Self-adjoint operators and nontrivial zeros of Dirichlet L-function

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Abstract

We give a kind of self-adjoint operator, whose spectrums are the set $S_ chi= {i(rho- frac{1}{2}) mid rho text{ is nontrivial zeros of *L*-function } L(chi,s) }$.

Full Text

Preamble

Self-Adjoint Operators and Nontrivial Zeros of Dirichlet L-Functions

Chaochao Sun

Abstract. We construct a self-adjoint operator whose spectrum is the set $S_\chi = \{i(\rho - \frac{1}{2}) \mid \rho$ is a nontrivial zero of the L-function $L(\chi, s)\}$.

Alain Connes first proved that the critical zeros of Hecke L-functions correspond to the spectrum of a suitable operator. His main method involves analyzing the relationship between L^2 of the adele class space and L^2 of the idele class space via the adelic Riemann-Roch theorem. In [4, Thm. 4.16], Connes, Consani, and Marcolli give a spectral realization of the zeros of Dirichlet L-functions as the action of $\mathbb{R}_+^*$ on a suitable space.

Motivated by Connes's spectral interpretation for zeros of L-functions, Ralf Meyer developed an alternative spectral interpretation for the poles and zeros in André Weil's explicit formula, though it is no longer directly related to the Riemann hypothesis. André Weil's explicit formula also has connections with Riesz potentials (see [6]). In [9, Corollary 4.2], R. Meyer proved that the eigenvalues of the transpose D_-^t of the operator D_- (induced by D on some function space) acting on a space of continuous linear functionals are exactly the nontrivial zeros of $\zeta(s)$.

Furthermore, Xian-Jin Li proved that every nontrivial zero of the zeta function is indeed an eigenvalue of D_-. His method was generalized to Dirichlet L-functions by Dongsheng Wu [13]. Liming Ge, Xian-Jin Li, Dongsheng Wu, and Boqing Xue [5] proved that the correspondence between the set of eigenvalues of D_- acting on H and the set of nontrivial zeros of $\zeta(s)$ is one-to-one.

Inspired by these results, we construct a suitable self-adjoint operator related to the nontrivial zeros of Dirichlet L-functions. Our approach follows the classical Hilbert-Pólya conjecture.

1. The Self-Adjoint Operator

Let $\mathbb{R}_+^\times = (0, \infty)$. Denote by $C^\infty(\mathbb{R}_+^\times)$ the set of smooth complex-valued functions on $\mathbb{R}_+^\times$. Define

$$H_0 = \left\{ f \in C^\infty(\mathbb{R}_+^\times) \,\middle|\, \forall m, n \in \mathbb{N}, \lim_{x\to\infty} x^m f^{(n)}(x) = 0 \text{ and } f^{(n)}(0) := \lim_{x\to 0^+} f^{(n)}(x) \text{ exists} \right\}.$$

Let

$$H_\cap = \left\{ f \in H_0 \,\middle|\, \int_0^\infty f(x)\,dx = 0,\ f(0) = 0,\ \text{and } f^{(2n+1)}(0) = 0 \text{ for } n \in \mathbb{N} \right\},$$

and

$$H_- = \{ f \in H_0 \mid f^{(n)}(0) = 0 \text{ for all } n \in \mathbb{N} \}.$$

Let χ be a primitive Dirichlet character. Define

$$H_\chi = \left\{ f \in H_0 \,\middle|\, f^{(2n+1)}(0) = 0 \text{ if } \chi(-1) = 1,\ \text{and } f^{(2n)}(0) = 0 \text{ if } \chi(-1) = -1,\ \forall n \in \mathbb{N} \right\}.$$

The inner product on H_0 is defined by

$$\langle f(x), g(x) \rangle = \int_0^\infty f(x)\overline{g(x)}\,dx.$$

Then H_0 is a unitary space (i.e., a complex inner product space). We define two self-adjoint operators D and M on H_0 by

$$Df(x) = -if'(x), \qquad Mf(x) = xf(x).$$

That is, for $f, g \in H_0$ we have $\langle Df, g\rangle = \langle f, Dg\rangle$ and $\langle Mf, g\rangle = \langle f, Mg\rangle$. These identities are straightforward to verify (see [?, Example 7.1.5, 7.1.6]). It is easy to check that

$$MD - DM = i. \tag{1.1}$$

Our key result is the following:

Theorem 1.1. $MD - \frac{i}{2}$ is a self-adjoint operator on H_0.

Proof. From equation (1.1), for $f, g \in H_0$ we have

$$\langle MDf, g\rangle = \langle f, MDg\rangle + i\langle f, g\rangle.$$

Therefore,

$$\langle (MD - \tfrac{i}{2})f, g\rangle = \langle MDf, g\rangle - \tfrac{i}{2}\langle f, g\rangle = \langle f, MDg\rangle + i\langle f, g\rangle - \tfrac{i}{2}\langle f, g\rangle = \langle f, MDg\rangle + \tfrac{i}{2}\langle f, g\rangle = \langle f, (MD - \tfrac{i}{2})g\rangle.$$

Hence, $MD - \frac{i}{2}$ is a self-adjoint operator.

Lemma 1.2. H_- is an invariant subspace of D and M; consequently, $MD - \frac{i}{2}$ is a self-adjoint operator on H_-.

Proof. It is easy to verify that for $f \in H_-$, we have $Df, Mf \in H_-$. Hence D and M are operators on H_-, and $MD - \frac{i}{2}$ is self-adjoint on this subspace.

For $f \in H_\cap$, define the operator Z by

$$(Zf)(x) = \sum_{n=1}^{\infty} f(nx),$$

and for $f \in H_\chi$, define the operator Z_χ by

$$(Z_\chi f)(x) = \sum_{n=1}^{\infty} \chi(n) f(nx).$$

Then we have $ZH_\cap \subset H_-$ and $Z_\chi H_\chi \subset H_-$ (see [13, Thm. 2.9]). Let $(ZH_\cap)^\perp$ and $(Z_\chi H_\chi)^\perp$ denote the orthogonal complements in H_-, i.e., the sets of vectors orthogonal to $ZH_\cap$ and $Z_\chi H_\chi$, respectively. We propose the following conjecture:

Conjecture 1.3.

$$H_- = (ZH_\cap)^\perp \oplus ZH_\cap = (Z_\chi H_\chi)^\perp \oplus Z_\chi H_\chi. \tag{1.2}$$

Theorem 1.4. Under Conjecture (1.3), we have canonical isomorphisms $(ZH_\cap)^\perp \cong H_-/ZH_\cap$ and $(Z_\chi H_\chi)^\perp \cong H_-/Z_\chi H_\chi$ that are unitary spaces. Moreover, $H_-/ZH_\cap$ and $H_-/Z_\chi H_\chi$ are unitary spaces, and $MD - \frac{i}{2}$ is a self-adjoint operator on them.

Proof. Since $ZH_\cap$ and $Z_\chi H_\chi$ are invariant subspaces of $MD-\frac{i}{2}$, we have $(ZH_\cap)^\perp$ and $(Z_\chi H_\chi)^\perp$ as invariant subspaces of $MD - \frac{i}{2}$. Furthermore, $MD - \frac{i}{2}$ is a self-adjoint operator on them.

Remark 1.5. $ZH_\cap$ and $Z_\chi H_\chi$ are invariant subspaces of MD, but they are not invariant under M or D alone.

Theorem 1.6. The Riemann hypothesis holds under Conjecture (1.3).

Proof. Let $S = \{i(\rho - \frac{1}{2}) \mid \rho \text{ is a nontrivial zero of } \zeta(s)\}$ and $S_\chi = \{i(\rho - \frac{1}{2}) \mid \rho \text{ is a nontrivial zero of } L(\chi, s)\}$. Then by Theorem 1.2 and Theorem 1.3 in [13], the spectrum of $MD - \frac{i}{2}$ on $H_-/ZH_\cap$ is S and on $H_-/Z_\chi H_\chi$ is S_χ. Since $MD-\frac{i}{2}$ is a self-adjoint operator, we have $S, S_\chi \subset \mathbb{R}$, which implies the Riemann hypothesis.

2. Connes's Method

Let $\mathbb{R}_+^*$ be the multiplicative group of positive real numbers and $L^2(\mathbb{R}_+^*)$ be the Hilbert space of square-integrable complex-valued functions on $\mathbb{R}_+^*$ with respect to the Haar measure d^*x on $\mathbb{R}_+^*$. We consider the smooth function space with compact support $C_c^\infty(\mathbb{R}_+^*)$. Let $C_c^\infty(\mathbb{R}_+^*)_0$ be the subspace of $C_c^\infty(\mathbb{R}_+^*)$ consisting of those $f \in C_c^\infty(\mathbb{R}_+^*)$ such that

$$\int_{\mathbb{R}_+^*} x f(x)\, d^*x = 0.$$

Let $L^2(\mathbb{R})_{\text{ev}}$ be the Hilbert space of square-integrable even functions on $\mathbb{R}$. The inner product in $L^2(\mathbb{R})_{\text{ev}}$ is normalized as

$$\langle \eta, \xi \rangle := \int_{-\infty}^{\infty} \eta(x)\overline{\xi(x)}\, dx = 2\int_0^\infty \eta(x)\overline{\xi(x)}\, dx.$$

Then $L^2(\mathbb{R})_{\text{ev}}$ is isomorphic to $L^2(\mathbb{R}_+^*)$ by the unitary isomorphism (see [3, Equation 17])

$$w : L^2(\mathbb{R})_{\text{ev}} \to L^2(\mathbb{R}_+^*), \quad (w\xi)(\lambda) := \lambda^{1/2}\xi(\lambda).$$

Let $S(\mathbb{R})$ be the Schwartz space of $\mathbb{R}$. Denote by $S(\mathbb{R}_+^*) := w(S(\mathbb{R}) \cap L^2(\mathbb{R})_{\text{ev}})$ the Schwartz space of $\mathbb{R}_+^*$.

Let $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to S^1$ be a Dirichlet character. Similar to the map C in [2, Chp. 5, §6], we define the following map (also appearing in [9, §6]):

$$\Sigma_\chi : C_c^\infty(\mathbb{R}_+^*)_0 \to L^2(\mathbb{R}_+^*), \quad f(x) \mapsto \sum_{n=1}^{\infty} \chi(n) f(nx).$$

Then $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ is a subspace of $L^2(\mathbb{R}_+^*)$.

Consider the regular representation U of $\mathbb{R}_+^*$ on $L^2(\mathbb{R}_+^*)$:

$$(U(\lambda)f)(x) := f(\lambda^{-1}x), \quad f \in L^2(\mathbb{R}_+^*),\ \lambda \in \mathbb{R}_+^*.$$

Let $U(\lambda)^t$ be the transpose of $U(\lambda)$, i.e., $(U(\lambda)^{tf})(x) = f(\lambda x)$. Then

$$\langle U(\lambda)f, g\rangle = \langle f, U(\lambda)^{tg}\rangle,$$

which follows from the change of variables $y = \lambda^{-1}x$:

$$\int_{\mathbb{R}_+^*} f(\lambda^{-1}x)\overline{g(x)}\, d^*x = \int_{\mathbb{R}_+^*} f(y)\overline{g(\lambda y)}\, d^*y.$$

Since $\mathbb{R}_+^*$ is a one-parameter group, the representation U is generated by a unique unbounded operator D (see [12, Thm. 6.2]) with

$$Df(x) = \lim_{h\to 0} \frac{f(e^{-h}x) - f(x)}{h} = -x\frac{d}{dx}f(x),$$

which is similar to the case appearing in [1, III, equation (26)]. By Theorem 1.1, we know that $i(D - \frac{1}{2})$ is self-adjoint. The orthogonal complement of $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ in $L^2(\mathbb{R}_+^*)$, denoted $H_\chi := (\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0)^\perp$, is closed; hence D can act on this subspace. We denote the restriction of D to H_χ by D_χ.

Since we are working with the orthogonal complement of an invariant subspace, we can assume that $i(D_\chi - \frac{1}{2})$ is a self-adjoint operator. The operator D acting on $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ satisfies

$$U^t(h)\eta = \eta \tag{2.1}$$

for some $h \in C^\infty(\mathbb{R}_+^*)$ such that $\hat{h}$ has compact support. Here $U^t(h)$ is an operator on $L^2(\mathbb{R}_+^*)$ defined by

$$U^t(h)f(x) = \int_{\mathbb{R}_+^*} h(\gamma)f(\gamma x)\, d^*\gamma.$$

Then we have the following theorem, following the idea of Connes's result [1, III. Thm. 1].

Theorem 2.1. For the Hilbert space H_χ, the operator D_χ has discrete spectrum, denoted $\operatorname{Sp} D_\chi$. In fact, $\operatorname{Sp} D_\chi$ is precisely the set of nontrivial zeros of the L-function $L(\chi, s)$.

Proof. Take $\psi \in S(\mathbb{R}_+^*)$. Then ψ belongs to the orthogonal complement of $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ if and only if

$$\int_{\mathbb{R}_+^*} (\Sigma_\chi f)(x)\psi(x)\, d^*x = 0, \quad \forall f \in C_c^\infty(\mathbb{R}_+^*)_0. \tag{2.2}$$

Consider the pairing $\mathbb{R}_+^* \times \mathbb{R} \to S^1$, $(r,t) \mapsto r^{it}$. Under this pairing, $\mathbb{R}$ can be viewed as the character group of $\mathbb{R}_+^*$. Consider the Fourier transform

$$\widehat{\psi}(t) = \int_{\mathbb{R}_+^*} \psi(x) x^{-it}\, d^*x$$

(see [3, Equation 21], or [11, Definition in §3.3, P. 102]), which is a meromorphic function over $\mathbb{C}$ whose only singularities are simple poles at a subset of non-positive integers [13, Lemma 2.1]. Then the inverse Fourier transform (see [11, Theorem 3.9]) is given by

$$\psi(x) = \int_{-\infty}^{\infty} \widehat{\psi}(t) x^{it}\, dt.$$

Substituting this formula into (2.2), we formally obtain

$$\int_{\mathbb{R}_+^*} \int_{-\infty}^{\infty} (\Sigma_\chi f)(x) x^{it} \widehat{\psi}(t)\, d^*x\, dt = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \chi(n) f(nx) \widehat{\psi}(t) x^{it}\, d^*x\, dt.$$

For $\operatorname{Im}(t) < -1$, we have

$$\int_{\mathbb{R}_+^*} \chi(n) f(nx) \widehat{\psi}(t) x^{it}\, d^*x = \widehat{\psi}(t) \int_{\mathbb{R}_+^*} f(y) y^{it}\, d^*y.$$

If χ is nontrivial, we have the analytic continuation for $t \in \mathbb{C}$:

$$\int_{\mathbb{R}_+^*} \chi(n) f(nx) \widehat{\psi}(t) x^{it}\, d^*x = L(\chi, it) \widehat{\psi}(t) \int_{\mathbb{R}_+^*} f(y) y^{it}\, d^*y.$$

Then by Fubini's theorem, we know that for nontrivial character χ,

$$\int_{\mathbb{R}_+^*} \int_{-\infty}^{\infty} (\Sigma_\chi f)(x) x^{it} \widehat{\psi}(t)\, d^*x\, dt = \int_{-\infty}^{\infty} L(\chi, it) \widehat{\psi}(t) \int_{\mathbb{R}_+^*} f(y) y^{it}\, d^*y\, dt.$$

If χ is trivial, then $L(\chi, it)$ has a pole at $t = -i$ and we must use the condition

$$\int_{\mathbb{R}_+^*} f(y) y^{i(-i)}\, d^*y = \int_{\mathbb{R}_+^*} f(y)\, dy = 0. \tag{2.4}$$

But the space of functions $f(x) \in C_c^\infty(\mathbb{R}_+^*)$ satisfying (2.4) is still dense in the Schwartz space $S(\mathbb{R}_+^*)$.

Combining (2.2) and the above computation, we obtain

$$L(\chi, it)\widehat{\psi}(t) = 0. \tag{2.5}$$

Since $L(\chi, it)$ is an analytic function of t, it is a multiplier of the algebra $S(\mathbb{R})$ of Schwartz functions in the variable t. Moreover, $|L(\chi, it)| = O(|t|^N)$ (see [10, 5.3]). Thus the product $L(\chi, it)\widehat{\psi}(t)$ is still a tempered distribution, and so is its Fourier transform. If the latter vanishes when tested against arbitrary smooth compactly supported functions, then $L(\chi, it)\widehat{\psi}(t)$ must vanish identically.

To understand equation (2.5), consider an equation for distributions $\alpha(t)$ on S^1 of the form

$$\phi(t)\alpha(t) = 0. \tag{2.6}$$

Assume $\phi(t) \in C^\infty(S^1)$ has finitely many zeros x_i of finite order n_i. Let J be the ideal of $C^\infty(S^1)$ generated by ϕ. Then for any $\psi \in J$, the order of ψ at x_i is no less than n_i [1, P. 86]. Thus the distributions $\delta_{x_i}, \delta'_{x_i}, \dots, \delta_{x_i}^{(n_i-1)}$ form a basis for the solution space of (2.6).

Now $\widehat{\psi}(t)$ is a distribution with compact support such that ψ is orthogonal to $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ and $L(\chi, it)\widehat{\psi}(t) = 0$. Therefore $\widehat{\psi}$ is a finite linear combination of distributions supported at the zeros of $L(\chi, it)$.

Conversely, let s be a zero of $L(\chi, s)$ and $k > 0$ its order. Define

$$\Delta_s(f) := \int_{\mathbb{R}_+^*} (\Sigma_\chi f)(x) x^s\, d^*x. \tag{2.7}$$

For $f \in C_c^\infty(\mathbb{R}_+^*)_0$, we have

$$\Delta_s(f) = \int_{\mathbb{R}_+^*} \sum_{n=1}^{\infty} \chi(n) f(nx) x^s\, d^*x = L(\chi, s) \int_{\mathbb{R}_+^*} f(y) y^s\, d^*y.$$

Hence $\Delta_s(f) = 0$ for $a = 0, 1, \dots, k-1$. Differentiating (2.7), we obtain

$$\Delta_s^{(a)}(f) = \int_{\mathbb{R}_+^*} (\Sigma_\chi f)(x) x^s (\log x)^a \, d^*x.$$

Thus η belongs to the orthogonal complement of $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ and satisfies (2.1) if and only if it is a finite linear combination of functions of the form

$$\eta_{t,a}(x) = x^{it}(\log x)^a, \quad \text{where } L^{(a)}(\chi, it) = 0, \; a < \text{order of the zero.}$$

The transpose of the representation U is thus given in this basis by

$$U(\lambda)^t \eta_{t,a}(x) = \eta_{t,a}(\lambda x) = (\lambda x)^{it}(\log(\lambda x))^a = \lambda^{it} \sum_{b=0}^{a} \binom{a}{b} (\log \lambda)^b \eta_{t,a-b}(x). \tag{2.8}$$

Multiplication by a function with bounded derivatives is a bounded operator in any Sobolev space. Using the density in the orthogonal complement of the range of Σ_χ of vectors satisfying (2.1), one can verify that if $L(\chi, is) \neq 0$, then is does not belong to the spectrum of D^t and hence of its transpose D_χ. This determines the spectrum of the operator D^t and consequently of D_χ.

Corollary 2.2. If Theorem 2.1 holds, then the Riemann hypothesis is true.

Proof. Since $i(D_\chi - \frac{1}{2})$ is a self-adjoint operator on the inner product space H_χ, the theorem follows from Theorem 2.1.

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