parameters involved and whose value may differ from line to line. In addition, we introduce some notation. Here and hereafter, $|E|_h$ will always denote the Haar measure of a measurable set $E \subset \mathbb{Q}_p^n$, and χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{Q}_p^n$.

2 Preliminaries

2.1 p-adic Function Spaces

Let $1 \leq q < \infty$. A measurable function f is said to belong to the p-adic Lebesgue space $L^q(\mathbb{Q}_p^n)$ if

$$\|f\|_{L^q(\mathbb{Q}p^n)} = \left(\int_{\mathbb{Q}p^n} |f(x)|^q \, dx \right)^{1/q} < \infty.$$

Moreover, for $q = \infty$, we denote $L^\infty(\mathbb{Q}_p^n)$ as the set of all measurable real-valued functions f on $\mathbb{Q}_p^n$ satisfying

$$\|f\|_{L^\infty(\mathbb{Q}_p^n)} = \text{ess sup}|f(x)| = \inf\big\{\lambda > 0 : |\{x \in \mathbb{Q}_p^n : |f(x)| > \lambda\}|_h = 0\big\} < \infty.$$

Here, if the limit exists, the integral in the above equation is defined as follows:

$$\int_{\mathbb{Q}p^n} |f(x)|^q \, dx = \lim_{\gamma\to\infty} \int_{B_\gamma(0)} |f(x)|^q \, dx = \lim_{\gamma\to\infty} \sum_{-\infty<k\leq\gamma} \int_{S_k(0)} |f(x)|^q \, dx.$$

A measurable function $q(\cdot)$ is called a variable exponent if $q(\cdot) : \mathbb{Q}_p^n \to (0,\infty)$. In [?], the following definitions are introduced.

Definition 2.1. Given a measurable function $q(\cdot)$ defined on $\mathbb{Q}_p^n$, we denote $q^- := \text{ess inf}\, q(x)$ and $q^+ := \text{ess sup}\, q(x)$. We denote by $\mathcal{P}(\mathbb{Q}_p^n)$ the set of all measurable functions $q(\cdot) : \mathbb{Q}_p^n \to (1, \infty)$ such that $1 < q^- \le q(x) \le q^+ < \infty$ for all $x \in \mathbb{Q}_p^n$.

Definition 2.2 (p-adic variable exponent Lebesgue spaces). Let $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$. Define the p-adic variable exponent Lebesgue spaces $L^{q(\cdot)}(\mathbb{Q}_p^n)$ as follows:

$$L^{q(\cdot)}(\mathbb{Q}_p^n) = \{f \text{ is measurable} : F_q(f/\eta) < \infty \text{ for some constant } \eta > 0\},$$

where $F_q(f) := \int_{\mathbb{Q}p^n} |f(x)|^{q(x)}\, dx$. The Lebesgue space $L^{q(\cdot)}(\mathbb{Q}_p^n)$ is a Banach function space with respect to the Luxemburg norm

$$\|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} = \inf\left\{\eta > 0 : F_q(f/\eta) = \int_{\mathbb{Q}p^n} \left(\frac{|f(x)|}{\eta}\right)^{q(x)} dx \le 1\right\}.$$

Definition 2.3 (log-Hölder continuity). Let $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$. Denote by $C^{\log}(\mathbb{Q}_p^n)$ the set of all $q(\cdot)$ which satisfy

$$\gamma(q^-(B_\gamma(x)) - q^+(B_\gamma(x))) \le C$$

for all $\gamma \in \mathbb{Z}$ and any $x \in \mathbb{Q}_p^n$, where C denotes a universal constant. The set $C_\infty^{\log}(\mathbb{Q}_p^n)$ consists of all $q(\cdot)$ which satisfy

$$|q(x) - q(y)| \le \frac{C}{\log_p(p + \min\{|x|_p, |y|_p\})}$$

for any $x, y \in \mathbb{Q}_p^n$, where C denotes a universal constant. Denote by $C^{\log}(\mathbb{Q}_p^n) = C_{\text{loc}}^{\log}(\mathbb{Q}_p^n) \cap C_\infty^{\log}(\mathbb{Q}_p^n)$ the set of all global log-Hölder continuous functions $q(\cdot)$.

Kim [**?**] gave the following definition of the p-adic version of BMO space.

Definition 2.4. Let $f \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$. If $\|M_p^\sharp(f)\|_{L^\infty(\mathbb{Q}_p^n)} < \infty$, then we say that f is a function of bounded mean oscillation on $\mathbb{Q}_p^n$. We denote the space of such functions by $\text{BMO}(\mathbb{Q}_p^n) = \{f \in L^1_{\text{loc}}(\mathbb{Q}_p^n) : M_p^\sharp(f) \in L^\infty(\mathbb{Q}_p^n)\}$. For $f \in \text{BMO}(\mathbb{Q}_p^n)$, we write

$$\|f\|_{\text{BMO}(\mathbb{Q}_p^n)} = \|M_p^\sharp(f)\|_{L^\infty(\mathbb{Q}_p^n)} = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|\, dy,$$

where $f_{B_\gamma(x)}$ is the average of f over $B_\gamma(x)$.

The following result introduces the basic definition of p-adic Lipschitz spaces [**?**].

Definition 2.5. Let $0 < \beta < 1$. The p-adic version of homogeneous Lipschitz spaces $\Lambda_\beta(\mathbb{Q}_p^n)$ is defined by

$$\Lambda_\beta(\mathbb{Q}_p^n) := \{f \in L^1_{\text{loc}}(\mathbb{Q}_p^n) : \|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} = \sup_{x,y\in\mathbb{Q}p^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|_p^\beta}.$$

Remark 7. (1) Assume that $1 \leq q < \infty$. The p-adic version of homogeneous Lipschitz spaces $\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)$ is defined by

$$\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n) : \|f\|_{\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)} = \sup_{B_\gamma(x)} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|^q \, dy \right)^{1/q}.$$

(2) (see Lemma 6 of [?]) By virtue of Definition 2.5, for all $0 < \beta < 1$ and $1 \leq q < \infty$, we have $\Lambda_\beta(\mathbb{Q}_p^n) \approx \mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)$ with equivalent norms.

It is well known that the classical Morrey spaces were introduced by Morrey in [?] to study certain problems in second-order elliptic partial differential equations.

Definition 2.6 (Classic Morrey space). The p-adic version of Morrey space $L^{q,\lambda}(\mathbb{Q}_p^n)$ is defined, for $1 \leq q \leq \infty$ and $0 \leq \lambda \leq n$, as the space of all $f \in L^q_{\mathrm{loc}}(\mathbb{Q}_p^n)$ with finite norm

$$\|f\|_{L^{q,\lambda}} = \sup_{B_\gamma(x)} |B_\gamma(x)|^{-\frac{\lambda}{qn}} \|f\|_{L^q(\mathbb{Q}p^n)} < \infty.$$

2.2 Auxiliary Propositions and Lemmas

In this section we state some auxiliary propositions and lemmas needed for proving our main theorems, describing only the partial results we require.

First, the p-adic version of Hölder's inequality can be found in [?].

Lemma 2.1 (Generalized Hölder's inequality on $\mathbb{Q}_p^n$). Let $\mathbb{Q}_p^n$ be an n-dimensional p-adic vector space. Suppose that $q_1(\cdot), q_2(\cdot), r(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$ satisfy $\frac{1}{r(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$ almost everywhere. Then there exists a positive constant C such that for all $f \in L^{q_1(\cdot)}(\mathbb{Q}_p^n)$ and $g \in L^{q_2(\cdot)}(\mathbb{Q}_p^n)$, the inequality

$$\|fg\|_{L^{r(\cdot)}(\mathbb{Q}_p^n)} \leq C\|f\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\|g\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}$$

holds.

The authors in [?] obtained the following Lemmas 2.2 and 2.3.

Lemma 2.2. Let $0 < \beta < 1$ and $0 < \alpha < \alpha + \beta < n$. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, then for any $x \in \mathbb{Q}_p^n$, we have

$$M^b_{\alpha,p}(f)(x) \leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M_{\alpha+\beta,p}(f)(x).$$

Lemma 2.3. Let $0 < \alpha < n$. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$, then for any $x \in \mathbb{Q}_p^n$ such that $M_{\alpha,p}(f)(x) < \infty$, we obtain

$$|[b, M_{\alpha,p}](f)(x)| \leq M_{\alpha,p}^b(f)(x).$$

The following result derives from [**?**].

Lemma 2.4. Assume $0 < \alpha < n$, $1 < r < n/\alpha$, and $0 < \lambda < n - r\alpha$. 1. If $1/q = 1/r - \alpha/(n-\lambda)$, then $M_{\alpha,p}$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$; 2. If $1/q = 1/r - \alpha/n$ and $\lambda/r = \kappa/q$, then $M_{\alpha,p}$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\kappa}(\mathbb{Q}_p^n)$.

The following result can be found in [**?**].

Lemma 2.5. Let $q(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$. Then there exists a positive constant C such that for any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, the equality

$$|B_\gamma(x)|_h = \|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_\gamma(x)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}$$

holds.

He and Li [**?**] gave the norm of characteristic functions.

Lemma 2.6. Let $1 \leq q < \infty$ and $0 < \lambda < n$. Then

$$\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} = |B_\gamma(x)|^{\frac{n-\lambda}{qn}}.$$

3 Proof of the Principal Results

Proof of Theorem 1.1. Since the implications (2) $\implies$ (3) and (5) $\implies$ (4) follow readily, and (2) $\implies$ (5) is similar to (3) $\implies$ (4), we only need to prove (1) $\implies$ (2), (3) $\implies$ (4), and (4) $\implies$ (1).

For (1) $\implies$ (2): For any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, using (3.2) of [**?**] and $M_p^\sharp(f)(x) \leq 2M_p(f)(x)$, we obtain

$$|[b, M_p^\sharp](f)(x)| \leq 4(b^-(x)M_p(f)(x) + M_p(b^- f)(x)) + 2M_{|b|_p} f(x).$$

Note that $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$ implies $|b| \in \mathrm{BMO}(\mathbb{Q}_p^n)$. For $q(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, using Minkowski's inequality and the boundedness of M_p and $M_{|b|_p}$ on $L^{q(\cdot)}(\mathbb{Q}_p^n)$ [**?**, **?**], we get

$$\|[b, M_p^\sharp](f)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq C(\|b^-\|_{L^\infty(\mathbb{Q}_p^n)} + \|b\|_{\mathrm{BMO}(\mathbb{Q}_p^n)})\|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.$$

For (3) $\implies$ (4): For any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$ and $y \in \mathbb{Q}_p^n$, we have from [**?**]

$$M_p^\sharp(\chi_{B_\gamma(x)})(y) = \frac{2(p-1)}{p^2}.$$

Using (3) and the above identity,

$$\|b-\frac{p^2}{2(p-1)}M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \le \frac{p^2}{2(p-1)}\|[b,M_p^\sharp](\chi_{B_\gamma(x)})\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \le C\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.$$

For (4) $\implies$ (1): For any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$ and $y \in \mathbb{Q}_p^n$, we have from [?]

$$|b_{B_\gamma(x)}| \le \frac{p^2}{2(p-1)}M_p^\sharp(b\chi_{B_\gamma(x)})(y).$$

Let $E = \{y \in B_\gamma(x) : b(y) \le b_{B_\gamma(x)}\}$. For any $y \in E$, we have

$$b(y) \le b_{B_\gamma(x)} \le \frac{p^2}{2(p-1)}M_p^\sharp(b\chi_{B_\gamma(x)})(y),$$

and thus

$$|b(y) - b_{B_\gamma(x)}| \le |b(y) - \frac{p^2}{2(p-1)}M_p^\sharp(b\chi_{B_\gamma(x)})(y)|.$$

Using (4) and Lemmas 2.1 and 2.5, we obtain

$$|B_\gamma(x)|_h \int_{B_\gamma(x)} |b(y)-b_{B_\gamma(x)}|\, dy \le |B_\gamma(x)|_h \int_{B_\gamma(x)} |b(y)-\frac{p^2}{2(p-1)}M_p^\sharp(b\chi_{B_\gamma(x)})(y)|\, dy,$$

which implies $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$.

Next, we further prove $b^- \in L^\infty(\mathbb{Q}_p^n)$. For any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, if $y \in B_\gamma(x)$, using the previous inequality we obtain

$$\frac{p^2}{2(p-1)}M_p^\sharp(b\chi_{B_\gamma(x)})(y) - b(y) \ge |b_{B_\gamma(x)}| - b^+(y) + b^-(y).$$

Integrating over $B_\gamma(x)$ and applying the p-adic version of the Lebesgue differentiation theorem as $\gamma \to -\infty$, we get $|b(y)| - b^+(y) + b^-(y) = 2b^-(y) \le C$, which yields $b^- \in L^\infty(\mathbb{Q}_p^n)$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Since the implications (2) $\implies$ (3) and (5) $\implies$ (4) follow readily, and (2) $\implies$ (5) is similar to (3) $\implies$ (4), we only need to prove (1) $\implies$ (2), (3) $\implies$ (4), and (4) $\implies$ (1).

For (1) $\implies$ (2): Suppose $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \ge 0$. For any $x \in \mathbb{Q}_p^n$, it follows from (3.1) and Theorem 2 of [?] that $[b, M_p^\sharp]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$.

For (3) $\implies$ (4): For any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, using assertion (3), the identity for $M_p^\sharp(\chi_{B_\gamma(x)})$, and Lemma 2.6, we obtain

$$\|b-\frac{p^2}{2(p-1)}M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} \le \frac{p^2}{2(p-1)}\|[b,M_p^\sharp](\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} \le C|B_\gamma(x)|^{\beta/(n-\lambda)}\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}.$$

For (4) $\implies$ (1): For any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$ and $y \in \mathbb{Q}_p^n$, we have from [?]

$$|b_{B_\gamma(x)}| \leq \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y).$$

Let $E = \{y \in B_\gamma(x) : b(y) \leq b_{B_\gamma(x)}\}$. For any $y \in E$,

$$|b(y) - b_{B_\gamma(x)}| \leq |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)|.$$

Using Lemma 2.6 and the fact that $1/r = 1/q + \beta/(n-\lambda)$, we obtain

$$\left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - b_{B_\gamma(x)}|^q \, dy\right)^{1/q} \leq C|B_\gamma(x)|^{\beta/n} \frac{\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} \leq C|B_\gamma(x)|^{\beta/n}.$$

Using Remark 7(2), this implies $b \in \Lambda_\beta(\mathbb{Q}_p^n)$.

Next, we need to prove $b \geq 0$, i.e., $b^- = 0$. On the one hand, note that from (3.3) of [?],

$$|B_\gamma(x)|_h \int_{B_\gamma(x)} |\frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y) - b(y)| \, dy \geq |b_{B_\gamma(x)}| - |B_\gamma(x)|_h \int_{B_\gamma(x)} b^+(y) \, dy + |B_\gamma(x)|_h \int_{B_\gamma(x)} b^-(y) \, dy.$$

On the other hand, applying Hölder's inequality and the previous estimate,

$$|B_\gamma(x)|_h \int_{B_\gamma(x)} |\frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y) - b(y)| \, dy \leq C|B_\gamma(x)|^{\beta/n} \|\chi_{B_\gamma(x)}\|_{L^{q',\lambda}(\mathbb{Q}_p^n)} \frac{\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}$$

Combining these inequalities and letting $\gamma \to -\infty$ with $y \in B_\gamma(x)$, by the p-adic version of the Lebesgue differentiation theorem we obtain $|b(y)| - b^+(y) + b^-(y) = 2b^-(y) = 0$, which completes the proof of Theorem 1.2.

Remark 8. For the proof of Theorem 1.2, note that the estimation method differs from Theorem 2.1 in [?], which is briefly described as follows: (i) In (1) $\Rightarrow$ (2), the approach not only requires b to be non-negative but is also relatively simple; (ii) In (2) $\Rightarrow$ (3), the norm estimate we deduce from boundedness differs from Theorem 2.1 of [?]\$; (iii) In (4) $\Rightarrow$ (1), we avoid using Hölder's inequality in combination with existing results, which is more helpful for understanding the proof.

Proof of Theorem 1.3. Since the implications (2) $\implies$ (3) and (5) $\implies$ (4) follow readily, and (2) $\implies$ (5) is similar to (3) $\implies$ (4), we only need to prove (1) $\implies$ (2), (3) $\implies$ (4), and (4) $\implies$ (1).

For (1) $\implies$ (2): Suppose $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$ and $b^- \in L^\infty(\mathbb{Q}_p^n)$. For any $x \in \mathbb{Q}_p^n$, it follows from (3.1) and Theorem 1.5 of [?] that $[b, M_p^\sharp]$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^n)$.

For (3) $\implies$ (4): For any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, using assertion (3) and the identity for $M_p^\sharp(\chi_{B_\gamma(x)})$, we obtain

$$\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} \leq \frac{p^2}{2(p-1)} \|[b, M_p^\sharp](\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} \leq C\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}.$$

For (4) $\implies$ (1): By virtue of Lemma 2.6 and the previous inequality, we have

$$\left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)|^q \, dy\right)^{1/q} \leq \frac{\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} \leq C.$$

It follows from Theorem 1.4 of [**?**] that $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$ and $b^- \in L^\infty(\mathbb{Q}_p^n)$. Thus we obtain Theorem 1.3.

Proof of Theorem 1.4. Using a similar approach to the proof of Theorem 1.5, Theorem 1.4 can be proven; hence we omit the proof.

Proof of Theorem 1.5. On the one hand, by Lemmas 2.2 and 2.4, we can obtain the result; the proof can also be found in Theorem 2.2 of [**?**].

On the other hand, if $M_{\alpha,p}^b$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$, then for all fixed p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$ and any $y \in B_\gamma(x)$, we have

$$\left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - b_{B_\gamma(x)}|^q \, dy\right)^{1/q} \leq C|B_\gamma(x)|^{\frac{n+\lambda}{nq} - 1 - \frac{\alpha+\beta}{n}} \|\chi_{B_\gamma(x)}\|_{L^{r,\lambda}(\mathbb{Q}_p^n)} \leq C,$$

where the last step uses the fact that $1/q = 1/r - (\alpha + \beta)/(n - \lambda)$. It follows from Remark 7(2) that $b \in \Lambda_\beta(\mathbb{Q}_p^n)$.

Remark 9. Note that the converse direction is different from Theorem 2.2 in [**?**], where we avoid using Hölder's inequality in combination with existing results, which is more helpful for understanding the proof.

For a fixed p-adic ball B^*, the fractional maximal function with respect to B^* of a locally integrable function f is given by

$$M_{\alpha,B^*,p}(f)(x) = \sup_{B_\gamma(x) \subset B^*} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y)| \, dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x)$ with $B_\gamma(x) \subset B^*$.

The following result plays a role in the proof of Theorem 1.7; for details, see [**?**].

Lemma 3.1. Let b be a locally integrable function on $\mathbb{Q}_p^n$, $0 < \beta < 1$, and $0 < \alpha < \alpha + \beta < n$. Then the following statements are equivalent: 1. $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$; 2. For all $1 \leq q < \infty$,

$$\left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - |B_\gamma(x)|_h M_{\alpha,B_\gamma(x),p}(b)(y)|^q \, dy\right)^{1/q} \leq C|B_\gamma(x)|^{\beta/n};$$

3. For some s with $1 \leq s < \infty$, condition (3.11) holds.

Proof of Theorem 1.6. Using a similar approach to the proof of Theorem 1.7, Theorem 1.6 can be proven; hence we omit the proof.

Proof of Theorem 1.7. Since $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$, for any $x \in \mathbb{Q}_p^n$, using Lemma 2.3 we obtain that $[b, M_{\alpha,p}]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$; the proof can also be found in Theorem 2.2 of [**?**].

Conversely, suppose $[b, M_{\alpha,p}] : L^{r,\lambda}(\mathbb{Q}_p^n) \to L^{q,\lambda}(\mathbb{Q}_p^n)$. For any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$ and all $y \in B_\gamma(x)$, the following estimates were obtained in [**?**]:

$$M_{\alpha,p}(\chi_{B_\gamma(x)})(y) = |B_\gamma(x)|^{\alpha/n}, \quad M_{\alpha,p}(b\chi_{B_\gamma(x)})(y) = M_{\alpha,B_\gamma(x),p}(b)(y).$$

Therefore,

$$\left(\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)} |b(y) - |B_\gamma(x)|_h M_{\alpha,B_\gamma(x),p}(b)(y)|^q \, dy\right)^{1/q} \leq C|B_\gamma(x)|^{\frac{n+\lambda}{nq}-1-\frac{\alpha+\beta}{n}} \|\chi_{B_\gamma(x)}\|_{L^{r,\lambda}(\mathbb{Q}_p^n)} \leq C,$$

where the last step uses $1/q = 1/r - (\alpha+\beta)/(n-\lambda)$. By Lemma 3.1, we get $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$.

Remark 10. Note that the estimation method differs from Theorem 2.3 in [**?**], where we avoid using Hölder's inequality in combination with existing results, which is more helpful for understanding the proof.

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Conflict of Interest

The authors state that there is no conflict of interest.

Data Availability Statement

All data generated or analyzed during this study are included in this published article.

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Note: Figure translations are in progress. See original paper for figures.