A Random Integration Algorithm for High-dimensional Function Spaces

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Abstract

We introduce a novel random integration algorithm that boasts both high convergence order and polynomial tractability for functions characterized by sparse frequencies or rapidly decaying Fourier coefficients. Specifically, for integration in periodic isotropic Sobolev space and the isotropic Sobolev space with compact support, our approach attains a near-optimal root mean square error. In contrast to previous nearly optimal algorithms, our method exhibits polynomial tractability,ensuring that the number of samples does not scale exponentially with increasing dimensions. Our integration algorithm also enjoys near-optimal bound for weighted Korobov space. Furthermore, the algorithm can be applied without the need for prior knowledge of weights, distinguishing it from component-by-component algorithms. For integration in the Wiener algebra, the sample complexity of our algorithm is independent of the decay rate of Fourier coefficients. The effectiveness of the integration is confirmed through numerical experiments.

Full Text

Preamble

A Random Integration Algorithm for High-dimensional Function Spaces

Liang Chen∗, Minqiang Xu†, Haizhang Zhang‡

Abstract

We introduce a novel random integration algorithm that exhibits a high convergence order for functions characterized by sparse frequencies or rapidly decaying Fourier coefficients. Specifically, for integration in periodic isotropic

Sobolev spaces and isotropic Sobolev spaces with compact support, our approach achieves a nearly optimal root mean square error (RMSE) bound. In contrast to previous nearly optimal algorithms, our method exhibits polynomial tractability. Our integration algorithm also enjoys nearly optimal bound for weighted Sobolev space. By incorporating the trick of change of variable, our algorithm is proven to achieve the semi-exponential convergence order for the integration of analytic functions, which marks a significant improvement over the previously obtained super-polynomial convergence order. Furthermore, for integration involving Wiener-type functions, the sample complexity of our algorithm remains independent of the decay rate of the Fourier coefficients.

Keywords: Numerical integration; Monte Carlo; Curse of dimensionality; Sample complexity; Information-Based complexity

MSC 2020: 65D30, 68Q25, 65C05, 42B10

Introduction

This paper is concerned with numerically integrating multivariate functions defined over the d-dimensional unit cube. Denote

$$\mathrm{INT}(f) := \int_{[0,1]^d} f(x)dx.$$

Evaluation of $\mathrm{INT}(f)$ amounts to estimating the value of $\hat{f}(0)$, where the Fourier transform is defined as

$$\hat{f}(w) = \int_{[0,1]^d} f(x)\exp(-2\pi i w\cdot x)dx.$$

For a given positive integer N, we introduce the following discrete Fourier transform

$$\tilde{f}^{(N)}(w) = \sum_{z\in\mathbb{Z}_N^d} f(z/N)\exp(-2\pi i w\cdot z/N), \quad w\in\mathbb{Z}^d$$

where $\mathbb{Z}_N^d := \mathbb{Z}^d \cap [0,N)^d$.

Inspired by the sparse Fourier transform (SFT) from signal processing [**?**, **?**, **?**], our methodology for estimating $\hat{f}(0)$ comprises two fundamental stages. First, we create a hash map that effectively disperses the frequencies, ensuring that the energy of frequencies near zero (excluding zero itself) is small. Second, we employ a low-pass filter to extract the low-frequency components.

Here, we utilize the hash mapping A_N developed in our previous paper [**?**],

$$A_N := \begin{pmatrix} I_{d-1} & 0 \\ v^\top & h_d \end{pmatrix}$$

where A_N is a d by d matrix, N is some prime number, I_{d-1} represents the identity matrix of order $d-1$, $v := (h_1, h_2, \ldots, h_{d-1})$, and (v, h_d) is drawn from the uniform distribution on the set $\mathbb{Z}^d \cap [1, N)^d$. Note that the distribution here differs from the setting in [**?**]. Subsequently, we construct the following low-pass filter:

$$f_{A_N,L,r}(x) := \sum_{|l|\leq L} f(\{A_N^\top(x - y_l)\})G_{r,l},$$

where $\{t\} \in [0, 1)$ denotes the fractional part of t (for instance, $\{3.2\} = 0.2$, $\{-1.3\} = 0.7$), for a vector v, $\{v\}$ signifies the operation of taking the fractional part of each component of the vector individually, $y_l = (0, \ldots, 0, l/N)^\top$, and $G_{r,l} = \exp(-\frac{l^2}{2r^2})$.

The comprehensive attributes of $G_{r,l}$ are elaborated in Lemma 2.3. By letting $z = A_N^\top x$, and substituting it into (Eq. (4)), we derive the following critical integration approximation of $\mathrm{INT}(f)$

$$I(f)^H_{N,L,r,z} := \sum_{|l|\leq L} f_{N,\eta}\left(\left\{\frac{z - lH_N}{N}\right\}\right) G_{r,l},$$

where H_N and z are drawn from the uniform distribution over $\mathbb{Z}^d \cap [1, N)^d$ and $\mathbb{Z}^d \cap [0, N)^d$, respectively.

When employing other recent high-dimensional SFT methods for estimating numerical integration, as cited in references [**?**, **?**, **?**], it is not feasible to achieve the desired error bounds for functions in Sobolev spaces. Previously, Gilbert et al. [**?**] introduced an algorithmic framework for high-dimensional SFT, which could potentially be adapted to develop integration algorithms that attain the desired error bound. However, such a framework would increase the upper bound by a factor of d^5, as analyzed on page 61 of [**?**]. We remark that (Eq. (5)) exhibits a certain similarity to the quadrature formula introduced in [**?**].

Utilizing (Eq. (5)), along with the local Monte Carlo sampling (see (Eq. (13))) and the median trick [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**], we shall construct a novel randomized integration algorithm. Our method introduces a fresh understanding in the construction of numerical integration algorithms from the perspectives of filtering and dispersing sparse frequencies. The algorithm possesses several theoretical advantages compared to existing algorithms. We summarize these advantages as follows:

- Our algorithm is nearly optimal for integration in the periodic isotropic Sobolev space as well as in the isotropic Sobolev space with compact support, while preserving polynomial tractability. Consequently, it is also nearly optimal and polynomially tractable for smooth functions that vanish on the boundary. In contrast, previous works [?, ?] have developed optimal integration algorithms for the mixed Sobolev space and the isotropic Sobolev space with compact support. However, these algorithms suffer from a drawback: their RMSE upper bounds are not less than $2^{\Theta(d)}M^{-s/d-1/2}$ (where d, s, and M denotes the dimension, the order, and the sample size, respectively, see Remark 3.2 in [?]. In [?], the corresponding constant is larger than $1/(d^{d/2}(2^{1/d}-1)^d)$, and the order of the Sobolev space is required to be greater than $d/2$, as observed from the proofs of Theorem 1 and Lemma 5 in [?]), rendering them not polynomially tractable in terms of sample complexity. Furthermore, our upper bound for integration in the isotropic Sobolev space with compact support is superior to that presented in [?, ?] when $s \leq \Theta(d)$ and $M \leq 2^{2\Theta(d)}$ (namely, $\log^{\Theta(1)} M \leq 2\Theta(d)$). Recently, [?] has proposed a nearly optimal integration algorithm for general smooth functions and smooth functions that vanish on the boundary of $[0,1]^d$ (see Theorem 3.2 in [?]). However, as acknowledged in the "Future work" section of [?], they still struggle to overcome the curse of dimensionality in terms of sample complexity. For more details, see Theorem 3.1, Remark 3.2, Corollary 3.3, and Remark 3.4 in Section 3.

- By incorporating the trick of change of variables, the numerical integration of the analytic functions over $[0,1]^d$ is proven to achieve semi-exponential convergence order, which marks a significant improvement over the previously obtained super-polynomial convergence order [?, ?, ?, ?] (our algorithmic analysis also applies to the infinitely differentiable functions defined in [?, ?]). For more details, see Theorem 4.2 and Remark 4.3 in Section 4.
- Our algorithm achieves a convergence rate of order $(\log(M))^2$ (where M denotes the sample size) for integration in the subspace of Wiener algebra. In comparison to previous studies [?, ?, ?, ?, ?, ?, ?], the sample complexity of our algorithm is independent of the decay rate of Fourier coefficients and the Hölder continuity of the functions, provided that we choose a sufficiently large N in (Eq. (5)). For more details, see Theorem 4.5 and Remarks 4.6, 4.7 in Section 4.
- Our algorithm is also nearly optimal for integration in weighted Sobolev space. Unlike the algorithms [?, ?, ?, ?, ?], our algorithm does not require prior information about weights, making it universally applicable to varying weights in the RMSE sense. The integration algorithms applicable to different weights under the worst-case scenario have been established in [?, ?, ?, ?, ?]. For more details see Theorem 5.1 and Remark 5.2 in Section 5. Very recently, [?] has presented a universal (with respect to both weights and order) nearly optimal algorithm for weighted Sobolev spaces in the scenario where the order $s > 1/2$, whereas our nearly optimal algo-

rithm is valid for the case where $s > 0$. These two works are independent of each other and different in terms of algorithms, the earliest version (for the "weighted Korobov space" of order $s > 0$) of our paper has already been uploaded to arXiv [**?**]. For more details, see Remark 5.3.

At the end of the introduction, we collect some notations that are frequently used in the paper.

- $\mathbb{N}$: the set of natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{Z}$: the set of all integers; For $N \in \mathbb{N}$, $\mathbb{Z}_N = \mathbb{Z} \cap [0, N)$.
- The torus is $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$. The symbol log refers to the natural logarithm.
- $\#U$: the cardinality of a set U; I_U: the indicator function of U.
- For two nonnegative functions h and g defined on a common domain, we express $h(\tau) = \Theta(g(\tau))$ if there exist positive absolute constants C_1 and C_2 such that $C_1 g(\tau) \le h(\tau) \le C_2 g(\tau)$ holds for all τ within the domain.
- $w \in [-b, b] \pmod N$: if there exists an integer a such that $w + aN \in [-b, b]$; $w \notin [-b, b] \pmod N$ if no such integer a exists.
- $\xi \sim \text{unif } \Omega$: ξ is a random variable uniformly distributed over the set Ω.
- For any function $f : \mathbb{Z}_N^d \to \mathbb{C}$, we define its expectation as $\mathbb{E}_z f(z) = \frac{1}{N^d} \sum_{z \in \mathbb{Z}_N^d} f(z)$.
- Given $H_N \in \mathbb{Z}^d \cap [1, N)^d$ and $B > 0$, $U_{H_N,B} := \{w \mid H_N \cdot w \in [-N/B, N/B] \pmod N\}$.
- For a vector $k = (k_1, k_2, \cdots, k_d) \in \mathbb{Z}^d \ \{0\}$, we define the support of k as $\text{supp}(k) = \{j \mid j \in \{1, 2, \ldots, d\}, k_j \neq 0\}$.
- The median of a set of complex numbers $\{a_j\}_{j=1}^k$ is defined as $\text{median}\{a_j\}_{j=1}^k = \text{median}\{\Re a_j\}_{j=1}^k + i \cdot \text{median}\{\Im a_j\}_{j=1}^k$, where $\text{median}\{\Re a_j\}_{j=1}^k$ and $\text{median}\{\Im a_j\}_{j=1}^k$ denote the medians of the real and imaginary parts of the set, respectively.
- In this paper, all hidden constants are considered to be absolute constants.
- Our code is available at https://github.com/Liang-CL-12/sparse-integration.

Integration Error from Sparsity of Frequencies

In this section, we characterize the integration error of our algorithm in terms of sparsity. The integration error to be established in Theorem 2.6 demonstrates that when the sample size is of order $n(\log(1/\epsilon))(\log(n/\epsilon))$, we can filter out the n nonzero frequency components with min-bandwidth (as defined in (Eq. (14))) bounded by M with the error ϵ. This suggests that the min-bandwidth of the sparse frequency components does not affect the integration error, provided that the sample size is slightly larger than the sparsity. For the remaining part, we achieve an error on the order of $(\log(1/\epsilon))^2/M$ (where M denotes the sample size), which nearly matches the error rate of standard Monte Carlo methods.

We first present some preliminary notions and results. For a function f, we consider the random integration algorithms with the random variable ω of the following form

$$A_{d,n,w}(f) = \phi_{d,n,w}(f(x_1), f(x_2), \cdots, f(x_n)),$$

where $\{x_j\}_{j=1}^n \subseteq [0,1]^d$ is a sample set and $\phi_{d,n,w}$ is a map $\mathbb{C}^n \to \mathbb{C}$.

For an accuracy $\epsilon \in (0,1)$, the information complexity of numerical integration in a function space F in the random setting is defined by

$$N(\epsilon, F, d) := \inf\left\{n \in \mathbb{N} \mid \exists A_{d,n,w} : E(F, A_{d,n,w}) \leq \epsilon\right\}$$

where

$$E(F, A_{d,n,w}) := \sup_{\|f\|_F \leq 1} \left(\mathbb{E}\left|\mathrm{Id}(f) - A_{d,n,w}(f)\right|^2\right)^{1/2}.$$

We say the random algorithms is polynomially tractable, if $N(\epsilon, F, d)$ is polynomial with respect to d and $1/\epsilon$. Furthermore, if $N(\epsilon, F, d)$ is independent of d, we say the algorithm is strong polynomially tractable. For further information on tractability in various settings, see [**?**, **?**, **?**] by Novak and Woźniakowski.

Next, we need some technical lemmas. According to Proposition 2.2 in [**?**], we have the following corollary.

Lemma 2.1. Let W be a given function space with semi-norm $\|\cdot\|_W$. If R_m is a random algorithm such that

$$\mathbb{P}\left\{|R_m(f) - \mathrm{INT}(f)| > \epsilon\right\} \leq \alpha$$

for all $\|f\|_W \leq 1$, where $0 < \epsilon, \alpha < 1/2$, then for an odd positive integer k, it holds

$$\mathbb{P}\left\{\left|\mathrm{median}\{R_{m,j}(f)\}_{j=1}^k - \mathrm{INT}(f)\right| > \epsilon\right\} \leq (4\alpha(1-\alpha))^{k/2},$$

where $\{R_{m,j}(f)\}_{j=1}^k$ is a set of k independent realizations of $R_m(f)$.

Based on Claim 7.2 in [**?**] (see the full version: arXiv:1201.2501v2), we have the following lemma. In order to ensure the paper's self-sufficiency, we present a proof.

Lemma 2.2. Let $G_{r,l,\infty} := \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(l+kN)^2}{2r^2}\right)$. For any $w \in \mathbb{Z}_N$, it holds

$$\sum_{l\in\mathbb{Z}_N} G_{r,l,\infty} \exp(2\pi i w l/N) = \sum_{k\in\mathbb{Z}} \exp\left(-2(\pi r(k + w/N))^2\right).$$

Proof. Recall the Fourier transform over $\mathbb{R}$

$$\hat{f}_{\mathbb{R}}(w) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi i w x) dx.$$

Let $g_r(x) = \frac{1}{\sqrt{2\pi} r} \exp(-\frac{x^2}{2r^2})$. It is well-known that $\hat{g}_r^{\mathbb{R}}(w) = \exp(-2(\pi r w)^2)$. Let $G_{r,\infty}(x) = \sum_{k \in \mathbb{Z}} g_r(x + kN)$. By Poisson's summation formula (see Theorem 2.28 in [**?**]),

$$G_{r,\infty}(x) = \sum_{w \in \mathbb{Z}} \exp(-2(\pi r w/N)^2) \exp(2\pi i w x/N).$$

Note that

$$\sum_{l \in \mathbb{Z}_N} \exp(2\pi i w' l/N) \exp(-2\pi i w l/N) = \begin{cases} N & \text{for } w' = w \pmod{N}, w', w \in \mathbb{Z}, \\ 0 & \text{for } w' \neq w \pmod{N}, w', w \in \mathbb{Z}. \end{cases}$$

Multiplying both sides of (Eq. (7)) by $\exp(2\pi i w x/N)$ and substituting $x = l$, we have

$$\sum_{l \in \mathbb{Z}_N} G_{r,l,\infty} \exp(2\pi i w l/N) = \sum_{k \in \mathbb{Z}} \exp\left(-2(\pi r(k + w/N))^2\right).$$

Lemma 2.3. Let $0 < \epsilon \leq 1/10$, B be an integer greater than 2, $r = B\sqrt{\log(1/\epsilon)}$, and $L = \lceil r\sqrt{2\log(1/\epsilon)} \rceil$. If N is an integer that satisfies $N \geq 3L$ then

$$\left| \sum_{|l| \leq L} G_{r,l} \exp(2\pi i w l/N) \right| \leq 10\epsilon \quad \text{for any } w \notin [-N/B, N/B] \pmod{N},$$

and

$$\left| 1 - \sum_{|l| \leq L} G_{r,l} \right| \leq 10\epsilon.$$

Proof. Given that $N \geq 3L$ and $L = \lceil r\sqrt{2\log(1/\epsilon)} \rceil$, we have for $l \in [-L, L]$,

$$|G_{r,l} - G_{r,l,\infty}| \leq 2\sum_{k=1}^{\infty} \exp\left(-\frac{(N - |l| + (k-1)N)^2}{2r^2}\right) \leq 2\sum_{k=1}^{\infty} \exp\left(-\frac{(N - L + (k-1)N)^2}{2r^2}\right).$$

Since $N \geq 3L$, we have $N - L \geq 2L$, and thus

$$|G_{r,l} - G_{r,l,\infty}| \leq 2\sum_{k=1}^{\infty} \exp\left(-\frac{(2L + (k-1)N)^2}{2r^2}\right) \leq 2\sum_{k=1}^{\infty} \exp\left(-\frac{4L^2 + 4L(k-1)N}{2r^2}\right).$$

Since $L/r \geq \sqrt{2\log(1/\epsilon)}$, we have $\exp(-\frac{3L^2}{2r^2}) \leq \epsilon^2$. It follows that for $l \in [-L, L]$,

$$|G_{r,l} - G_{r,l,\infty}| \leq \frac{\epsilon^2}{1 - \exp(-4)}.$$

Denote the set $\mathbb{Z}_N \quad ([0, L] \cup [N - L, N - 1])$ by M. Note that

$$\sum_{l \in M} G_{r,l,\infty} \leq 2 \sum_{L < l \leq N/2} G_{r,l,\infty} \leq 4 \sum_{L < l \leq N} \exp\left(-\frac{l^2}{2r^2}\right) \leq 4 \sum_{l > L} \exp\left(-\frac{l^2}{2r^2}\right).$$

Since $L/r \geq \sqrt{2\log(1/\epsilon)}$ and $N > 3L$, we have

$$\sum_{l \in M} G_{r,l,\infty} \leq 4\int_{L}^{\infty} \exp\left(-\frac{x^2}{2r^2}\right) dx \leq 4r \exp\left(-\frac{L^2}{2r^2}\right) \leq 4r\epsilon^2 \leq 8\epsilon.$$

Therefore, by employing (Eq. (10)), $0 < \epsilon \leq 1/10$, and $L = \lceil r\sqrt{2\log(1/\epsilon)}\rceil$, we derive that

$$\left|\sum_{|l| \leq L} G_{r,l} \exp(2\pi i w l/N)\right| \leq \sum_{|l| \leq L} |G_{r,l} - G_{r,l,\infty}| + \left|\sum_{|l| \leq L} G_{r,l,\infty} \exp(2\pi i w l/N)\right| \leq \epsilon^2(2L+1) + 8\epsilon \leq 9\epsilon.$$

Using Lemma 2.2 and taking $w = 0$ in (Eq. (6)), we have

$$\left|\sum_{|l| \leq L} G_{r,l,\infty} - 1\right| = \left|\sum_{k \in \mathbb{Z}} \exp(-2(\pi r k)^2) - 1\right| \leq \epsilon.$$

Combining (Eq. (11)), (Eq. (12)), and the triangle inequality yields

$$\left|1 - \sum_{|l| \leq L} G_{r,l}\right| \leq 10\epsilon.$$

On the other hand, using (Eq. (6)) and considering the condition $r = B\sqrt{\log(1/\epsilon)}$, we derive that for $w \notin [-N/B, N/B] \pmod N$,

$$\left|\sum_{|l|\leq L} G_{r,l,\infty} \exp(2\pi i w l/N)\right| = \left|\sum_{k\in\mathbb{Z}} \exp(-2(\pi r(k + w/N))^2)\right| \leq \epsilon.$$

Combining the above inequality and (Eq. (11)), we complete the proof of (Eq. (8)).

The lemma presented below is closely related to Lemma 3.2 in [**?**] and Lemma 4 in [**?**].

Lemma 2.4. Let N be a prime number and B be a positive integer with $N \geq 3B > 6$. Then, for any $w \in \mathbb{Z}^d$ with $w \neq 0 \pmod N$, it holds $H_N \cdot w \in [-N/B, N/B] \pmod N$ with probability at most $4/B$ over the randomness of H_N, where $H_N := (h_1, h_2, \dots, h_d)$ is drawn from the uniform distribution over $\mathbb{Z}^d \cap [1, N)^d$.

Proof. Since $w \neq 0 \pmod N$, it must have a non-zero component. Without loss of generality, let us assume that $w_d \neq 0 \pmod N$. Then, fixing arbitrary values $h_1, \dots, h_{d-1}$, with probability at most $4/B$ (no greater than $\frac{\#\{w\in\mathbb{Z}|w\in[-N/B,N/B]\}}{N}$) over the randomness of h_d, we have $\sum_{i=1}^{d} h_i w_i \in [-N/B, N/B] \pmod N$, thereby completing the proof.

Now, we introduce the local Monte Carlo sampling. For any function $f : [0,1]^d \to \mathbb{C}$, we introduce a new function $f_{N,\eta}$ defined as

$$f_{N,\eta}(z/N) := f(z/N + \eta_z), \quad z \in \mathbb{Z}_N^d,$$

where $\eta = (\eta_z)_{z\in\mathbb{Z}_N^d}$, and each η_z is independently and identically distributed from the uniform distribution on $[0, 1/N)^d$.

Given a function f with $\|f\|_{L_2([0,1]^d)} \leq 1$, we use a_ξ to denote the Fourier coefficient of f. Then f can be expressed as

$$f(x) = \sum_{\xi\in\mathbb{Z}^d} a_\xi \exp(2\pi i \xi \cdot x).$$

For $M > 0$, let

$$U \subseteq \{w = (w_1, w_2, \dots, w_d) \in \mathbb{Z}^d \quad \{0\} \mid \min_{j\in\mathrm{supp}(w)} |w_j| \leq M\},$$

and define $g(x)$ and $R(x)$ as

$$g(x) = \sum_{\xi \in U} a_\xi \exp(2\pi i \xi \cdot x), \quad R(x) = \sum_{\xi \notin (U \cup \{0\})} a_\xi \exp(2\pi i \xi \cdot x).$$

We call M the min-bandwidth of g. Then the function f can be rewritten as

$$f(x) = a_0 + g(x) + R(x), \quad x \in [0,1]^d.$$

By the definition of $f_{N,\eta}$ in (Eq. (13)), we obtain

$$f_{N,\eta}(z/N) = a_0 + g(z/N + \eta_z) + R(z/N + \eta_z).$$

For convenience, we let

$$a_0^* = \tilde{f}_{N,\eta}^{(N)}(0) = a_0 + \tilde{g}_{N,\eta}^{(N)}(0) + \tilde{R}_{N,\eta}^{(N)}(0),$$

and

$$R_2(z/N) = R_{N,\eta}(z/N) + g_{N,\eta}(z/N) - g(z/N) - \tilde{g}_{N,\eta}^{(N)}(0) - \tilde{R}_{N,\eta}^{(N)}(0).$$

Note that $\tilde{R}_2(0) = 0$, and $f_{N,\eta}$ can be rewritten as

$$f_{N,\eta}(z/N) = a_0^* + g(z/N) + R_2(z/N).$$

Lemma 2.5. Let f and $f_{N,\eta}$ be functions defined in (Eq. (16)) and (Eq. (19)). Let $K = \#U$ and $\|f\|_{L_2([0,1]^d)} \leq 1$. We have over the randomness of η that

$$\mathbb{P}\left\{|a_0^* - a_0| \leq \epsilon\right\} > 1 - \epsilon^2 N^d,$$

and

$$\mathbb{P}\left\{\sum_{z \in \mathbb{Z}_N^d} |R_2(z/N)|^2 \leq \|R\|_{L_2([0,1]^d)}^2 + \Theta\left(\frac{\sigma^2 N^{d/2}}{n}\right)\right\} \geq 1 - \sigma^2 - \epsilon^2 N^d.$$

Proof. Since $\mathbb{E}_\eta \tilde{f}_{N,\eta}^{(N)}(0) = a_0$ and $\|f\|_{L_2([0,1]^d)} \leq 1$, we have

$$\mathbb{E}_\eta \left|\tilde{f}_{N,\eta}^{(N)}(0) - a_0\right|^2 \leq \frac{1}{N^d}.$$

Then by Chebyshev's inequality,

$$\mathbb{P}\{|a_0^* - a_0| \leq \epsilon\} = \mathbb{P}\left\{\left|\tilde{f}_{N,\eta}^{(N)}(0) - a_0\right| \leq \epsilon\right\} \geq 1 - \frac{\mathbb{E}_\eta \left|\tilde{f}_{N,\eta}^{(N)}(0) - a_0\right|^2}{\epsilon^2} > 1 - \epsilon^2 N^d.$$

Notice that

$$R_2(z/N) = R_{N,\eta}(z/N) + (g_{N,\eta}(z/N) - g(z/N)) - \left(\tilde{g}_{N,\eta}^{(N)}(0) + \tilde{R}_{N,\eta}^{(N)}(0)\right).$$

Next, we will estimate the first two terms in (Eq. (22)) respectively. Note that

$$\sum_{z \in \mathbb{Z}_N^d} |R_{N,\eta}(z/N)|^2 = \|R\|_{L_2([0,1]^d)}^2.$$

Then for a positive number σ^2, the application of Markov's inequality yields

$$\mathbb{P}\left\{\sum_{z \in \mathbb{Z}_N^d} |R_{N,\eta}(z/N)|^2 \leq \frac{\|R\|_{L_2([0,1]^d)}^2}{\sigma^2/2}\right\} > 1 - \sigma^2/2.$$

Analogous to (Eq. (20)), we have $\mathbb{E}_\eta|\tilde{g}_{N,\eta}^{(N)}(w) - a_w|^2 \leq \frac{1}{N^d}$ for $w \in U$. Applying Parseval's identity, we have

$$\mathbb{E}_\eta \mathbb{E}_z |g_{N,\eta}(z/N) - g(z/N)|^2 \leq \frac{2K}{N^d}.$$

Applying Markov's inequality and the above inequalities, we obtain the following two inequalities:

$$\mathbb{P}\left\{\mathbb{E}_z |g_{N,\eta}(z/N) - g(z/N)|^2 \leq \frac{2K}{\sigma^2 N^d}\right\} \geq 1 - \sigma^2,$$

and

$$\mathbb{P}\left\{\sum_{z \in \mathbb{Z}_N^d} |g_{N,\eta}(z/N) - g(z/N)|^2 \leq \frac{2K}{\sigma^2 N^{d/2}}\right\} \geq 1 - \sigma^2 N^{d/2}.$$

By (Eq. (21)), (Eq. (23)), and (Eq. (25)),

$$\sum_{z \in \mathbb{Z}_N^d} |R_2(z/N)|^2 \leq \|R\|_{L_2([0,1]^d)}^2 + \Theta(\epsilon^2) + \sigma^2 N^{d/2}$$

holds with probability at least $1-\sigma^2-\epsilon^2 N^d$.

To establish the theoretical analysis framework for our algorithm, we meticulously design a function space that is comprised of the function f specifically defined in (16):

$$F_{K,M,\lambda_1,\lambda_2} := \left\{ f(x) = a_0 + g(x) + R(x), x \in [0,1]^d \;\middle|\; \begin{array}{l} g(x) = \sum_{\xi \in U} a_\xi \exp(2\pi i \xi \cdot x), \\ U \subseteq \{w = (w_1, w_2, \dots, w_d) \in \mathbb{Z}^d \quad \{0\} \mid \min_{j \in \mathrm{supp}(w)} |w_j| \leq M\} \\ \sum_{\xi \in U} |a_\xi| \leq \lambda_1, \#U = K, \\ R(x) = \sum_{w \in \mathbb{Z}^d \ (U \cup \{0\})} b_w \exp(2\pi i w \cdot x), \|R\|^2_{L_2([0,1]^d)} \leq \lambda_2 \end{array} \right.$$

It is obvious that $\mathrm{INT}(f) = \hat{f}(0) = a_0$ for any $f \in F_{K,M,\lambda_1,\lambda_2}$.

Theorem 2.6. Let $f \in F_{K,M,\lambda_1,\lambda_2}$ with $\|f\|_{L_2([0,1]^d)} \leq 1$. Given $0 < \sigma, \sigma_2 \leq 1/10$, $n \geq K+1$, $0 < \epsilon < \sigma/n$, we take $B = \lceil 4n/\sigma \rceil$, $r = B\sqrt{\log(1/\epsilon)}$, $L = \lceil r\sqrt{2\log(1/\epsilon)} \rceil$, and let $N = \max\{\Theta(1/(\sigma\epsilon)^{2/d}), \Theta((K/(\sigma_2\epsilon^2))^{2/d}), M+1, 3L\}$ be a prime number. Then it holds that over the randomness of H_N, z, and η,

$$\left| I(f_{N,\eta})^H_{N,L,r,z} - \mathrm{INT}(f) \right| \leq \Theta\left((\lambda_1 + 1)\epsilon + \sqrt{\frac{\lambda_2 \log(1/\epsilon)}{L\sigma\sigma_2}} \right)$$

with probability at least $1 - \sigma - \sigma_2$.

Taking the median of outputs $\{I(f_{N,\eta_j})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^t$, which are obtained by independently randomizing H_N, z, and η for t times, where t is an odd number greater than or equal to 7, we have

$$\left| \mathrm{median}\{I(f_{N,\eta_j})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^t - \mathrm{INT}(f) \right| \leq \Theta\left((\lambda_1 + 1)\epsilon + \sqrt{\frac{\lambda_2 \log(1/\epsilon)}{L\sigma\sigma_2}} \right)$$

with probability at least $1 - (2\sigma + \sigma_2)^t$.

Furthermore, the RMSE of the presented integration method is bounded by

$$\left(\mathbb{E}_{\{H_{N_j}, z_j, \eta_j\}_{j=1}^t} \left| \mathrm{median}\{I(f_{N,\eta_j})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^t - \mathrm{INT}(f) \right|^2 \right)^{1/2} \leq \Theta\left((\lambda_1 + 1)\epsilon + \sqrt{\frac{\lambda_2 \log(1/\epsilon)}{L\sigma\sigma_2}} + (2\sigma + \sigma_2)^t \right).$$

Proof. For any function $f \in F_{K,M,\lambda_1,\lambda_2}$, f can be expressed as $f(x) = a_0 + g(x) + R(x)$, with the conditions $\sum_{\xi\in U} |a_\xi| \le \lambda_1$, $\mathrm{INT}(R) = 0$, and $\|R\|^2_{L_2([0,1]^d)} \le \lambda_2$.

As presented in (Eq. (19)), we have

$$f_{N,\eta}(z/N) = a_0^* + g(z/N) + R_2(z/N), \quad z \in \mathbb{Z}_N^d.$$

To bound the integration error, by (Eq. (5)) and that $a_0 = \mathrm{INT}(f)$, it suffices to estimate the following three terms:

$$\left| a_0^* \sum_{|l|\le L} G_{r,l} - a_0 \right|, \quad \left| \sum_{|l|\le L} g\left(\left\{ \frac{z - lH_N}{N} \right\} \right) G_{r,l} \right|, \quad \left| \sum_{|l|\le L} R_2\left(\left\{ \frac{z - lH_N}{N} \right\} \right) G_{r,l} \right|.$$

By choosing a prime number $N = \max\{\Theta(1/(\sigma\epsilon)^{2/d}), \Theta((K/(\sigma_2\epsilon^2))^{2/d}), M + 1, 3L\}$ and making use of Lemma 2.5, we have

$$\mathbb{P}\{|a_0^* - a_0| \le \Theta(\epsilon)\} \ge 1 - \sigma.$$

Moreover, the following inequality holds with probability at least $1 - \sigma - \sigma_2$,

$$\sum_{z\in\mathbb{Z}_N^d} |R_2(z/N)|^2 = \sum_{z\in\mathbb{Z}_N^d} \left| \tilde{R}_2(z/N) \right|^2 \le \lambda_2 + \Theta(\epsilon^2).$$

The application of (Eq. (18)) and (Eq. (15)) (recall the condition $N \ge M + 1$) yields that: $\tilde{R}^{(N)}(0) = 0$, $\tilde{g}^{(N)}(0) = 0$, and $\tilde{f}^{(N)}_{N,\eta}(0) = a_0^*$.

From Lemma 2.3 and (Eq. (34)), we deduce

$$\left| a_0^* \sum_{|l|\le L} G_{r,l} - a_0 \right| \le \Theta(\epsilon)$$

with probability at least $1 - \sigma$.

We will analyze the last two terms of (Eq. (33)) separately. Let $n \ge K + 1$, $B = \lceil 4n/\sigma \rceil$, $r = B\sqrt{\log(1/\epsilon)}$, and $L = r\sqrt{2\log(1/\epsilon)}$. By Lemma 2.4, for $w^{(j)} \in U$,

$$\mathbb{P}\{w^{(j)} \in U_{H_N,B}\} \le \sigma/n, \quad 1 \le j \le K,$$

which leads to

$$\mathbb{P}\{w^{(j)} \notin U_{H_N,B} \text{ holds for every } j = 1, 2, \dots, K\} \geq 1 - \sigma.$$

Furthermore, by Lemma 2.3, it holds

$$\left| \sum_{|l| \leq L} g\left(\left\{\frac{z - lH_N}{N}\right\}\right) G_{r,l} \right| \leq \lambda_1 \epsilon$$

for every $z \in \mathbb{Z}_N^d$ with probability at least $1 - \sigma$.

Next, we analyze R_2. For each $\xi \in \mathbb{Z}_N^d \ \{0\}$, we let $\tilde{R}_2(\xi) = c_\xi$ and denote

$$t_\xi := \sum_{|l| \leq L} \exp\left(2\pi i \xi \cdot \left\{\frac{z - lH_N}{N}\right\}\right) G_{r,l}.$$

The application of Lemma 2.3 leads to $\sum_{|l| \leq L} G_{r,l} \leq 2$. Therefore,

$$\sum_{\xi \in \mathbb{Z}_N^d \ \{0\}} |c_\xi|^2 t_\xi \leq \Theta(1) \|R_2\|_{L_2([0,1]^d)}^2.$$

In addition, by using Lemma 2.4, we derive

$$\mathbb{P}\{\xi \in U_{H_N,B}\} \leq \Theta(1/B).$$

The combination of (Eq. (39)) and (Eq. (40)) yields

$$\mathbb{E}_{H_N}\left[t_\xi \mid \xi \in U_{H_N,B}\right] \mathbb{P}\{\xi \in U_{H_N,B}\} \leq \Theta(1/B).$$

On the other hand, by using (Eq. (8)), we have

$$\mathbb{E}_{H_N}\left[t_\xi \mid \xi \notin U_{H_N,B}\right] \mathbb{P}\{\xi \notin U_{H_N,B}\} \leq \Theta(\epsilon^2).$$

Therefore, it follows that for $H_N \sim \text{unif } \mathbb{Z}^d \cap [1, N)^d$,

$$\mathbb{E}_{H_N} t_\xi \leq \Theta(1/B + \epsilon^2).$$

Upon observing that for every distinct $\xi, \eta \in \mathbb{Z}_N^d \ \{0\}$,

$$\sum_{z \in \mathbb{Z}_N^d} \sum_{|l| \leq L} \exp\left(2\pi i \xi \cdot \left\{\frac{z - lH_N}{N}\right\}\right) G_{r,l} \sum_{|l'| \leq L} \exp\left(-2\pi i \eta \cdot \left\{\frac{z - l'H_N}{N}\right\}\right) G_{r,l'} = 0,$$

and $\tilde{R}_2(\xi) = c_\xi$, by combining (Eq. (35)) and (Eq. (42)), we immediately arrive at the conclusion that with probability at least $1-\sigma-\sigma_2$ over the randomness of η,

$$\left|\sum_{|l|\le L} R_2\left(\left\{\frac{z-lH_N}{N}\right\}\right) G_{r,l}\right| \le \Theta\left(\frac{\lambda_2}{n\sigma\sigma_2}\right),$$

where $z \sim \text{unif } \mathbb{Z}_N^d$.

Recalling the conditions $B = \lceil 4n/\sigma \rceil$ and $\epsilon < 1/n$, we obtain the following inequality by Markov's inequality,

$$\left|\sum_{|l|\le L} R_2\left(\left\{\frac{z-lH_N}{N}\right\}\right) G_{r,l}\right| \le \Theta\left(\sqrt{\frac{\lambda_2}{n\sigma\sigma_2}}\right)$$

with probability at least $1-2\sigma-\sigma_2$, where the probability is determined by the random variables z, H_N, and η.

Summarizing (Eq. (32)), (Eq. (36)), (Eq. (37)), and (Eq. (45)), and applying Markov's inequality, we obtain

$$\left|I(f_{N,\eta})^H_{N,L,r,z} - \text{INT}(f)\right| \le \Theta\left((\lambda_1+1)\epsilon + \sqrt{\frac{\lambda_2}{n\sigma\sigma_2}}\right)$$

with probability at least $1-4\sigma-\sigma_2$.

Replacing σ by $\sigma/4$, we complete the proof of (Eq. (28)).

By randomly selecting H_N, z, and η to compute $I(f_{N,\eta})^H_{N,L,r,z}$ t times, we obtain t different results. Extracting the median of the obtained results, the estimation (29) can be naturally derived from Lemma 2.1. Denote $\text{median}\{I(f_{N,\eta_j})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^t - \text{INT}(f)$ by Y. Then

$$\mathbb{P}\{200k \le |Y| \le 200(k+1)\} \le \binom{t}{k}(2\sigma+\sigma_2)^k(1-2\sigma-\sigma_2)^{t-k}.$$

Therefore,

$$\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}|Y|^2 \le \Theta\left((\lambda_1+1)\epsilon + \sqrt{\frac{\lambda_2}{n\sigma\sigma_2}} + (2\sigma+\sigma_2)^t\right)^2.$$

This completes the proof.

Integration in Isotropic Sobolev Spaces

We are now prepared to develop an estimation for the periodic isotropic Sobolev spaces of order $s \geq 0$.

$$H^s(\mathbb{T}^d) := \left\{ f \in L_2(\mathbb{T}^d) \middle| \|f\|_{H^s(\mathbb{T}^d)} := \sum_{w \in \mathbb{Z}^d} \left(1 + \sum_{j=1}^{d} |2\pi w_j|^2\right)^{s/2} |\hat{f}(w)|^2 < \infty \right\}.$$

Theorem 3.1. Let $f \in H^s(\mathbb{T}^d)$ with $\|f\|_{H^s(\mathbb{T}^d)} \leq 1$. Let $0 < \sigma \leq 1/10$, $n \geq 1$, $\epsilon < \sigma/n$, $B = \lceil 4n/\sigma \rceil$, $r = B\sqrt{\log(1/\epsilon)}$, $L = \lceil r\sqrt{2\log(1/\epsilon)} \rceil$ and $N = \max\{\Theta(1/(\sigma\epsilon)^{2/d}), \Theta((n/\epsilon^2)^{2/d}), 3L\}$ be a prime number. It holds with probability at least $1 - \sigma - 1/10$ over the randomness of H_N, z, and η that

$$\left|I(f_{N,\eta})^{H}_{N,L,r,z} - \mathrm{INT}(f)\right| \leq \Theta\left(\frac{(\log(1/\epsilon))^{s/d+1/2}}{(L\sigma)^{s/d+1/2}}\right).$$

Let $\mathrm{median}\{I(f_{N,\eta_j})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^{t}$ be defined as in Theorem 2.6. It holds with probability at least $1 - (2\sqrt{\sigma} + 1/10)^t$ that

$$\left|\mathrm{median}\{I(f_{N,\eta_j})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^{t} - \mathrm{INT}(f)\right| \leq \Theta\left(\frac{(\log(1/\epsilon))^{s/d+1/2}}{(L\sigma)^{s/d+1/2}}\right).$$

Furthermore, the RMSE is bounded by

$$\left(\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^{t}}\left|\mathrm{median}\{I(f_{N,\eta})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^{t} - \mathrm{INT}(f)\right|^2\right)^{1/2} \leq \Theta\left(\frac{(\log(1/\epsilon))^{s/d+1/2}}{(L\sigma)^{s/d+1/2}} + L\epsilon + (2\sqrt{\sigma} + 1/10)^t\right).$$

Remark 3.2. We use $M := (2L+1)t$ to represent the total sample size employed by our algorithm. Substituting $\sigma = 1/10$, $t = \log^{5/2}(n^{s/d+1/2})$, and $\epsilon = 1/n^{s/d+3/2}$ into (Eq. (50)), the RMSE error is bounded above by

$$\Theta\left(\frac{(3 + 4s/d)^{4s/d+3}(\log M)^{2s/d+1}}{M^{s/d+1/2}}\right).$$

When the RMSE error is equal to ϵ, it follows that the sample size M is polynomial with respect to $1/\epsilon$. This demonstrates that the random algorithm has strong polynomial tractability. The upper bound in (Eq. (51)) (up to an absolute constant factor) also applies to integration in the isotropic Sobolev space with compact support (see Corollary 3.3 and Remark 3.4). In contrast to

[**?**, **?**], where the upper bounds for integration algorithms in this space are at least $2^{\Theta(d)}M^{-s/d-1/2}$, our bound outperforms these results when $s \leq \Theta(d)$ and $M \leq 2^{2\Theta(d)}$ (i.e., $\log^{\Theta(1)} M \leq 2\Theta(d)$).

Additionally, in the absence of prior knowledge of the order s, setting $\epsilon = 1/(n \log n)$ and $t = \log^{5/2}(n \log n)$ yields an algorithm universally applicable to all s.

Proof. Let n be a positive integer, and define $\beta_{n,d} = n^{1/d}/2\pi$. Decompose any function $f \in H^s(\mathbb{T}^d)$ as

$$f(x) = a_0 + g(x) + R(x),$$

where

$$g(x) = \sum_{w \in \mathbb{Z}^d \cap (-\beta_{n,d}, \beta_{n,d})^d \ \{0\}} a_w \exp(2\pi i w \cdot x), \quad R(x) = \sum_{w \notin \mathbb{Z}^d \cap (-\beta_{n,d}, \beta_{n,d})^d} a_w \exp(2\pi i w \cdot x).$$

Since $f \in H^s(\mathbb{T}^d)$, it follows that $\|R\|_{L_2([0,1]^d)} \leq \Theta(n^{-s/d}\|f\|_{H^s})$. Additionally,

$$W := \#\{w \mid w \in \mathbb{Z}^d \cap (-\beta_{n,d}, \beta_{n,d})^d \ \ \{0\}\} \leq n - 1.$$

By the Cauchy-Schwarz inequality,

$$\sum_{w \in \mathbb{Z}^d \cap (-\beta_{n,d}, \beta_{n,d})^d \ \{0\}} |a_w| \leq \sqrt{n-1}.$$

Let $B = 4n/\sigma$. Theorem 3.1 follows immediately by letting $\sigma_2 = 1/10$ and $M = \beta_{n,d}$ in Theorem 2.6.

Given a bounded measurable set Ω with volume 1, we denote the isotropic Sobolev spaces with compact support in Ω by $\mathring{H}^s(\Omega)$ (see the definition on page 1189 in [**?**] for details). Specifically,

$$\mathring{H}^s(\Omega) := \{f \in \mathring{L}_2(\Omega) \mid D^\alpha f \in L_2(\mathbb{R}^d) \text{ for } \alpha \in \mathbb{N}_0^d \text{ and } |\alpha|_1 \leq s\}$$

equipped with the norm

$$\|f\|_{\mathring{H}^s(\Omega)} := \|f\|_{L_2(\Omega)} + \sum_{|\alpha|_1 = s} \|D^\alpha f\|_{L_2(\Omega)},$$

where $\mathring{L}_2(\Omega) := \{f \in L_2(\mathbb{R}^d) \mid \mathrm{supp}(f) \subseteq \Omega\}$, $D^\alpha f$ denotes the α-order weak partial derivative of a function f.

For any $f \in \mathring{H}^s(\Omega)$, define the function

$$F(x) = \sum_{k \in \mathbb{Z}^d} f(k+x).$$

It holds

$$\int_\Omega f(x)dx = \int_{[0,1]^d} F(x)dx, \quad \int_{\mathbb{R}^d} |D^\alpha f(x)|^2 dx = \int_{[0,1]^d} |D^\alpha F(x)|^2 dx \quad \text{for every } \alpha \in \mathbb{N}_0^d.$$

By the Parseval identity and Minkowski inequality, we have $\|F\|_{H^s(\mathbb{T}^d)} \leq \|f\|_{\mathring{H}^s(\Omega)}$.

Consequently, the error bounds (48), (49), and (50) derived earlier remain valid for the integral $\mathrm{INT}(F) = \int_\Omega f(x)dx$.

A relevant question concerns the number of samples required to determine the value of $F(x)$. More precisely, based on (Eq. (5)), how many samples are needed to obtain the values of the set $\{F_{N,\eta}(\{(z - lH_N)/N\})\}_{l=-L}^{L}$? Define

$$J_\Omega(x) := \sum_{k \in \mathbb{Z}^d} I_\Omega(k+x).$$

With a certain probability, only $\Theta(L)$ samples are required. In fact,

$$2L + 1 = \sum_{l=-L}^{L} \mathbb{E}_{\eta,z} J_\Omega \left(\left\{ \frac{z - lH_N}{N} \right\} + \eta_z \right).$$

By Markov's inequality,

$$\mathbb{P} \left\{ \sum_{l=-L}^{L} \#\{k \mid k \in \mathbb{Z}^d, (z - lH_N)/N + \eta_z + k \in \Omega\} \geq 200L + 100 \right\} \leq 1/100.$$

In other words, with a probability of at least 99/100, the number of samples required to obtain $\{F_{N,\eta}(\{(z - lH_N)/N\})\}_{l=-L}^{L}$ is at most $200L + 100$.

When searching the sets

$$V_l := \{(z - lH_N)/N + \eta_z + k \mid k \in \mathbb{Z}^d, (z - lH_N)/N + \eta_z + k \in \Omega\}, \quad -L \leq l \leq L,$$

no samples of the function f are required. If the number of elements in the union $\bigcup_{-L\le l\le L} V_l$ exceeds $200L+100$, we promptly output the result (denoted as $I^*(F_{N,\eta})^H_{N,L,r,z}$) as zero. Otherwise, set $I^*(F_{N,\eta})^H_{N,L,r,z} = I(F_{N,\eta})^H_{N,L,r,z}$.

This ensures that the sample complexity of our algorithm remains polynomial. Consequently, we have the following corollary.

Corollary 3.3. Let $f \in \mathring{H}^s(\Omega)$ with $\|f\|_{\mathring{H}^s(\Omega)} \le 1$. Let $F(x) = \sum_{k\in\mathbb{Z}^d} f(k+x)$. Assume the same parameter settings as those in Theorem 3.1. Then it holds with probability at least $1-\sigma-1/10-1/100$ over the randomness of H_N, z, and η that

$$\left|I^*(F_{N,\eta})^H_{N,L,r,z} - \mathrm{INT}(F)\right| \le \Theta\left(\frac{(L\delta/\log(1/\epsilon))^{s/d+1/2}}{M^{s/d+1/2}}\right).$$

By obtaining $\mathrm{median}\{I^*(F_{N,\eta_j})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^t$ as in Theorem 2.6, we have with probability at least $1-(2\sqrt{\sigma}+1/10+1/100)^t$ that

$$\left|\mathrm{median}\{I^*(F_{N,\eta_j})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^t - \mathrm{INT}(F)\right| \le \Theta\left(\frac{(L\delta/\log(1/\epsilon))^{s/d+1/2}}{M^{s/d+1/2}}\right).$$

Furthermore, the RMSE is bounded by

$$\left(\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}\left|\mathrm{median}\{I^*(F_{N,\eta})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^t - \mathrm{INT}(F)\right|^2\right)^{1/2} \le \Theta\left(\frac{(L\delta/\log(1/\epsilon))^{s/d+1/2}}{M^{s/d+1/2}} + L\epsilon + (2\sqrt{\sigma}+1/10\right.$$

Remark 3.4. We adopt the definition of the isotropic Sobolev space from [**?**], which differs subtly from that in [**?**]. Nevertheless, the theorem remains applicable to the compactly supported isotropic Sobolev space as defined in [**?**].

Remark 3.5. The analysis of the algorithm's sample complexity and polynomial tractability is parallel to that in Theorem 3.1 and Remark 3.2.

Integration of Analytic and Wiener-type Functions

In this section, we estimate the integral error for analytic and Wiener-type functions. We derive semi-exponential convergence rates for the numerical integration of analytic functions in the asymptotic sense.

To this end, we first present a lemma to estimate the coefficients of higher-order derivatives of composite functions. Faà di Bruno's formula [**?**] states that the k-th order derivative of a composite function $g\circ f$ admits the following representation:

$$(g \circ f)^{(k)}(x) = \sum_{p=1}^{k} \sum_{\substack{p_1+\cdots+p_k=p \\ p_1+2p_2+\cdots+kp_k=k}} a(p, p_1, \dots, p_k) g^{(p)}(f(x)) \prod_{j=1}^{k} (f^{(j)}(x))^{p_j},$$

where $a(p, p_1, \dots, p_k)$ are non-negative constants dependent only on $p, p_1, \dots, p_k$.

Lemma 4.1. Let $k \in \mathbb{N}$ and $a(p, p_1, \dots, p_k)$ be defined as in (Eq. 52). It holds

$$\sum_{p=1}^{k} \sum_{\substack{p_1+\cdots+p_k=p \\ p_1+2p_2+\cdots+kp_k=k}} a(p, p_1, \dots, p_k) \leq 4^k.$$

Proof. For $1 \leq q \leq k$, consider $g_q(x) = x^q/q!$ and $f(x) = e^x$. By Faà di Bruno's formula,

$$\frac{q^q}{q!} = (g_q \circ f)^{(q)}|_{x=0} = \sum_{\substack{p_1+\cdots+p_k=q \\ p_1+2p_2+\cdots+kp_k=k}} a(p, p_1, \dots, p_k) \frac{q!}{(q-p)!} \leq q! \sum_{\substack{p_1+\cdots+p_k=q \\ p_1+2p_2+\cdots+kp_k=k}} a(p, p_1, \dots, p_k).$$

Summing over q from 1 to k and using Stirling's formula, we obtain:

$$\sum_{p=1}^{k} \sum_{\substack{p_1+\cdots+p_k=p \\ p_1+2p_2+\cdots+kp_k=k}} a(p, p_1, \dots, p_k) \leq \sum_{q=1}^{k} \frac{q^q}{q!q!} \leq 4^k.$$

Now, we present the main idea of the error analysis. For any analytic function f defined on $[0,1]^d$, we first use the trick of change of variable (see [**?**, **?**, **?**]) to construct a smooth function Tf, which vanishes on the boundary of the hypercube $[0,1]^d$ and satisfies the integral identity $\int_{[0,1]^d} f(t)dt = \int_{[0,1]^d} Tf(t)dt$. Specifically, we utilize the function introduced in [**?**]:

$$\psi(y) = \begin{cases} \frac{\int_0^y \exp(-1/(\xi(1-\xi)))d\xi}{\int_0^1 \exp(-1/(\xi(1-\xi)))d\xi} & \text{if } y \in [0,1], \\ 0 & \text{if } y < 0, \\ 1 & \text{if } y > 1. \end{cases}$$

For $t = (t_1, t_2, \dots, t_d) \in \mathbb{R}^d$, let $f_\psi(t) = f(\psi(t_1), \psi(t_2), \dots, \psi(t_d))$, and define

$$Tf(t) := \left(\prod_{j=1}^{d} \psi'(t_j) \right) f_\psi(t).$$

It follows that

$$\int_{[0,1]^d} f(t)dt = \int_{[0,1]^d} Tf(t)dt.$$

For arbitrary analytic function f defined on $[0,1]^d$, applying the results proposed in [?] (see page 2273), we know that there exist constants $C_{f,1} > 1$ and $C_{f,2} > 1$ depending only on f, such that

$$\left\| \frac{\partial^{J_1+\cdots+J_d} f}{\partial x_1^{J_1} \cdots \partial x_d^{J_d}} \right\|_{L_\infty([0,1]^d)} \le C_{f,1} (C_{f,2})^J J!,$$

for all $t \in [0,1]^d$, where $\{J_1, J_2, \dots, J_d\} \subseteq \mathbb{N}_0$ and $J = J_1 + \cdots + J_d$.

We consider the more general case, i.e., Theorem 4.2 applies to the functions that satisfy the following inequality

$$\left\| \frac{\partial^{J_1+\cdots+J_d} f}{\partial x_1^{J_1} \cdots \partial x_d^{J_d}} \right\|_{L_1([0,1]^d)} \le C_{f,1} (C_{f,2})^J J!,$$

for all $t \in [0,1]^d$, where $C_{f,1} > 1$ and $C_{f,2} > 1$ are some constants depending only on f, $\{J_1, J_2, \dots, J_d\} \subseteq \mathbb{N}_0$ and $J = J_1 + \cdots + J_d$. Both analytic functions defined on $[0,1]^d$ and the infinitely differentiable functions defined in [?, ?] satisfy (Eq. (54)). By demonstrating that the Fourier coefficients of Tf exhibit semi-exponential decay and using Theorem 2.6, we establish the following result.

Theorem 4.2. Assume that f satisfies (Eq. (54)). Let the parameters satisfy $0 < \sigma \le 1/10$, $n > 3$, $0 < \epsilon < \sigma/n$, $B = \lceil 4n/\sigma \rceil$, $r = B\sqrt{\log(1/\epsilon)}$, and $L = \lceil r\sqrt{2\log(1/\epsilon)} \rceil$. Additionally, let N be a prime number such that $N = \max\{\Theta((n/(\ln(L)\epsilon^2))^{2/d}), \Theta(1/(\sigma\epsilon)^{2/d}), 3L\}$. Then with respect to the randomness of H_N, z, and η, the following inequality holds with a probability of at least $1 - 1/(\ln L)$:

$$\left| I(Tf_{N,\eta})^H_{N,L,r,z} - \mathrm{INT}(f) \right| \le C_{f,0} \exp\left(-C_{f,*} L^{C^*/d}\right),$$

where $C_{f,0}$ represents a positive constant dependent on both the function f and the dimension d, $C_{f,*}$ denotes a positive constant dependent on f, and C^* is a positive absolute constant.

Let $\mathrm{median}\{I(Tf_{N,\eta_j})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^t$ be defined as in Theorem 2.6. Then the following inequality holds with a probability of at least $1 - (2\sqrt{1/(\ln L)})^t$:

$$\left| \mathrm{median}\{I(Tf_{N,\eta_j})^{H_{N_j}}_{N,L,r,z_j}\}_{j=1}^t - \mathrm{INT}(f) \right| \le C_{f,0} \exp\left(-C_{f,*} L^{C^*/d}\right).$$

Furthermore,

$$\left(\mathbb{E}_{\{H_{N_j},z_j,\eta_j\}_{j=1}^t}\left|\text{median}\{I(Tf_{N,\eta})_{N,L,r,z_j}^{H_{N_j}}\}_{j=1}^t-\text{INT}(f)\right|^2\right)^{1/2}\leq C_{f,0}\exp\left(-C_{f,*}L^{C^*/d}\right)+(2\sqrt{1/(\ln L)})^t.$$

Proof. We start the proof by estimating $\|\partial^k(Tf)\|_{L_1([0,1]^d)}$. Let $(g\circ h)(y):=\exp(-1/(y(1-y)))$. For every $k\in\mathbb{N}$, using Lemma 4.1, we have

$$\sup_{y\in(0,1)}\left|\frac{d^k}{dy^k}(g\circ h)(y)\right|\leq c_1(4k/e)^{2k}k!,$$

where $c_1=2e^2/\int_0^1\exp(-1/(\xi(1-\xi)))d\xi$ represents an absolute constant. For the last inequality in (Eq. (58)), we have used the fact that the function x^{2k}/e^x attains its maximum on the interval $(1,+\infty)$ at $x=2k$.

Combining (Eq. (54)), (Eq. (58)) with Lemma 4.1, we deduce that

$$\left\|\frac{\partial^k}{\partial x_j^k}(Tf)\right\|_{L_1([0,1]^d)}\leq 4^kC_{f,1}(C_{f,2})^kk!(c_1+1)^k(4k)^{2k}k!\quad\text{for every }k\in\mathbb{N}\text{ and }1\leq j\leq d.$$

From (Eq. (58)) and (Eq. (59)), by applying Leibniz's formula and Stirling's formula, we can prove that there exist absolute constants C_1 and C_2 such that

$$\left\|\frac{\partial^k}{\partial x_j^k}(Tf)\right\|_{L_1([0,1]^d)}\leq C_{f,1}(C_1C_{f,2}k)^{C_2(k+d)}\quad\text{for every }k\in\mathbb{N}\text{ and }1\leq j\leq d.$$

Employing (Eq. (60)) and integration by parts, we obtain

$$|\widehat{Tf}(w)|\leq C_{f,1}(C_1C_{f,2}k)^{C_2(k+d)}/\|w\|_\infty^k\quad\text{for every }k\in\mathbb{N},$$

where $w=(w_1,w_2,\dots,w_d)\in\mathbb{N}_0^d\ \{0\}$ and $\|w\|_\infty:=\max_{1\leq j\leq d}\{|w_1|,|w_2|,\dots,|w_d|\}$.

By choosing $k=\max\{\lfloor(\|w\|_\infty/e)^{1/(2C_2)}\rfloor,d+1\}$, when $(\|w\|_\infty/e)^{1/(2C_2)}\geq d+1$, we derive

$$|\widehat{Tf}(w)|\leq C_{f,1}\exp\left(-\frac{\|w\|_\infty^{1/(2C_2)}}{C_1C_{f,2}}\right).$$

Consequently, there exist positive constants C_3 and $C_{f,3}$ such that

$$|\widehat{Tf}(w)| \leq C_{f,1} e^{-\|w\|_\infty^{C_3}} \quad \text{for all } \|w\|_\infty \geq d+1.$$

Next, let $\beta > 1$, $\rho > 1$, and $\lambda > 0$. Through a series of deductions, we obtain

$$\sum_{m=1}^{\infty} m^d \rho^{-(m\beta)^\lambda} \leq C_{\rho,\lambda,d} \rho^{-\beta^\lambda},$$

where $C_{\rho,\lambda,d} := \sum_{m=1}^{\infty} m^d \rho^{-m^\lambda+1}$. Therefore, for $\beta > 1$, there exists a constant $C_{f,4,d}$ depending on f and d, and a constant $C_{f,5}$ depending on f, such that

$$\left(\sum_{\|w\|_\infty > \beta} |\widehat{Tf}(w)|^2 \right)^{1/2} \leq C_{f,4,d} \beta^{d/2} e^{-\beta^{C_3}} C_{f,5}.$$

Let $K \geq 2$ be a positive integer, and define $\beta_{K,d} = ((K-1)^{1/d} - 1)/2$. We decompose Tf as

$$Tf(x) = a_0 + g(x) + R(x),$$

where

$$g(x) = \sum_{w \in \mathbb{Z}^d \cap [-\beta_{K,d}, \beta_{K,d}]^d \ \{0\}} a_w \exp(2\pi i w \cdot x), \quad R(x) = \sum_{w \notin \mathbb{Z}^d \cap [-\beta_{K,d}, \beta_{K,d}]^d} a_w \exp(2\pi i w \cdot x).$$

Then

$$\#\{w \mid w \in \mathbb{Z}^d \cap [-\beta_{K,d}, \beta_{K,d}]^d \ \ \{0\}\} \leq K,$$

and

$$\|R\|_{L_2([0,1]^d)} \leq C_{f,4,d} e^{-(K-1)^{C_3/d}} C_{f,5}.$$

By choosing $\sigma = \sigma_2 = 1/(2 \ln L)$ in Theorem 2.6, we successfully complete the proof.

Remark 4.3. By substituting $\epsilon = \exp(-L^{C^*/d})$ into (Eq. (55)), we demonstrate that the error exhibits semi-exponential behavior in relation to L, a property that holds with a probability of at least $1 - 1/\ln L$. This result is comparable to Theorem 2 presented in [**?**]. Furthermore, the results obtained from Eq. (57) can be compared with Corollary 3 in [**?**]. In [**?**], the upper bound for the convergence rate is articulated as $C_f M^{-C\log(M)/d}$, where C_f is a constant dependent on f and d, and C is an absolute constant such that $C < 0.21$. Asymptotically, our convergence order significantly surpasses the findings reported in [?], despite the fact that the constant in the exponential term of our error bounds is contingent upon f. Moreover, our algorithm also achieves sub-exponential convergence for the infinitely differentiable functions as defined in [?, ?].

Remark 4.4. From the standpoint of sample complexity, our theorem establishes the existence of M samples that can achieve a semi-exponential convergence rate. However, to achieve high precision with this method, it also involves approximating the integral $\int_0^y \exp(-1/(\xi(1-\xi)))d\xi$.

For future investigations, we propose an alternative framework: for any specified sample size M, we will employ B-splines to construct a k_M-order smooth function ψ_{k_M} as a substitute for the

Note: Figure translations are in progress. See original paper for figures.