Characterization for Lipschitz Spaces via commutators of Some Maximal Functions on p-adic Orlicz Spaces

Authors: Yunpeng Chang, Jianglong Wu, Yunpeng Chang

Date: 2024-03-01T00:00:00+00:00

Abstract

In this paper, the main aim is to demonstrate the boundedness for commutators of fractional maximal function and sharp maximal function in the context of the p-adic version of Orlicz spaces, where the symbols of the commutators belong to the p-adic version of Lipschitz space, whereby some new characterizations for Λβ(Qnp) spaces are given.

Full Text

Characterization of Lipschitz Spaces via Commutators of Some Maximal Functions on p-adic Orlicz Spaces

Yunpeng Chang and **Jianglong Wu**
Department of Mathematics, Mudanjiang Normal University, Mudanjiang, 157011, China

Corresponding author

Abstract: In this paper, our main aim is to demonstrate the boundedness of commutators of fractional maximal functions and sharp maximal functions in the context of p-adic Orlicz spaces, where the symbols of the commutators belong to the p-adic Lipschitz space. This yields new characterizations for $\Lambda_\beta(\mathbb{Q}_p^n)$ spaces.

Keywords: commutator; Lipschitz space; Orlicz space; p-adic field; fractional maximal function; sharp maximal function

AMS (2020) Subject Classification: 26A33, 11E95, 42B35, 11S80

Introduction and Main Results

Research on p-adic fields spans multiple disciplines, including mathematics, theoretical physics, and computer science. In mathematics, p-adic fields constitute an important branch of number theory, primarily investigating the properties and structures of p-adic numbers [?, ?]. In theoretical physics, p-adic theory finds wide application in quantum mechanics, string theory, and cosmology. Specifically, p-adic numbers can describe particle states and interactions in quantum mechanics [?, ?, ?], while in string theory they characterize string vibration patterns and interactions. Additionally, p-adic numbers are employed to describe the geometric structure and evolution of the universe in cosmology. In computer science, p-adic theory is applied in cryptography, computer graphics, and data encryption [?, ?, ?]. In cryptography, p-adic numbers provide methods for encryption and decryption, while in computer graphics they can generate images with complex geometric shapes. In summary, research on p-adic fields involves multiple disciplines with broad applications and profound research significance.

Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the fields of integers and rational numbers, respectively. For a fixed prime number p, the p-adic field $\mathbb{Q}_p$, originally introduced by K. Hensel in 1908, is the completion of $\mathbb{Q}$ with respect to the non-Archimedean p-adic absolute value. For $x \in \mathbb{Q}$, write $x = p^\gamma \frac{a}{b}$ where $\gamma \in \mathbb{Z}$, and a, b are non-zero integers not divisible by p. The p-adic absolute value is defined as $|x|_p = p^{-\gamma}$.

It is well known that the non-Archimedean p-adic absolute value shares many properties with the Archimedean absolute value, such as positive definiteness, multiplicativity, and the non-Archimedean inequality. Specifically, these properties are:

1. $|x|_p \geq 0$, and $|x|_p = 0$ if and only if $x = 0$;
2. $|xy|_p = |x|_p |y|_p$;
3. $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. If $|x|_p \neq |y|_p$, then equality holds, and the converse is also true.

Combining properties (1) and (3), we obtain the same triangle inequality as for the Archimedean absolute value, namely $|x + y|_p \leq |x|_p + |y|_p$.

From standard p-adic analysis, any non-zero p-adic number x can be written as

$$x = p^\gamma (a_0 + a_{1p} + a_{2p}^2 + \cdots) = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad a_j = 0, \dots, p-1,$$

where $|x|_p = p^{-\gamma}$ when $a_\gamma \neq 0$. Naturally, the above p-adic series converges.

Next, we consider the n-dimensional p-adic linear space $\mathbb{Q}_p^n$. When $n = 1$, this reduces to the case described above. For any n-dimensional vector $x = (x_1, x_2, \dots, x_n)$ where $x_i \in \mathbb{Q}_p$ $(i = 1, \dots, n)$, the p-adic absolute value is given by

$$|x|_p = \max_{1 \leq j \leq n} |x_j|_p.$$

The p-adic ball is denoted by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p \le p^\gamma\},$$

with center $a \in \mathbb{Q}_p^n$ and radius p^γ where $\gamma \in \mathbb{Z}$. The corresponding p-adic sphere is denoted by

$$S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p = p^\gamma\} = B_\gamma(a) \setminus B_{\gamma-1}(a).$$

In particular, if $a = 0$ and $\gamma = 0$, then $B_0(0)$ and $S_0(0)$ are called the p-adic unit ball and p-adic unit sphere, respectively. Moreover, when $a = 0$, we typically omit the center in the notation for p-adic balls and spheres.

From the definition of p-adic balls and spheres, we observe the following relationship: $B_\gamma(a) = \bigcup_{k\le\gamma} S_k(a)$. For any $a_0 \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$, it is not difficult to obtain the equalities

$$\mathbb{Q}_p^n \setminus \{0\} = \bigcup_{\gamma\in\mathbb{Z}} S_\gamma, \quad a_0 + B_\gamma = B_\gamma(a_0), \quad a_0 + S_\gamma = S_\gamma(a_0) = B_\gamma(a_0) \setminus B_{\gamma-1}(a_0).$$

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists a Haar measure on $\mathbb{Q}_p^n$. It is easy to see that the unique Haar measure dx on $\mathbb{Q}_p^n$ (up to a positive constant multiple) satisfies translation invariance (i.e., $d(x+a) = dx$). Here we normalize the measure so that

$$\int_{B_0} dx = |B_0|_h = 1,$$

where $|B_0|_h$ denotes the Haar measure of the p-adic unit ball. In general, for any $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$, we have

$$\int_{B_\gamma(a)} dx = |B_\gamma(a)|_h = p^{n\gamma},$$

and

$$\int_{S_\gamma(a)} dx = |S_\gamma(a)|_h = p^{n\gamma}(1-p^{-n}) = |B_\gamma(a)|_h - |B_{\gamma-1}(a)|_h.$$

For more details about p-adic analysis, we refer readers to [**?**, **?**] and the references therein.

As is well known, the study of commutators has attracted considerable attention due to their many applications in partial differential equations and harmonic analysis. Let T be a classical singular integral operator. The Coifman-Rochberg-Weiss type commutator $[b,T]$ generated by T and a suitable function b is defined by

$$[b,T]f = bT(f) - T(bf).$$

It is well known that $b \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded on $L^s(\mathbb{R}^n)$ for $1 < s < \infty$, as proved by Coifman, Rochberg, and Weiss [**?**] (see also [**?**]). In [**?**], Janson proved that the necessary and sufficient condition for the boundedness of commutators from $L^s(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ ($1 < s < \frac{n}{\beta}$ and $\frac{1}{q} = \frac{1}{s} - \frac{\beta}{n}$ with $0 < \beta < 1$) is $b \in \Lambda_\beta(\mathbb{R}^n)$, thus providing characterizations of the Lipschitz space $\Lambda_\beta(\mathbb{R}^n)$.

Let $0 < \alpha < n$. We define the p-adic fractional maximal operator as

$$M^p_\alpha(f)(x) = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y)|\, dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$. Then the maximal commutator of M^p_α with b is given by

$$M^p_{\alpha,b}(f)(x) = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(x) - b(y)||f(y)|\, dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$. Furthermore, assuming $b : \mathbb{Q}_p^n \to \mathbb{R}$ and $f : \mathbb{Q}_p^n \to \mathbb{R}$ are measurable functions, the nonlinear commutators of the fractional maximal operator can be defined as follows:

$$[b, M^p_\alpha]f(x) = b(x)M^p_\alpha(f)(x) - M^p_\alpha(bf)(x).$$

It is easy to see that $M^p_{\alpha,b}$ is positive and sublinear, while $[b, M^p_\alpha]$ is neither positive nor sublinear. The fractional maximal operator and its commutators generated by functions of different forms have been studied by many authors and applied to various contexts; see, for example, [**?**, **?**, **?**, **?**]. Specifically, for the case where the symbol $b \in \mathrm{BMO}(\mathbb{R}^n)$, Bastero et al. [**?**] obtained the boundedness of $[b, M]$ on $L^q(\mathbb{R}^n)$ for $1 < q < \infty$. Zhang and Wu further considered the same problem for commutators of fractional maximal functions in [**?**], obtaining results on variable Lebesgue spaces, as well as for the commutator $[b, M^\sharp]$. When the symbol $b \in \Lambda_\beta(\mathbb{R}^n)$, Guliyev, Deringoz, and Hasanov [**?**] established the boundedness of fractional maximal commutators and nonlinear commutators of fractional maximal functions on Orlicz spaces. In [**?**], Zhang et al. extended some results of [**?**] to nonnegative Lipschitz functions and obtained characterizations for the boundedness of the maximal commutator $M_{\alpha,b}$.

The commutators of fractional maximal operators $[b, M^p_\alpha]$ have been studied by many authors in the p-adic setting; see, for instance, [**?**, **?**, **?**, **?**, **?**]. Recently, He and Li [**?**] gave necessary and sufficient conditions for the boundedness of some commutators of maximal functions on p-adic linear spaces. Moreover, Wu and Chang [**?**, **?**] not only extended these results to commutators of fractional maximal functions but also to p-adic variable Lebesgue spaces, as well as to the commutator $[b, M^\sharp]$. However, the study of Orlicz spaces in the p-adic setting remains quite limited, which appears worthy of further investigation.

Motivated by the aforementioned literature, the purpose of this article is to obtain boundedness results for commutators of fractional maximal functions and sharp maximal functions in the context of p-adic Orlicz spaces, where the symbols of the commutators belong to p-adic Lipschitz spaces, thereby providing new characterizations for $\Lambda_\beta(\mathbb{Q}_p^n)$ spaces. Our results are stated as follows.

For a fixed p-adic ball B^* and $\alpha > 0$, the fractional maximal function with respect to B^* of a locally integrable function f is given by

$$M^p_{\alpha,B^*}(f)(x) = \sup_{B_\gamma(x)\subset B^*} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y)|\, dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset B^*$. When $\alpha = 0$, $M^p_0 = M^p$.

Naturally, we need to consider the following boundedness result for the nonlinear commutators $[b, M^p_\alpha]$ and $M^p_{\alpha,b}$.

Theorem 1.1. Assume that $b \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ and $b \geq 0$. Suppose that Φ and Ψ are Young functions satisfying $\Phi \in Y \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n})$. Then the following statements are equivalent:

1. $b \in \Lambda_\beta(\mathbb{Q}_p^n)$;
2. $[b, M^p_\alpha]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$;
3. There exists a constant $C > 0$ such that

$$|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - M^p_{B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \leq C; \qquad (1)$$

4. There exists a constant $C > 0$ such that

$$\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - M^p_{B_\gamma(x)}(b)(y)|\, dy \leq C. \qquad (2)$$

Remark 1. Theorem 1.1 obtains new characterizations of non-negative Lipschitz functions. Similar conclusions were shown in Lebesgue spaces and variable exponent Lebesgue spaces in [**?**, **?**]. Letting $\Psi^{-1}(|B_\gamma(x)|_h^{-1}) = \|\chi_{B_\gamma(x)}\|^{-1}_{L^\Psi(\mathbb{Q}_p^n)}$ (see Lemma 2.3 below), then (1.1) is equivalent to the following inequality:

$$\frac{1}{|B_\gamma(x)|_h} \frac{\|(b - M^p_{B_\gamma(x)}(b))\chi_{B_\gamma(x)}\|_{L^\Psi(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^\Psi(\mathbb{Q}_p^n)}} \leq C,$$

which can be compared with known results. For instance, let $q(\cdot)$ be a variable exponent satisfying the condition that the Hardy-Littlewood maximal operator is bounded on $L^{q(\cdot)}(\mathbb{Q}_p^n)$. Then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$ if and only if

$$\frac{1}{|B_\gamma(x)|_h} \frac{\|(b - M^p_{B_\gamma(x)}(b))\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \leq C.$$

For the case $\Phi(t) = t^r$ and $\Psi(t) = t^q$, we obtain the following result (see also [**?**]):

Corollary 1.1. Assume that $b \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ and $b \geq 0$. If $1 < r < \frac{n}{\alpha+\beta}$ and $\frac{1}{q} = \frac{1}{r} - \frac{\alpha+\beta}{n}$, then the following statements are equivalent:

1. $b \in \Lambda_\beta(\mathbb{Q}_p^n)$;
2. $[b, M_\alpha^p]$ is bounded from $L^r(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$;
3. There exists a positive constant C such that

$$\frac{1}{|B_\gamma(x)|_h} \left(\int_{B_\gamma(x)} |b(y) - M^p_{B_\gamma(x)}(b)(y)|^q \, dy \right)^{1/q} \leq C;$$

4. There exists a positive constant C such that

$$\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - M^p_{B_\gamma(x)}(b)(y)| \, dy \leq C.$$

For the case $\alpha = 0$ in Theorem 1.1, we have the following result, which extends Theorem 4 in [**?**] to Orlicz spaces.

Corollary 1.2. Assume that $b \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ and $b \geq 0$. Suppose that Φ and Ψ are Young functions satisfying $\Phi \in Y \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma\beta}\Phi^{-1}(p^{-\gamma n})$. Then the following statements are equivalent:

1. $b \in \Lambda_\beta(\mathbb{Q}_p^n)$;
2. $[b, M^p]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$;
3. There exists a constant $C > 0$ such that

$$\Phi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - M^p_{B_\gamma(x)}(b)(\cdot)\|_{L^\Phi(B_\gamma(x))} \leq C;$$

4. There exists a constant $C > 0$ such that

$$\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - M^p_{B_\gamma(x)}(b)(y)| \, dy \leq C.$$

Theorem 1.2. Assume that $b \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ and $b \geq 0$. Suppose that Φ and Ψ are Young functions satisfying $\Phi \in Y \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n})$. Then the following statements are equivalent:

1. $b \in \Lambda_\beta(\mathbb{Q}_p^n)$;
2. $M^p_{\alpha,b}$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$;
3. There exists a positive constant C such that

$$|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - b_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))} \leq C; \tag{3}$$

4. There exists a positive constant C such that

$$\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - b_{B_\gamma(x)}| \, dy \leq C. \tag{4}$$

Remark 2. Theorem 1.2 obtains new characterizations of Lipschitz functions. Similar results can be found in p-adic variable exponent Lebesgue spaces in [**?**].

For the case $\alpha = 0$ in Theorem 1.2, we have the following result, which extends Theorem 1 in [**?**] to Orlicz spaces.

Corollary 1.3. Assume that $b \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$, $0 < \beta < 1$, and suppose that Φ and Ψ are Young functions satisfying $\Phi \in Y \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma\beta}\Phi^{-1}(p^{-\gamma n})$. Then the following statements are equivalent:

1. $b \in \Lambda_\beta(\mathbb{Q}_p^n)$;
2. M_b^p is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$;
3. There exists a positive constant C such that

$$|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - b_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))} \leq C;$$

4. There exists a positive constant C such that

$$\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - b_{B_\gamma(x)}| \, dy \leq C.$$

To introduce further results, we also need the p-adic version of the sharp maximal function. For a locally integrable function f on $\mathbb{Q}_p^n$, define in [**?**]:

$$M^{\sharp,p}(f)(x) = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}| \, dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$ and $f_{B_\gamma(x)} = \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} f(y) \, dy$.

The following theorem introduces the commutator of the sharp maximal function with a Lipschitz function b in Orlicz spaces.

Theorem 1.3. Assume that $b \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ and $b \geq 0$. Suppose that Φ and Ψ are Young functions satisfying $\Phi \in Y \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma\beta}\Phi^{-1}(p^{-\gamma n})$. Then the following statements are equivalent:

1. $b \in \Lambda_\beta(\mathbb{Q}_p^n)$;
2. $[b, M^{\sharp,p}]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$;
3. There exists a positive constant C such that

$$|B_\gamma(x)|_h \Phi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - 2(p-1) M^{\sharp,p}(b\chi_{B_\gamma(x)})(\cdot)\|_{L^\Phi(B_\gamma(x))} \leq C; \quad (5)$$

4. There exists a positive constant C such that

$$\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - 2(p-1) M^{\sharp,p}(b\chi_{B_\gamma(x)})(y)| \, dy \leq C. \quad (6)$$

Remark 3. Theorem 1.3 obtains new characterizations of non-negative Lipschitz functions that differ from Theorem 1.2. Similar conclusions were shown in Lebesgue spaces in [**?**, **?**].

For the case $\Phi(t) = t^r$, we obtain the following result (see also [**?**]):

Corollary 1.4. Assume that $b \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ and $b \geq 0$. If $1 < r < \frac{n}{\beta}$ and $\frac{1}{q} = \frac{1}{r} - \frac{\beta}{n}$, then the following statements are equivalent:

1. $b \in \Lambda_\beta(\mathbb{Q}_p^n)$;
2. $[b, M^{\sharp,p}]$ is bounded from $L^r(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$;
3. There exists a positive constant C such that

$$\frac{1}{|B_\gamma(x)|_h} \left(\int_{B_\gamma(x)} |b(y) - 2(p-1) M^{\sharp,p}(b\chi_{B_\gamma(x)})(y)|^q \, dy \right)^{1/q} \leq C;$$

4. There exists a positive constant C such that

$$\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - 2(p-1) M^{\sharp,p}(b\chi_{B_\gamma(x)})(y)| \, dy \leq C.$$

Throughout this paper, the letter C always denotes a constant independent of the main parameters involved and whose value may differ from line to line. In addition, we introduce some notation. Here and hereafter, $|E|_h$ will always denote the Haar measure of a measurable set $E \subset \mathbb{Q}_p^n$, and χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{Q}_p^n$.

Preliminaries

2.1 p-adic Function Spaces

Assume that $1 \leq q < \infty$. We denote by $L^q(\mathbb{Q}_p^n)$ the p-adic Lebesgue space, defined as the space of all functions f with finite norm

$$\|f\|_{L^q(\mathbb{Q}p^n)} = \left(\int_{\mathbb{Q}p^n} |f(x)|^q \, dx \right)^{1/q}.$$

For $q = \infty$, we denote by $L^\infty(\mathbb{Q}_p^n)$ the set of all measurable real-valued functions f on $\mathbb{Q}_p^n$ satisfying

$$\|f\|_{L^\infty(\mathbb{Q}_p^n)} = \operatorname{ess\,sup} |f(x)| = \inf\{\lambda > 0 : |\{x \in \mathbb{Q}_p^n : |f(x)| > \lambda\}|_h = 0\} < \infty.$$

Here, if the limit exists, the integral in the above equation is defined as follows:

$$\int_{\mathbb{Q}p^n} |f(x)|^q \, dx = \lim_{\gamma \to \infty} \int_{B_\gamma(0)} |f(x)|^q \, dx = \lim_{\gamma \to \infty} \sum_{-\infty < k \leq \gamma} \int_{S_k(0)} |f(x)|^q \, dx.$$

In particular, since $\mathbb{Q}_p^n = \bigcup_{\gamma=-\infty}^{\infty} S_\gamma$ and $d(tx) = |t|_p^n dx$ for $t \in \mathbb{Q}_p \setminus \{0\}$, if $f \in L^1(\mathbb{Q}_p^n)$, then

$$\int_{\mathbb{Q}p^n} f(x)\,dx = \sum_{\gamma=-\infty}^{\infty} \int_{S_\gamma} f(x)\,dx, \quad \int_{\mathbb{Q}p^n} f(tx)\,dx = \int_{\mathbb{Q}p^n} f(x)\,dx.$$

A function $\Phi : [0,\infty) \to [0,\infty]$ that is left-continuous, convex, and increasing, and satisfies $\lim_{t\to 0^+} \Phi(t) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$, is called a Young function.

Definition 2.1. Let Φ be a Young function. 1. Denote by Y the set of all functions $\Phi : [0,\infty) \to [0,\infty]$ such that $0 < \Phi(t) < \infty$ for $0 < t < \infty$. 2. Denote by ∇_2 the set of all functions $\Phi : [0,\infty) \to [0,\infty]$ such that for some $K > 1$,

$$\Phi(t) \le \frac{\Phi(Kt)}{2}, \quad t \ge 0.$$

If $\Phi \in Y$, it is easy to show that Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to $[0,\infty)$. Thus there exists an inverse function

$$\Phi^{-1}(s) = \inf\{t \ge 0 : \Phi(t) > s\}, \quad s \in [0,\infty].$$

Definition 2.2. Given a Young function Φ, the Young complement $\bar{\Phi}$ is defined by

$$\bar{\Phi}(x) = \begin{cases} \sup_{0\le y<\infty}(xy - \Phi(y)) & \text{if } y \in [0,\infty), \\ \infty & \text{if } y = \infty. \end{cases}$$

We now define the p-adic Orlicz space. The Orlicz space $L^\Phi(\mathbb{Q}_p^n)$ is defined as the set of all measurable functions $g : \mathbb{Q}_p^n \to \mathbb{R}$ such that for some $\beta > 0$,

$$\int_{\mathbb{Q}p^n} \Phi\left(\frac{|g(x)|}{\beta}\right) dx < \infty.$$

When Φ is a Young function (a convex Orlicz function), the quantity

$$\|f\|_\Phi = \inf\left\{\alpha > 0 : \int_{\mathbb{Q}p^n} \Phi\left(\frac{|f(x)|}{\alpha}\right) dx \le 1\right\}$$

is well known as the Luxemburg norm (see [**?**]).

If $\Phi(t) = t^q$ with $1 \le q < \infty$, then $L^\Phi(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$. If $\Phi(t) = 0$ for $0 \le t \le 1$ and $\Phi(t) = \infty$ for $t > 1$, then $L^\Phi(\mathbb{Q}_p^n) = L^\infty(\mathbb{Q}_p^n)$.

2.2 Auxiliary Propositions and Lemmas

In this section we state some auxiliary propositions and lemmas needed for proving our main theorems, describing only the partial results we require.

The following lemma provides the basic definition of p-adic Lipschitz spaces; the third part can be found in [?].

Lemma 2.1. Assume $0 < \beta < 1$ and let $\mathbb{Q}_p^n$ be an n-dimensional p-adic linear space. 1. The p-adic version of homogeneous Lipschitz spaces $\Lambda_\beta(\mathbb{Q}_p^n)$ is defined by

$$\Lambda_\beta(\mathbb{Q}_p^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n) : \|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} = \sup_{x,y\in\mathbb{Q}p^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|_p^\beta}.$$

2. If $1 \leq q < \infty$, the p-adic version of Lipschitz spaces $\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)$ is defined by

$$\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n) : \|f\|_{\mathrm{Lip}_\beta{}^q(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{Lip}_\beta{}^q(\mathbb{Q}_p^n)} = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h^{1/q}} \left(\int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|^q \, dy \right)^{1/q}.$$

3. The homogeneous Lipschitz space $\Lambda_\beta(\mathbb{Q}_p^n)$ is equivalent to the above $\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)$ spaces; that is,

$$\|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \approx \|f\|_{\mathrm{Lip}_\beta{}^q(\mathbb{Q}_p^n)}.$$

The following result in Euclidean space can be found in [?]; by a similar argument we obtain the p-adic version of Hölder' s inequality on Orlicz spaces.

Lemma 2.2. Let $\mathbb{Q}_p^n$ be an n-dimensional p-adic linear space. For a Young function Φ with its complementary function Ψ, assume that $f \in L^\Phi(\mathbb{Q}_p^n)$ and $g \in L^\Psi(\mathbb{Q}_p^n)$. Then there exists a positive constant C such that

$$\int_{\mathbb{Q}p^n} |f(x)g(x)| \, dx \leq C\|f\|_{L^\Phi(\mathbb{Q}_p^n)}\|g\|_{L^\Psi(\mathbb{Q}_p^n)}.$$

By elementary calculations we obtain the following results for the characteristic function $\chi_{B_\gamma(x)}$.

Lemma 2.3. Assume that Φ is a Young function and $B_\gamma(x)$ is a set in $\mathbb{Q}_p^n$ with finite Haar measure. Then

$$\|\chi_{B_\gamma(x)}\|_{L^\Phi(\mathbb{Q}_p^n)} = \frac{1}{\Phi^{-1}(|B_\gamma(x)|_h^{-1})}.$$

The following results follow from Lemmas 2.2 and 2.3; we omit the proofs.

Lemma 2.4. For a p-adic ball $B_\gamma(x)$ and a Young function Φ, the following inequality holds:

$$\int_{B_\gamma(x)} |f(y)|\,dy \leq C|B_\gamma(x)|_h \Phi^{-1}(|B_\gamma(x)|_h^{-1})\|f\|_{L^\Phi(B_\gamma(x))}.$$

The following results can be found in [?] for Euclidean spaces; by a similar argument we obtain the p-adic version and omit the proof.

Lemma 2.5. Assume that $0 < \alpha < n$, and let Φ, Ψ be Young functions with $\Phi \in Y \cap \nabla_2$. Then for all $\gamma \in \mathbb{Z}$, there exists a positive constant C independent of γ such that the fractional maximal operator M_α^p is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$ if and only if

$$p^{\gamma\alpha}\Phi^{-1}(p^{-\gamma n}) \leq C\Psi^{-1}(p^{-\gamma n}).$$

The following lemma (see [?], Lemma 2.11) can be obtained.

Lemma 2.6. Let $b \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$. For any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$:

1. If $0 \leq \alpha < n$, then for all $y \in B_\gamma(x)$,

$$M_\alpha^p(b\chi_{B_\gamma(x)})(y) = M^p_{\alpha,B_\gamma(x)}(b)(y) \quad \text{and} \quad M_\alpha^p(\chi_{B_\gamma(x)})(y) = M^p_{\alpha,B_\gamma(x)}(\chi_{B_\gamma(x)})(y) = |B_\gamma(x)|_h^{\alpha/n}.$$

2. For any $y \in B_\gamma(x)$,

$$|b_{B_\gamma(x)}| \leq |B_\gamma(x)|_h^{-\alpha/n} M^p_{\alpha,B_\gamma(x)}(b)(y).$$

3. Let $E = \{y \in B_\gamma(x) : b(y) \leq b_{B_\gamma(x)}\}$ and $F = B_\gamma(x) \setminus E = \{y \in B_\gamma(x) : b(y) > b_{B_\gamma(x)}\}$. Then the following equality holds trivially:

$$\int_E |b(y) - b_{B_\gamma(x)}|\,dy = \int_F |b(y) - b_{B_\gamma(x)}|\,dy.$$

Finally, the following results play a key role in proving our main theorems. For details, see Lemma 2.7 in [?].

Lemma 2.7. Assume $0 < \beta < 1$ and let b be a locally integrable function on $\mathbb{Q}_p^n$. Then the following assertions are equivalent:

1. $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$;
2. For all $1 \leq s < \infty$, there exists a positive constant C such that

$$\frac{1}{|B_\gamma(x)|_h^{1/s}} \left(\int_{B_\gamma(x)} |b(y) - M^p_{B_\gamma(x)}(b)(y)|^s\,dy \right)^{1/s} \leq C; \tag{7}$$

3. (2.1) holds for some $1 \le s < \infty$.

Lemma 2.8. Let $0 \le \alpha < n$, $0 < \beta < 1$, $0 < \alpha + \beta < n$, and let f be a locally integrable function on $\mathbb{Q}_p^n$. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \ge 0$, then for any $x \in \mathbb{Q}_p^n$ with $M_\alpha^p(f)(x) < \infty$, we have

$$|[b, M_\alpha^p](f)(x)| \le \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M_{\alpha+\beta}^p(f)(x).$$

To obtain our theorems, we need to prove the following result.

Lemma 2.9. Assume that $b \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$ and $b \ge 0$. Suppose that Φ and Ψ are Young functions satisfying $\Phi \in Y \cap \nabla_2$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n})$. Then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ is necessary for the boundedness of $[b, M_\alpha^p]$ from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.

Proof. For any fixed p-adic ball $B_\gamma(x)$ and any $y \in B_\gamma(x)$, by applying Lemma 2.6 we have

$$M_\alpha^p(b\chi_{B_\gamma(x)})(y) = M_{\alpha,B_\gamma(x)}^p(b)(y) \quad \text{and} \quad M_\alpha^p(\chi_{B_\gamma(x)})(y) = |B_\gamma(x)|_h^{\alpha/n}.$$

Since $[b, M_\alpha^p]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$, by Lemma 2.3 we obtain

$$\begin{aligned}
&|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - |B_\gamma(x)|_h^{-\alpha/n} M_{\alpha,B_\gamma(x)}^p(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \\
&= |B_\gamma(x)|_h^{1-\alpha/n-\beta} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|[b, M_\alpha^p](\chi_{B_\gamma(x)})\|_{L^\Psi(B_\gamma(x))} \\
&\le C|B_\gamma(x)|_h^{1-\alpha/n-\beta} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|\chi_{B_\gamma(x)}\|_{L^\Phi(B_\gamma(x))}.
\end{aligned}$$

In view of the proof of Lemma 3.1 in [**?**], we have

$$\int_{B_\gamma(x)} |b(y) - b_{B_\gamma(x)}|\, dy \le \int_{B_\gamma(x)} |b(y) - |B_\gamma(x)|_h^{-\alpha/n} M_{\alpha,B_\gamma(x)}^p(b)(y)|\, dy.$$

Thus, by applying Lemma 2.2 we get

$$\int_{B_\gamma(x)} |b(y) - b_{B_\gamma(x)}|\, dy \le C|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - |B_\gamma(x)|_h^{-\alpha/n} M_{\alpha,B_\gamma(x)}^p(b)(\cdot)\|_{L^\Psi(B_\gamma(x))}.$$

It follows from Lemma 2.1 that $b \in \Lambda_\beta(\mathbb{Q}_p^n)$.

Proof of the Main Results

Proof of Theorem 1.1. Since the equivalences (1) $\iff$ (4) follow directly from Lemma 2.7, we only need to prove (1) $\implies$ (2), (2) $\implies$ (3), and (3) $\implies$ (4).

(1) $\Longrightarrow$ (2): Since $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$, by Lemma 2.8 we have for almost every $x \in \mathbb{Q}_p^n$ and all $f \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$,

$$|[b, M_\alpha^p](f)(x)| \leq \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M_{\alpha+\beta}^p(f)(x).$$

It follows from Lemma 2.5 that $[b, M_\alpha^p]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.

(2) $\Longrightarrow$ (3): We divide the proof into two cases according to the range of α.

Case 1. If $\alpha = 0$, then for any fixed p-adic ball $B_\gamma(x)$ and any $y \in B_\gamma(x)$, we claim that (see also [**?**])

$$M^p(\chi_{B_\gamma(x)})(y) = \chi_{B_\gamma(x)}(y), \quad M^p(b\chi_{B_\gamma(x)})(y) = M^p_{B_\gamma(x)}(b)(y).$$

Consequently,

$$b(y) - M^p_{B_\gamma(x)}(b)(y) = b(y)M^p(\chi_{B_\gamma(x)})(y) - M^p(b\chi_{B_\gamma(x)})(y) = [b, M^p](\chi_{B_\gamma(x)})(y).$$

Using the condition $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n}) = p^{\beta\gamma}\Phi^{-1}(p^{-\gamma n})$ and Lemma 2.3, we obtain

$$\begin{aligned}
&|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1})\|b(\cdot) - M^p_{B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \\
&= |B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1})\|[b, M^p](\chi_{B_\gamma(x)})(\cdot)\|_{L^\Psi(B_\gamma(x))} \\
&\leq C|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1})\|\chi_{B_\gamma(x)}\|_{L^\Phi(B_\gamma(x))}.
\end{aligned}$$

Thus we obtain (1.1).

Case 2. If $0 < \alpha < n$, then for any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$,

$$\begin{aligned}
&|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1})\|b(\cdot) - M^p_{B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \\
&\leq |B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1})\|b(\cdot) - |B_\gamma(x)|_h^{-\alpha/n} M^p_{\alpha, B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \\
&\quad + |B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1})\|M^p_{B_\gamma(x)}(b)(\cdot) - |B_\gamma(x)|_h^{-\alpha/n} M^p_{\alpha, B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \\
&:= I_1 + I_2.
\end{aligned}$$

We first consider I_1. For any $y \in B_\gamma(x)$, Lemma 2.6 gives

$$M^p_\alpha(b\chi_{B_\gamma(x)})(y) = M^p_{\alpha, B_\gamma(x)}(b)(y) \quad \text{and} \quad M^p_\alpha(\chi_{B_\gamma(x)})(y) = M^p_{\alpha, B_\gamma(x)}(\chi_{B_\gamma(x)})(y) = |B_\gamma(x)|_h^{\alpha/n}.$$

Thus, for any $y \in B_\gamma(x)$,

$$\begin{aligned}
b(y) - |B_\gamma(x)|_h^{-\alpha/n} M^p_{\alpha, B_\gamma(x)}(b)(y) &= |B_\gamma(x)|_h^{-\alpha/n}\left(b(y)|B_\gamma(x)|_h^{\alpha/n} - M^p_{\alpha, B_\gamma(x)}(b)(y)\right) \\
&= |B_\gamma(x)|_h^{-\alpha/n}\left(b(y)M^p_\alpha(\chi_{B_\gamma(x)})(y) - M^p_\alpha(b\chi_{B_\gamma(x)})(y)\right) \\
&= |B_\gamma(x)|_h^{-\alpha/n}[b, M^p_\alpha](\chi_{B_\gamma(x)})(y).
\end{aligned}$$

Since $[b, M_\alpha^p]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$, by Lemma 2.3 and the hypothesis $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n})$, we have

$$\begin{aligned}
I_1 &= |B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - |B_\gamma(x)|_h^{-\alpha/n} M^p_{\alpha, B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \\
&= |B_\gamma(x)|_h^{1-\alpha/n-\beta} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|[b, M_\alpha^p](\chi_{B_\gamma(x)})\|_{L^\Psi(B_\gamma(x))} \\
&\le C|B_\gamma(x)|_h^{1-\alpha/n-\beta} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|\chi_{B_\gamma(x)}\|_{L^\Phi(B_\gamma(x))}.
\end{aligned}$$

Next, we estimate I_2. For any $y \in B_\gamma(x)$, using (3.1) we have

$$\begin{aligned}
&|M^p_{B_\gamma(x)}(b)(y) - |B_\gamma(x)|_h^{-\alpha/n} M^p_{\alpha, B_\gamma(x)}(b)(y)| \\
&= |M^p(\chi_{B_\gamma(x)})(y) M^p(b\chi_{B_\gamma(x)})(y) - |B_\gamma(x)|_h^{-\alpha/n} M^p_{\alpha, B_\gamma(x)}(b)(y)| \\
&\le |M^p(\chi_{B_\gamma(x)})(y) M^p(b\chi_{B_\gamma(x)})(y) - M_\alpha^p(b\chi_{B_\gamma(x)})(y)| \\
&\quad + |M_\alpha^p(b\chi_{B_\gamma(x)})(y) - |b(y)| M_\alpha^p(\chi_{B_\gamma(x)})(y)| \\
&\quad + ||b(y)| M_\alpha^p(\chi_{B_\gamma(x)})(y) - |b(y)| M^p(\chi_{B_\gamma(x)})(y)| \\
&\quad + ||b(y)| M^p(\chi_{B_\gamma(x)})(y) - M^p(b\chi_{B_\gamma(x)})(y)| \\
&= |B_\gamma(x)|_h^{-\alpha/n} \big|[|b|, M_\alpha^p]\chi_{B_\gamma(x)}(y)\big| + \big|[|b|, M^p]\chi_{B_\gamma(x)}(y)\big|.
\end{aligned}$$

Since $[b, M_\alpha^p]$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$, Lemma 2.9 implies that $|b| \in \Lambda_\beta(\mathbb{Q}_p^n)$. By Lemma 2.6 and Lemma 2.8, for any $y \in B_\gamma(x)$,

$$\begin{aligned}
&\big|[|b|, M_\alpha^p]\chi_{B_\gamma(x)}(y)\big| \le \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M^p_{\alpha+\beta}(\chi_{B_\gamma(x)})(y) \le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} |B_\gamma(x)|_h^{(\alpha+\beta)/n}, \\
&\big|[|b|, M^p]\chi_{B_\gamma(x)}(y)\big| \le \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M^p_{\beta}(\chi_{B_\gamma(x)})(y) \le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} |B_\gamma(x)|_h^{\beta/n}.
\end{aligned}$$

Using Lemma 2.3, we obtain

$$\begin{aligned}
I_2 &\le |B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|M^p_{B_\gamma(x)}(b)(\cdot) - |B_\gamma(x)|_h^{-\alpha/n} M^p_{\alpha, B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \\
&\le C\big(\Psi^{-1}(|B_\gamma(x)|_h^{-1}) |B_\gamma(x)|_h^{1-\beta/n} \|[|b|, M_\alpha^p]\chi_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))} \\
&\quad + |B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|[|b|, M^p]\chi_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))}\big) \\
&\le C\big(\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) |B_\gamma(x)|_h^{1-\beta/n} \|\chi_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))} \\
&\quad + \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \Psi^{-1}(|B_\gamma(x)|_h^{-1}) |B_\gamma(x)|_h \|\chi_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))}\big) \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}.
\end{aligned}$$

Combining the estimates for I_1 and I_2 yields (1.1) since $B_\gamma(x)$ is arbitrary.

(3) $\implies$ (4): To deduce (1.2) from (1.1), assume (1.1) holds. For any fixed p-adic ball $B_\gamma(x)$, by Lemma 2.2 and (1.1) we obtain

$$\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - M^p_{B_\gamma(x)}(b)(y)|\, dy \le C|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - M^p_{B_\gamma(x)}(b)(\cdot)\|_{L^\Psi(B_\gamma(x))} \le C.$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Since the implications (1) $\iff$ (4) follow readily from Corollary 1.3 in [**?**], and (3) $\iff$ (4) follows directly from Lemma 2.2, we only need to prove (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3).

(1) $\Longrightarrow$ (2): Since $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, for any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$,

$$\begin{aligned} M^p_{\alpha,b}(f)(x) &= \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(x)-b(y)||f(y)|\,dy \\ &\le \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |x-y|_p^\beta |f(y)|\,dy \\ &\le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h^{1-(\alpha+\beta)/n}} \int_{B_\gamma(x)} |f(y)|\,dy \\ &= C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M^p_{\alpha+\beta}(f)(x). \end{aligned}$$

Thus, by Lemma 2.5 we conclude that $M^p_{\alpha,b}$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$.

(2) $\Longrightarrow$ (3): For any fixed p-adic ball $B_\gamma(x)$ and every $y \in B_\gamma(x)$,

$$\begin{aligned} |b(y) - b_{B_\gamma(x)}| &\le \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y)-b(z)|\,dz \\ &= \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y)-b(z)|\chi_{B_\gamma(x)}(z)\,dz \\ &\le M^p_{\alpha,b}(\chi_{B_\gamma(x)})(y). \end{aligned}$$

Since $M^p_{\alpha,b}$ is bounded from $L^\Phi(\mathbb{Q}_p^n)$ to $L^\Psi(\mathbb{Q}_p^n)$ and $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma(\alpha+\beta)}\Phi^{-1}(p^{-\gamma n})$, applying Lemma 2.3 yields

$$\begin{aligned} |B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - b_{B_\gamma(x)}\|_{L^\Psi(B_\gamma(x))} &\le |B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|M^p_{\alpha,b}(\chi_{B_\gamma(x)})(\cdot)\|_{L^\Psi(B_\gamma(x))} \\ &\le C|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|\chi_{B_\gamma(x)}\|_{L^\Phi(B_\gamma(x))} \\ &\le C, \end{aligned}$$

which implies (1.3). This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Since the implications (1) $\iff$ (4) follow readily from Theorem 1.4 in [**?**], we only need to prove (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), and (3) $\Longrightarrow$ (4).

(1) $\Longrightarrow$ (2): Assume $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \ge 0$. For any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, the following estimate was obtained in [**?**]:

$$|[b, M^{\sharp,p}]f(x)| \le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M^p_\beta(f)(x).$$

Therefore, by Lemma 2.5 we conclude that $[b, M^{\sharp,p}]$ is bounded from $L^{\Phi}(\mathbb{Q}_p^n)$ to $L^{\Psi}(\mathbb{Q}_p^n)$.

(2) $\implies$ (3): Assume that $[b, M^{\sharp,p}]$ is bounded from $L^{\Phi}(\mathbb{Q}_p^n)$ to $L^{\Psi}(\mathbb{Q}_p^n)$. To prove (1.5), note that for any fixed p-adic ball $B_\gamma(x)$ (see also [?]),

$$M^{\sharp,p}(\chi_{B_\gamma(x)})(y) = 2(p-1) \quad \text{for } y \in B_\gamma(x).$$

Then for all $y \in B_\gamma(x)$,

$$\begin{aligned} b(y) - 2(p-1)M^{\sharp,p}(b\chi_{B_\gamma(x)})(y) &= 2(p-1)\left(\frac{b(y)}{2(p-1)} - M^{\sharp,p}(b\chi_{B_\gamma(x)})(y)\right) \\ &= 2(p-1)\left(b(y)M^{\sharp,p}(\chi_{B_\gamma(x)})(y) - M^{\sharp,p}(b\chi_{B_\gamma(x)})(y)\right) \\ &= 2(p-1)[b, M^{\sharp,p}](\chi_{B_\gamma(x)})(y). \end{aligned}$$

Since $[b, M^{\sharp,p}]$ is bounded from $L^{\Phi}(\mathbb{Q}_p^n)$ to $L^{\Psi}(\mathbb{Q}_p^n)$, applying Lemma 2.3 and noting that $\Psi^{-1}(p^{-\gamma n}) \approx p^{\gamma\beta}\Phi^{-1}(p^{-\gamma n})$, we have

$$\begin{aligned} &|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - 2(p-1)M^{\sharp,p}(b\chi_{B_\gamma(x)})(\cdot)\|_{L^{\Psi}(B_\gamma(x))} \\ &= 2(p-1)|B_\gamma(x)|_h \Psi^{-1}(|B_\gamma(x)|_h^{-1}) \|[b, M^{\sharp,p}](\chi_{B_\gamma(x)})\|_{L^{\Psi}(B_\gamma(x))} \\ &\leq C|B_\gamma(x)|_h \Phi^{-1}(|B_\gamma(x)|_h^{-1}) \|\chi_{B_\gamma(x)}\|_{L^{\Phi}(\mathbb{Q}_p^n)} \leq C. \end{aligned}$$

Thus we obtain (1.5).

(3) $\implies$ (4): Assume (1.5) holds. To prove (1.6), for any fixed p-adic ball $B_\gamma(x)$, by Lemma 2.2 and (1.5) we have

$$\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - 2(p-1)M^{\sharp,p}(b\chi_{B_\gamma(x)})(y)| \, dy \leq C|B_\gamma(x)|_h \Phi^{-1}(|B_\gamma(x)|_h^{-1}) \|b(\cdot) - 2(p-1)M^{\sharp,p}(b\chi_{B_\gamma(x)})(\cdot)\|_{L^{\Psi}$$

which implies (1.6) since the constant C is independent of $B_\gamma(x)$. This completes the proof of Theorem 1.3.

Funding Information

This work was partly supported by the Fundamental Research Funds for Education Department of Heilongjiang Province (No. 1453ZD031, 2019-KYYWF-0909, SJGY20220609), the Reform and Development Foundation for Local Colleges and Universities of the Central Government (No. 2020YQ07), and MNU (No. KCSZKC-2022026, KCSZAL-2022013).

Conflict of Interest

The authors state that there is no conflict of interest.

Data Availability Statement

All data generated or analyzed during this study are included in this published article.

Author Contributions

All authors contributed equally to the writing of this article. All authors read the final manuscript and approved its submission.

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Note: Figure translations are in progress. See original paper for figures.