Estimates for fractional integral operator and its commutators on grand p-adic Herz-Morrey spaces

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Abstract

In this article, the main aim is to introduce the grand variable Herz space over the p-adic fields and demonstrate the boundedness for fractional integral operator, fractional maximal operator in the context of the grand p-adic version of Herz-Morrey spaces with variable exponent, as well as the Lipschitz estimates for the commutators of fractional integral operator, fractional maximal operator, and sharp maximal function on the grand p-adic version of Herz-Morrey spaces with variable exponent.

Full Text

Estimates for Fractional Integral Operator and Its Commutators on Grand p-adic Herz-Morrey Spaces

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Abstract: This article introduces grand variable Herz spaces over p-adic fields and establishes the boundedness of fractional integral operators and fractional maximal operators on grand p-adic Herz-Morrey spaces with variable exponent. Additionally, we obtain Lipschitz estimates for commutators of fractional integral operators, fractional maximal operators, and sharp maximal functions on these spaces.

Keywords: p-adic field; fractional integral operator; grand space; variable exponent Herz-Morrey space; commutator

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Introduction and Main Results

Over the past decades, p-adic analysis has experienced rapid and substantial development, driven by its powerful applications across various fields. In physics, p-adic analysis finds application in the theory of p-adic strings, complex disordered systems such as spin glasses, and quantum mechanics [?, ?, ?]. Beyond physics, it holds significant importance in biology and geology; specifically, mathematical methods from p-adic analysis can reveal certain biological phenomena [?, ?], and the theory can address fractal problems in geology [?, ?].

We begin by introducing some fundamental concepts concerning p-adic fields. Let $\mathbb{Z}$ denote the ring of integers. For a fixed prime number p, the p-adic field $\mathbb{Q}_p$, originally introduced by K. Hensel in 1897, is the completion of the rational number field $\mathbb{Q}$ with respect to the non-Archimedean p-adic absolute value. For a rational number $x = p^\gamma \frac{a}{b}$, where $x \in \mathbb{Q}$, $\gamma \in \mathbb{Z}$, and a, b are non-zero integers not divisible by p, the p-adic absolute value is defined as $|x|_p = p^{-\gamma}$.

The non-Archimedean p-adic absolute value shares many properties with the Archimedean absolute value, including positive definiteness, multiplicativity, and the non-Archimedean inequality. Specifically, these properties are:

1. $|x|_p \geq 0$, and $|x|_p = 0$ if and only if $x = 0$;
2. $|xy|_p = |x|_p |y|_p$;
3. $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. If $|x|_p \neq |y|_p$, then equality holds, and the converse is also true.

Combining properties (1) and (3), we obtain the triangle inequality analogous to the Archimedean case: $|x + y|_p \leq |x|_p + |y|_p$.

From standard p-adic analysis, any non-zero p-adic number x can be expressed as:

$$x = p^\gamma (a_0 + a_{1p} + a_{2p}^2 + \cdots) = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad a_j = 0, \dots, p-1,$$

where $|x|_p = p^{-\gamma}$ when $a_\gamma \neq 0$. This representation converges naturally.

Next, we consider the n-dimensional p-adic linear space $\mathbb{Q}_p^n$. For any vector $x = (x_1, x_2, \dots, x_n)$ with $x_i \in \mathbb{Q}_p$ $(i = 1, \dots, n)$, the p-adic absolute value is defined as:

$$|x|_p = \max_{1 \leq j \leq n} |x_j|_p.$$

The p-adic ball is denoted by:

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\},$$

where $a \in \mathbb{Q}_p^n$ is the center and p^γ with $\gamma \in \mathbb{Z}$ is the radius. The corresponding p-adic sphere is:

$$S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\} = B_\gamma(a) \setminus B_{\gamma-1}(a).$$

In particular, when $a = 0$ and $\gamma = 0$, $B_0(0)$ and $S_0(0)$ are called the p-adic unit ball and p-adic unit sphere, respectively. Moreover, when $a = 0$, we typically omit the center in the notation for balls and spheres. From these definitions, we observe the following relationships: for any $a_0 \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$,

$$B_\gamma(a_0) = \bigcup_{k \leq \gamma} S_k(a_0), \quad \mathbb{Q}_p^n \setminus \{0\} = \bigcup_{\gamma \in \mathbb{Z}} S_\gamma,$$

and $a_0 + B_\gamma = B_\gamma(a_0)$, $a_0 + S_\gamma = S_\gamma(a_0) = B_\gamma(a_0) \setminus B_{\gamma-1}(a_0)$.

For simplicity, we define the characteristic function $\chi_k = \chi_{S_k} = \chi_{B_k \setminus B_{k-1}}$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists a unique Haar measure dx on $\mathbb{Q}_p^n$ (up to a positive constant multiple) that is translation invariant (i.e., $d(x+a) = dx$). Normalizing the measure so that:

$$\int_{B_0} dx = |B_0|_h = 1,$$

where $|B_0|_h$ denotes the Haar measure of the p-adic unit ball. More generally, for any $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$,

$$\int_{B_\gamma(a)} dx = |B_\gamma(a)|_h = p^{n\gamma}, \quad \int_{S_\gamma(a)} dx = |S_\gamma(a)|_h = p^{n\gamma}(1-p^{-n}) = |B_\gamma(a)|_h - |B_{\gamma-1}(a)|_h.$$

For further details on p-adic analysis, we refer readers to [?, ?] and the references therein.

Operator theory has attracted considerable attention due to its numerous applications in partial differential equations and harmonic analysis, where the primary concern is boundedness across different function spaces. In this article, we focus on the following p-adic fractional integral operator.

For $0 < \alpha < n$, we define the p-adic fractional integral operator as:

$$I_\alpha^p(f)(x) = \int_{\mathbb{Q}p^n} \frac{f(y)}{|x-y|_p^{n-\alpha}} \, dy.$$

As a major branch of harmonic analysis, function spaces with variable exponents generalize classical function spaces. For instance, variable exponent Lebesgue spaces extend classical Lebesgue spaces, and Herz-Morrey spaces generalize Herz spaces. On one hand, Cortés and Rafeiro [?] introduced p-adic variable exponent Lebesgue spaces and established many properties and applications. The boundedness of fractional integral and maximal operators was obtained in [?]. On the other hand, Sarfraz, Aslam, Zaman, and Jarad [?] derived estimates for fractional integral operators on p-adic Herz-Morrey spaces. Recently, grand function spaces with variable exponents have shown positive development trends. In Euclidean spaces, the boundedness of fractional integral operators on grand

Herz-Morrey spaces was given in [**?**]. Sultan et al. [**?**] defined grand p-adic Herz-Morrey spaces with variable exponent and obtained boundedness for an intrinsic square function. Consequently, studies of p-adic grand Herz-Morrey spaces with variable exponents remain relatively scarce and warrant further investigation.

Inspired by the aforementioned literature and, in some cases, by standard harmonic analysis on Euclidean space, our primary focus is the p-adic field $\mathbb{Q}_p$. The purpose of this paper is to investigate the boundedness of fractional integral operators and fractional maximal operators on p-adic Herz-Morrey spaces with variable exponent, as well as Lipschitz estimates for commutators of fractional integral operators, fractional maximal operators, and sharp maximal functions on grand p-adic Herz-Morrey spaces with variable exponent.

We first present the following boundedness result for the fractional integral operator I_α^p on p-adic vector spaces.

Theorem 1.1. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $0 < \alpha < n$, and $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\alpha$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If $\eta(\cdot)$ satisfies the following conditions: 1. $\lambda + \alpha - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\lambda + \alpha - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then I_α^p is bounded from $M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$.

Remark 1. For corresponding results in Euclidean space, see [**?**].

When $\eta(\cdot), q_1(\cdot)$, and $q_2(\cdot)$ in Theorem 1.1 are constant exponents, these results remain novel.

Corollary 1.1. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $0 < \alpha < n$, and q_1, q_2 satisfy $1 < q_1 < n/\alpha$ and $1/q_2 = 1/q_1 - \alpha/n$. If $\alpha + \lambda - n/q_1 < \eta < n/q_1'$, then I_α^p is bounded from $M\dot{K}_{\lambda,q_1}^{\eta,u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2}^{\eta,u}(\mathbb{Q}_p^n)$.

For the case $\lambda = 0$ in Corollary 1.1, we obtain boundedness of the p-adic fractional integral operator on grand Herz spaces.

Corollary 1.2. Let $1 \leq u < \infty$, $0 < \alpha < n$, and q_1, q_2 satisfy $1 < q_1 < n/\alpha$ and $1/q_2 = 1/q_1 - \alpha/n$. If $\alpha - n/q_1 < \eta < n/q_1'$, then I_α^p is bounded from $\dot{K}_{q_1}^{\eta,u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2}^{\eta,u}(\mathbb{Q}_p^n)$.

For the case $\lambda = 0$ in Theorem 1.1, we obtain boundedness of the fractional integral operator on grand p-adic variable Herz spaces, which is also new.

Corollary 1.3. Let $1 \leq u < \infty$, $0 < \alpha < n$, and $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\alpha$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If $\eta(\cdot)$ satisfies: 1. $\alpha - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\alpha - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then I_α^p is bounded from $\dot{K}_{q_1(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$.

When $\eta(\cdot)$ is a constant exponent, this result is also new in p-adic vector spaces.

Corollary 1.4. Let $1 \leq u < \infty$, $0 < \alpha < n$, and $q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\alpha$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If η satisfies: 1.

$\alpha - n/q_1(\cdot) < \eta < n/q_1'(\cdot)$, 2. $\alpha - n/q_1(\infty) < \eta < n/q_1'(\infty)$,

then I_α^p is bounded from $\dot{K}_{q_1(\cdot)}^{\eta,u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2(\cdot)}^{\eta,u}(\mathbb{Q}_p^n)$.

For $0 < \alpha < n$, we define the p-adic fractional maximal operator as:

$$M_\alpha^p(f)(x) = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y)|\, dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$.

The maximal commutator of M_α^p with symbol b is given by:

$$M_{\alpha,b}^p(f)(x) = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(x) - b(y)||f(y)|\, dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$.

Furthermore, for measurable functions $b : \mathbb{Q}_p^n \to \mathbb{R}$ and $f : \mathbb{Q}_p^n \to \mathbb{R}$, the nonlinear commutator of the fractional maximal operator is defined as:

$$[b, M_\alpha^p]f(x) = b(x)M_\alpha^p(f)(x) - M_\alpha^p(bf)(x).$$

When $\alpha = 0$, we have $[b, M^p] = [b, M_0^p]$ and $M^p = M_0^p$.

Motivated by Sobolev inequalities, we establish the boundedness of the fractional maximal operator on grand p-adic Herz-Morrey spaces with variable exponent.

Theorem 1.2. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $0 < \alpha < n$, and $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\alpha$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If $\eta(\cdot)$ satisfies: 1. $\alpha + \lambda - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\alpha + \lambda - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then M_α^p is bounded from $M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$.

When $\eta(\cdot), q_1(\cdot)$, and $q_2(\cdot)$ in Theorem 1.2 are constant exponents, these results remain new.

Corollary 1.5. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $0 < \alpha < n$, and q_1, q_2 satisfy $1 < q_1 < n/\alpha$ and $1/q_2 = 1/q_1 - \alpha/n$. If $\alpha + \lambda - n/q_1 < \eta < n/q_1'$, then M_α^p is bounded from $M\dot{K}_{\lambda,q_1}^{\eta,u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2}^{\eta,u}(\mathbb{Q}_p^n)$.

For $\lambda = 0$ in Corollary 1.5, we obtain boundedness of the p-adic fractional maximal operator on grand Herz spaces.

Corollary 1.6. Let $1 \leq u < \infty$, $0 < \alpha < n$, and q_1, q_2 satisfy $1 < q_1 < n/\alpha$ and $1/q_2 = 1/q_1 - \alpha/n$. If $\alpha - n/q_1 < \eta < n/q_1'$, then M_α^p is bounded from $\dot{K}_{q_1}^{\eta,u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2}^{\eta,u}(\mathbb{Q}_p^n)$.

For $\lambda = 0$ in Theorem 1.2, we obtain boundedness of the fractional maximal operator on grand p-adic variable Herz spaces, which is also new.

Corollary 1.7. Let $1 \leq u < \infty$, $0 < \alpha < n$, and $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\alpha$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If $\eta(\cdot)$ satisfies: 1. $\alpha - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\alpha - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then M_α^p is bounded from $\dot{K}_{q_1(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$.

When $\eta(\cdot)$ is a constant exponent, this result is also new in p-adic vector spaces.

Corollary 1.8. Let $1 \leq u < \infty$, $0 < \alpha < n$, and $q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\alpha$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If η satisfies: 1. $\alpha - n/q_1(\cdot) < \eta < n/q_1'(\cdot)$, 2. $\alpha - n/q_1(\infty) < \eta < n/q_1'(\infty)$,

then M_α^p is bounded from $\dot{K}_{q_1(\cdot)}^{\eta,u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2(\cdot)}^{\eta,u}(\mathbb{Q}_p^n)$.

For measurable functions $b : \mathbb{Q}_p^n \to \mathbb{R}$ and $f : \mathbb{Q}_p^n \to \mathbb{R}$, the commutator of the fractional integral operator is defined as:

$$[b, I_\alpha^p](f)(x) = b(x)I_\alpha^p f(x) - I_\alpha^p(bf)(x) = \int_{\mathbb{Q}_{p^n}} \frac{(b(x) - b(y))f(y)}{|x-y|_p^{n-\alpha}}\, dy. \tag{1.1}$$

We now present Lipschitz estimates for the (nonlinear) commutators of fractional integral and maximal operators on grand p-adic Herz-Morrey spaces with variable exponent.

Theorem 1.3. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $0 < \alpha < \alpha+\beta < n$, and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$. For $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/(\alpha+\beta)$, and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, if $\eta(\cdot)$ satisfies: 1. $\lambda + \alpha + \beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\lambda + \alpha + \beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then $[b, I_\alpha^p]$ is bounded from $M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$.

When $\eta(\cdot), q_1(\cdot)$, and $q_2(\cdot)$ in Theorem 1.3 are constant exponents, these results remain new.

Corollary 1.9. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $0 < \alpha < \alpha + \beta < n$, and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$. For q_1, q_2 with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n$, if $\lambda + \alpha + \beta - n/q_1 < \eta < n/q_1'$, then $[b, I_\alpha^p]$ is bounded from $M\dot{K}_{\lambda,q_1}^{\eta,u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2}^{\eta,u}(\mathbb{Q}_p^n)$.

For $\lambda = 0$ in Corollary 1.9, we obtain boundedness of the commutator of the p-adic fractional integral operator on grand Herz spaces.

Corollary 1.10. Let $1 \leq u < \infty$, $0 < \alpha < \alpha + \beta < n$, and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$. For q_1, q_2 with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n$, if $\alpha + \beta - n/q_1 < \eta < n/q_1'$, then $[b, I_\alpha^p]$ is bounded from $\dot{K}_{q_1}^{\eta,u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2}^{\eta,u}(\mathbb{Q}_p^n)$.

For $\lambda = 0$ in Theorem 1.3, we obtain boundedness of the commutator of the fractional integral operator on grand p-adic variable Herz spaces, which is also new.

Corollary 1.11. Let $1 \leq u < \infty$, $0 < \alpha < \alpha+\beta < n$, and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$. For $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/(\alpha+\beta)$, and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, if $\eta(\cdot)$ satisfies: 1. $\alpha+\beta-n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\alpha+\beta-n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then $[b, I_\alpha^p]$ is bounded from $\dot{K}_{q_1(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$.

When $\eta(\cdot)$ is a constant exponent, this result is also new in p-adic vector spaces.

Corollary 1.12. Let $1 \leq u < \infty$, $0 < \alpha < \alpha+\beta < n$, and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$. For $q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/(\alpha+\beta)$, and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, if η satisfies: 1. $\alpha+\beta-n/q_1(\cdot) < \eta < n/q_1'(\cdot)$, 2. $\alpha+\beta-n/q_1(\infty) < \eta < n/q_1'(\infty)$,

then $[b, I_\alpha^p]$ is bounded from $\dot{K}_{q_1(\cdot)}^{\eta,u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2(\cdot)}^{\eta,u}(\mathbb{Q}_p^n)$.

Naturally, we present the following result, which is also new.

Theorem 1.4. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $0 < \alpha < \alpha+\beta < n$, and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$. For $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/(\alpha+\beta)$, and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, if $\eta(\cdot)$ satisfies: 1. $\lambda+\alpha+\beta-n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\lambda+\alpha+\beta-n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then $M_{\alpha,b}^p$ is bounded from $M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$.

When $\eta(\cdot), q_1(\cdot)$, and $q_2(\cdot)$ in Theorem 1.4 are constant exponents, these results remain new.

Corollary 1.13. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $0 < \alpha < \alpha+\beta < n$, and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$. For q_1, q_2 with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n$, if $\lambda+\alpha+\beta-n/q_1 < \eta < n/q_1'$, then $M_{\alpha,b}^p$ is bounded from $M\dot{K}_{\lambda,q_1}^{\eta,u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2}^{\eta,u}(\mathbb{Q}_p^n)$.

For $\lambda = 0$ in Corollary 1.13, we obtain boundedness of the commutator of the p-adic fractional maximal operator on grand Herz spaces.

Corollary 1.14. Let $1 \leq u < \infty$, $0 < \alpha < \alpha+\beta < n$, and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$. For q_1, q_2 with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n$, if $\alpha+\beta-n/q_1 < \eta < n/q_1'$, then $M_{\alpha,b}^p$ is bounded from $\dot{K}_{q_1}^{\eta,u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2}^{\eta,u}(\mathbb{Q}_p^n)$.

For $\lambda = 0$ in Theorem 1.4, we obtain boundedness of the commutator of the fractional maximal operator on grand p-adic variable Herz spaces, which is also new.

Corollary 1.15. Let $1 \leq u < \infty$, $0 < \alpha < \alpha+\beta < n$, and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$. For $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/(\alpha+\beta)$, and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, if $\eta(\cdot)$ satisfies: 1. $\alpha+\beta-n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\alpha+\beta-n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then $M_{\alpha,b}^p$ is bounded from $\dot{K}_{q_1(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$.

When $\eta(\cdot)$ is a constant exponent, this result is also new in p-adic vector spaces.

Corollary 1.16. Let $1 \le u < \infty$, $0 < \alpha < \alpha + \beta < n$, and $b \in \Lambda_{\beta}(\mathbb{Q}_p^n)$ with $0 < \beta < 1$. For $q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/(\alpha+\beta)$, and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, if η satisfies: 1. $\alpha+\beta-n/q_1(\cdot) < \eta < n/q_1'(\cdot)$, 2. $\alpha+\beta-n/q_1(\infty) < \eta < n/q_1'(\infty)$,

then $M^p_{\alpha,b}$ is bounded from $\dot{K}^{\eta,u}_{q_1(\cdot)}(\mathbb{Q}_p^n)$ to $\dot{K}^{\eta,u}_{q_2(\cdot)}(\mathbb{Q}_p^n)$.

For $\alpha = 0$, the following results follow from Theorem 1.4 and are also new.

Corollary 1.17. Let $1 \le u < \infty$, $0 < \lambda < \infty$, and $b \in \Lambda_{\beta}(\mathbb{Q}_p^n)$ with $0 < \beta < 1$. For $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\beta$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$, if $\eta(\cdot)$ satisfies: 1. $\lambda+\beta-n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\lambda+\beta-n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then $[b, M^p]$ is bounded from $M\dot{K}^{\eta(\cdot),u}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)$ to $M\dot{K}^{\eta(\cdot),u}_{\lambda,q_2(\cdot)}(\mathbb{Q}_p^n)$.

Theorem 1.5. Let $1 \le u < \infty$, $0 < \lambda < \infty$, $0 < \alpha < \alpha+\beta < n$, $b \in \Lambda_{\beta}(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \ge 0$. For $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/(\alpha+\beta)$, and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, if $\eta(\cdot)$ satisfies: 1. $\lambda+\alpha+\beta-n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\lambda+\alpha+\beta-n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then $[b, M^p_{\alpha}]$ is bounded from $M\dot{K}^{\eta(\cdot),u}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)$ to $M\dot{K}^{\eta(\cdot),u}_{\lambda,q_2(\cdot)}(\mathbb{Q}_p^n)$.

When $\eta(\cdot), q_1(\cdot)$, and $q_2(\cdot)$ in Theorem 1.5 are constant exponents, these results remain new.

Corollary 1.18. Let $1 \le u < \infty$, $0 < \lambda < \infty$, $0 < \alpha < \alpha+\beta < n$, $b \in \Lambda_{\beta}(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \ge 0$. For q_1, q_2 with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n$, if $\lambda+\alpha+\beta-n/q_1 < \eta < n/q_1'$, then $[b, M^p_{\alpha}]$ is bounded from $M\dot{K}^{\eta,u}_{\lambda,q_1}(\mathbb{Q}_p^n)$ to $M\dot{K}^{\eta,u}_{\lambda,q_2}(\mathbb{Q}_p^n)$.

For $\lambda = 0$ in Corollary 1.18, we obtain boundedness of the nonlinear commutator of the p-adic fractional maximal operator on grand Herz spaces.

Corollary 1.19. Let $1 \le u < \infty$, $0 < \alpha < \alpha+\beta < n$, $b \in \Lambda_{\beta}(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \ge 0$. For q_1, q_2 with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n$, if $\alpha+\beta-n/q_1 < \eta < n/q_1'$, then $[b, M^p_{\alpha}]$ is bounded from $\dot{K}^{\eta,u}_{q_1}(\mathbb{Q}_p^n)$ to $\dot{K}^{\eta,u}_{q_2}(\mathbb{Q}_p^n)$.

For $\lambda = 0$ in Theorem 1.5, we obtain boundedness of the nonlinear commutator of the fractional maximal operator on grand p-adic variable Herz spaces, which is also new.

Corollary 1.20. Let $1 \le u < \infty$, $0 < \alpha < \alpha+\beta < n$, $b \in \Lambda_{\beta}(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \ge 0$. For $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/(\alpha+\beta)$, and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, if $\eta(\cdot)$ satisfies: 1. $\alpha+\beta-n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\alpha+\beta-n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then $[b, M^p_{\alpha}]$ is bounded from $\dot{K}^{\eta(\cdot),u}_{q_1(\cdot)}(\mathbb{Q}_p^n)$ to $\dot{K}^{\eta(\cdot),u}_{q_2(\cdot)}(\mathbb{Q}_p^n)$.

When $\eta(\cdot)$ is a constant exponent, this result is also new in p-adic vector spaces.

Corollary 1.21. Let $1 \leq u < \infty$, $0 < \alpha < \alpha + \beta < n$, $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \geq 0$. For $q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/(\alpha+\beta)$, and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, if η satisfies: 1. $\alpha + \beta - n/q_1(\cdot) < \eta < n/q_1'(\cdot)$, 2. $\alpha + \beta - n/q_1(\infty) < \eta < n/q_1'(\infty)$,

then $[b, M_\alpha^p]$ is bounded from $\dot{K}_{q_1(\cdot)}^{\eta,u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2(\cdot)}^{\eta,u}(\mathbb{Q}_p^n)$.

For $\alpha = 0$, the following results follow from Theorem 1.5 and are also new.

Corollary 1.22. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \geq 0$. For $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\beta$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$, if $\eta(\cdot)$ satisfies: 1. $\lambda + \beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\lambda + \beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then $[b, M^p]$ is bounded from $M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$.

To introduce the next theorem, we first define the p-adic version of the sharp maximal function. For a locally integrable function f on $\mathbb{Q}_p^n$, following [**?**], we define:

$$M_p^\sharp(f)(x) = \sup_{B_\gamma(x)} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|\, dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$ and $f_{B_\gamma(x)} = \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} f(y)\, dy$.

The commutator generated by $M_p^\sharp$ and $b \in L_{\text{loc}}(\mathbb{Q}_p^n)$ is:

$$[b, M_p^\sharp](f)(x) = b(x) M_p^\sharp(f)(x) - M_p^\sharp(bf)(x).$$

Theorem 1.6. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \geq 0$. For $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\beta$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$, if $\eta(\cdot)$ satisfies: 1. $\lambda + \beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\lambda + \beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then $[b, M_p^\sharp]$ is bounded from $M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$.

When $\eta(\cdot), q_1(\cdot)$, and $q_2(\cdot)$ in Theorem 1.6 are constant exponents, these results remain new.

Corollary 1.23. Let $1 \leq u < \infty$, $0 < \lambda < \infty$, $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \geq 0$. For q_1, q_2 with $1 < q_1 < n/\beta$ and $1/q_2 = 1/q_1 - \beta/n$, if $\lambda + \beta - n/q_1 < \eta < n/q_1'$, then $[b, M_p^\sharp]$ is bounded from $M\dot{K}_{\lambda,q_1}^{\eta,u}(\mathbb{Q}_p^n)$ to $M\dot{K}_{\lambda,q_2}^{\eta,u}(\mathbb{Q}_p^n)$.

For $\lambda = 0$ in Corollary 1.23, we obtain boundedness of the nonlinear commutator of the p-adic sharp maximal function on grand Herz spaces.

Corollary 1.24. Let $1 \leq u < \infty$, $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \geq 0$. For q_1, q_2 with $1 < q_1 < n/\beta$ and $1/q_2 = 1/q_1 - \beta/n$, if $\beta - n/q_1 < \eta < n/q_1'$, then $[b, M_p^\sharp]$ is bounded from $\dot{K}_{q_1}^{\eta,u}(\mathbb{Q}_p^n)$ to $\dot{K}_{q_2}^{\eta,u}(\mathbb{Q}_p^n)$.

For $\lambda = 0$ in Theorem 1.5, we obtain boundedness of the commutator of the sharp maximal function on grand p-adic variable Herz spaces, which is also new.

Corollary 1.25. Let $1 \le u < \infty$, $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \ge 0$. For $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\beta$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$, if $\eta(\cdot)$ satisfies: 1. $\beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, 2. $\beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

then $[b, M_p^\sharp]$ is bounded from $\dot{K}^{\eta(\cdot),u}_{q_1(\cdot)}(\mathbb{Q}_p^n)$ to $\dot{K}^{\eta(\cdot),u}_{q_2(\cdot)}(\mathbb{Q}_p^n)$.

When $\eta(\cdot)$ is a constant exponent, this result is also new in p-adic vector spaces.

Corollary 1.26. Let $1 \le u < \infty$, $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ with $0 < \beta < 1$, and $b \ge 0$. For $q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\beta$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$, if η satisfies: 1. $\beta - n/q_1(\cdot) < \eta < n/q_1'(\cdot)$, 2. $\beta - n/q_1(\infty) < \eta < n/q_1'(\infty)$,

then $[b, M_p^\sharp]$ is bounded from $\dot{K}^{\eta,u}_{q_1(\cdot)}(\mathbb{Q}_p^n)$ to $\dot{K}^{\eta,u}_{q_2(\cdot)}(\mathbb{Q}_p^n)$.

Throughout this paper, the letter C denotes a constant independent of the main parameters, whose value may vary from line to line. Additionally, we establish some notation: $|E|_h$ always denotes the Haar measure of a measurable set $E \subset \mathbb{Q}_p^n$, and χ_E denotes its characteristic function.

Preliminaries

2.1 p-adic Function Spaces

For $1 \le q < \infty$, we denote by $L^q(\mathbb{Q}_p^n)$ the p-adic Lebesgue space of all locally q-integrable functions with finite norm:

$$\|f\|_{L^q(\mathbb{Q}{p^n})} = \left(\int_{\mathbb{Q}{p^n}} |f(x)|^q \, dx \right)^{1/q}.$$

For $q = \infty$, $L^\infty(\mathbb{Q}_p^n)$ denotes the space of all measurable real-valued functions f on $\mathbb{Q}_p^n$ satisfying:

$$\|f\|_{L^\infty(\mathbb{Q}_p^n)} = \operatorname{ess\,sup} |f(x)| = \inf\{\lambda > 0 : |\{x \in \mathbb{Q}_p^n : |f(x)| > \lambda\}|_h = 0\}.$$

When the limit exists, the integral above is defined as:

$$\int_{\mathbb{Q}{p^n}} |f(x)|^q \, dx = \lim_{\gamma \to \infty} \int_{B_\gamma(0)} |f(x)|^q \, dx = \lim_{\gamma \to \infty} \sum_{-\infty < k \le \gamma} \int_{S_k(0)} |f(x)|^q \, dx.$$

In particular, since $\mathbb{Q}_p^n = \bigcup_{\gamma=-\infty}^{\infty} S_\gamma$ and $d(tx) = |t|_p^n dx$ for $t \in \mathbb{Q}_p \setminus \{0\}$, if $f \in L^1(\mathbb{Q}_p^n)$, then:

$$\int_{\mathbb{Q}{p^n}} f(x) \, dx = \sum_{k \in \mathbb{Z}} \int_{S_k} f(x) \, dx, \quad \int_{\mathbb{Q}{p^n}} f(tx) \, dx = \int_{\mathbb{Q}{p^n}} f(x) \, dx.$$

A measurable function $q(\cdot)$ is called a variable exponent if $q(\cdot) : \mathbb{Q}_p^n \to (0, \infty)$. The following definitions introduce notation for p-adic variable exponent Lebesgue spaces, following [**?**].

Definition 2.1. Given a measurable function $q(\cdot)$ on $\mathbb{Q}_p^n$, we denote:

$$q_- := \operatorname{ess\,inf} q(x), \quad q_+ := \operatorname{ess\,sup} q(x).$$

1. $q'(\cdot)$ denotes the conjugate exponent satisfying $1/q(x) + 1/q'(x) = 1$ almost everywhere. 2. Let $\mathcal{P}(\mathbb{Q}_p^n)$ denote the set of all measurable functions $q(\cdot) : \mathbb{Q}_p^n \to (1, \infty)$ such that $1 \le q_- \le q(x) \le q_+ < \infty$ for all $x \in \mathbb{Q}_p^n$.

Definition 2.2 (p-adic variable exponent Lebesgue spaces). Let $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$. The p-adic variable exponent Lebesgue space $L^{q(\cdot)}(\mathbb{Q}_p^n)$ is defined as:

$$L^{q(\cdot)}(\mathbb{Q}_p^n) = \{f \text{ measurable} : \rho_q(f/\eta) < \infty \text{ for some } \eta > 0\},$$

where $\rho_q(f) := \int_{\mathbb{Q}p^n} |f(x)|^{q(x)} \, dx$.

The space $L^{q(\cdot)}(\mathbb{Q}_p^n)$ is a Banach function space equipped with the Luxemburg norm:

$$\|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} = \inf \left\{ \eta > 0 : \rho_q(f/\eta) = \int_{\mathbb{Q}p^n} \left(\frac{|f(x)|}{\eta} \right)^{q(x)} dx \le 1 \right\}.$$

Cortés and Rafeiro [**?**] introduced the following class of exponents.

Definition 2.3 (log-Hölder continuity). Let $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$ be a measurable function. 1. Denote by $C_0^{\log}(\mathbb{Q}_p^n)$ the set of all $q(\cdot)$ satisfying:

$$\gamma(q_-(B_\gamma(x)) - q_+(B_\gamma(x))) \le C, \quad \forall \gamma \in \mathbb{Z}, x \in \mathbb{Q}_p^n,$$

where C is a universal constant. 2. The set $C_\infty^{\log}(\mathbb{Q}_p^n)$ consists of all $q(\cdot)$ satisfying:

$$|q(x) - q(y)| \le \frac{C}{\log_p(p + \min\{|x|_p, |y|_p\})}, \quad \forall x, y \in \mathbb{Q}_p^n,$$

where C is a universal constant. 3. Let $C^{\log}(\mathbb{Q}_p^n) = C_0^{\log}(\mathbb{Q}_p^n) \cap C_\infty^{\log}(\mathbb{Q}_p^n)$ denote the set of all globally log-Hölder continuous functions $q(\cdot)$.

Finally, we introduce p-adic Herz-Morrey spaces with variable exponent and grand p-adic Herz-Morrey spaces with variable exponent [**?**, **?**].

Definition 2.4. Let $\eta(\cdot) \in L^\infty(\mathbb{Q}_p^n)$, $1 \le u < \infty$, $s(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, and $0 \le \lambda < \infty$. The homogeneous p-adic Herz-Morrey space with variable exponent $M\dot{K}_{\lambda,s(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)$ is defined as:

$$M\dot{K}_{\lambda,s(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n) = \{g \in L_{\text{loc}}^{s(\cdot)}(\mathbb{Q}_p^n \setminus \{0\}) : \|g\|_{M\dot{K}_{\lambda,s(\cdot)}^{\eta(\cdot),u}(\mathbb{Q}_p^n)} < \infty\},$$

where the norm is given by:

$$\|g\|_{M\dot{K}^{\eta(\cdot),u}_{\lambda,s(\cdot)}(\mathbb{Q}_p^n)} = \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{\eta(0)ku}\|g\chi_k\|^u_{L^{s(\cdot)}(\mathbb{Q}_p^n)}\right\}^{1/u}.$$

Definition 2.5. Let $\eta(\cdot) \in L^\infty(\mathbb{Q}_p^n)$, $1 \le u < \infty$, $s(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, and $\theta > 0$. The homogeneous grand p-adic Herz-Morrey space with variable exponent $M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,s(\cdot)}(\mathbb{Q}_p^n)$ is defined as:

$$M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,s(\cdot)}(\mathbb{Q}_p^n) = \{g \in L^{s(\cdot)}_{\mathrm{loc}}(\mathbb{Q}_p^n \setminus \{0\}) : \|g\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,s(\cdot)}} < \infty\},$$

where the norm is:

$$\|g\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,s(\cdot)}} = \sup_{\epsilon>0} \epsilon^{\theta/u(1+\epsilon)}\left\{\sum_{k=-\infty}^{k_0} p^{\eta(0)ku(1+\epsilon)}\|g\chi_k\|^{u(1+\epsilon)}_{L^{s(\cdot)}(\mathbb{Q}_p^n)}\right\}^{1/u(1+\epsilon)}.$$

When $\lambda = 0$, the grand Herz-Morrey space $M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,s(\cdot)}(\mathbb{Q}_p^n)$ reduces to the grand p-adic variable Herz space $\dot{K}^{\eta(\cdot),u),\theta}_{s(\cdot)}(\mathbb{Q}_p^n)$, whose definition in the Euclidean context can be found in [**?**].

Definition 2.6. Let $\eta(\cdot) \in L^\infty(\mathbb{Q}_p^n)$, $1 \le u < \infty$, $s(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, and $\theta > 0$. The homogeneous grand p-adic Herz space with variable exponent $\dot{K}^{\eta(\cdot),u),\theta}_{s(\cdot)}(\mathbb{Q}_p^n)$ is defined as:

$$\dot{K}^{\eta(\cdot),u),\theta}_{s(\cdot)}(\mathbb{Q}_p^n) = \{g \in L^{s(\cdot)}(\mathbb{Q}_p^n) : \|g\|_{\dot{K}^{\eta(\cdot),u),\theta}_{s(\cdot)}} < \infty\},$$

where the norm is:

$$\|g\|_{\dot{K}^{\eta(\cdot),u),\theta}_{s(\cdot)}} = \sup_{\epsilon>0} \epsilon^{\theta/u(1+\epsilon)}\left\{\sum_{k=-\infty}^{\infty} p^{\eta(0)ku(1+\epsilon)}\|g\chi_k\|^{u(1+\epsilon)}_{L^{s(\cdot)}(\mathbb{Q}_p^n)}\right\}^{1/u(1+\epsilon)}.$$

Definition 2.7. Let $0 < \beta < 1$. The p-adic version of the homogeneous Lipschitz space $\Lambda_\beta(\mathbb{Q}_p^n)$ is defined by:

$$\Lambda_\beta(\mathbb{Q}_p^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n) : \|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} < \infty\},$$

where:

$$\|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} = \sup_{x,y\in\mathbb{Q}{p}^n, x\neq y} \frac{|f(x) - f(y)|}{|x-y|_p^\beta}.$$

2.2 Auxiliary Propositions and Lemmas

This section presents auxiliary results needed for proving our main theorems, describing only the partial results we require.

First, the p-adic version of Hölder' s inequality can be found in [**?**].

Lemma 2.1 (Generalized Hölder' s inequality on $\mathbb{Q}_p^n$). Let $\mathbb{Q}_p^n$ be an n-dimensional p-adic vector space. Suppose $q_1(\cdot), q_2(\cdot), r(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$ satisfy $1/r(\cdot) = 1/q_1(\cdot) + 1/q_2(\cdot)$ almost everywhere. Then there exists a positive constant C such that for all $f \in L^{q_1(\cdot)}(\mathbb{Q}_p^n)$ and $g \in L^{q_2(\cdot)}(\mathbb{Q}_p^n)$:

$$\|fg\|_{L^{r(\cdot)}(\mathbb{Q}_p^n)} \leq C\|f\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\|g\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}.$$

For proving the main theorems, the following estimates for norms of characteristic functions are derived from [**?**], with the second part being crucial.

Lemma 2.2 (Norms of characteristic functions). If $q(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$, then:

$$\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{q(x,\gamma)},$$

where:

$$q(x,\gamma) = \begin{cases} q(x), & \text{if } \gamma < 0, \\ q(\infty), & \text{if } \gamma \geq 0. \end{cases}$$

The following lemma provides the boundedness of fractional integral operators on p-adic variable exponent Lebesgue spaces; see [**?**] for details.

Lemma 2.3. Assume $0 < \alpha < n$. For all $r(\cdot), q(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $r_+ < n/\alpha$, and $1/q(\cdot) = 1/r(\cdot) - \alpha/n$, we have:

$$I_\alpha^p : L^{r(\cdot)}(\mathbb{Q}_p^n) \to L^{q(\cdot)}(\mathbb{Q}_p^n).$$

The following result from [**?**] shows that the fractional maximal operator can be controlled by the fractional integral operator.

Lemma 2.4. Let $0 < \alpha < n$. For all $x \in \mathbb{Q}_p^n$, there exists a positive constant C such that:

$$|M_\alpha^p(f)(x)| \leq C|I_\alpha^p(|f|)(x)|.$$

Lemmas 2.5 and 2.6 were obtained in [**?**].

Lemma 2.5. Let $0 < \beta < 1$ and $0 < \alpha < \alpha + \beta < n$. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, then for any $x \in \mathbb{Q}_p^n$:

$$M_{\alpha,b}^p \leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M_{\alpha+\beta}^p(f)(x).$$

Lemma 2.6. Let $0 < \alpha < n$. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$, then for any $x \in \mathbb{Q}_p^n$ such that $|[b, M_\alpha^p](f)(x)| < \infty$, we have:

$$|[b, M_\alpha^p](f)(x)| \leq M_{\alpha,b}^p(f)(x).$$

Finally, the following result follows from Definition 2.5 and the proof of Theorem 3.4 in [?].

Lemma 2.7. Let $\eta(\cdot) \in L^{\infty}(\mathbb{Q}_p^n)$, $1 \leq u < \infty$, and $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$. Then for all $l \in \mathbb{Z}$, there exists a constant $C > 0$ such that if $f \in M\dot{K}_{\lambda,q(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$:

$$\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{l(\lambda-\eta(\cdot))}\|f\|_{M\dot{K}_{\lambda,q(\cdot)}^{\eta(\cdot),u),\theta}}.$$

Using Lemma 2.2, we deduce the following conclusion, which simplifies the proof of the main theorem.

Lemma 2.8. Let $0 < \alpha < n$. For all $q_1(\cdot), q_2(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\alpha$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$, and for any $k, l \in \mathbb{Z}$:

$$\frac{|B_k(x)|_h\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)}}{|B_l(x)|_h\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)}} \leq C\begin{cases} p^{(l-k)n/q_1'(\cdot)-l\alpha}, & \text{if } k<0, l<0, \\ p^{ln/q_1'(\infty)-ln/q_2(\infty)-l\alpha}, & \text{if } k\geq 0, l<0, \\ p^{(l-k)n/q_1'(\cdot)-l\alpha}, & \text{if } k<0, l\geq 0, \\ p^{ln/q_1'(\infty)-ln/q_2(\infty)-l\alpha}, & \text{if } k\geq 0, l\geq 0. \end{cases}$$

Proof. We divide the proof into four cases based on the ranges of k and l.

Case 1: If $k, l < 0$, for any fixed p-adic sphere $S_k, S_l \subset \mathbb{Q}_p^n$, using Lemma 2.2 and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$:

$$|B_k(x)|_h\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{-kn}p^{q_2(\cdot)}p^{q_1'(\cdot)} = Cp^{(l-k)n/q_1'(\cdot)}.$$

Case 2: If $k < 0, l \geq 0$, similarly using Lemma 2.2:

$$|B_l(x)|_h\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{-ln}p^{q_2(\cdot)}p^{q_1'(\infty)} \leq Cp^{ln/q_1'(\infty)-ln/q_2(\infty)-l\alpha}.$$

Case 3: If $k \geq 0, l < 0$, for any fixed p-adic ball $B_k(x) \subset \mathbb{Q}_p^n$:

$$|B_k(x)|_h\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{-kn}p^{q_2(\infty)}p^{q_1'(\cdot)} = Cp^{(l-k)n/q_1'(\cdot)}.$$

Case 4: If $k \geq 0, l \geq 0$, for any fixed p-adic ball $B_l(x) \subset \mathbb{Q}_p^n$:

$$|B_l(x)|_h\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{-ln}p^{q_2(\infty)}p^{q_1'(\infty)} \leq Cp^{ln/q_1'(\infty)-ln/q_2(\infty)-l\alpha}.$$

Combining Cases 1-4 completes the proof of Lemma 2.8.

Proofs of the Principal Results

Proof of Theorem 1.1. For $f \in M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)$, we write $f(z_1) = \sum_{l=-\infty}^{\infty} f(z_1)\chi_l(z_1)$. Assume k_0 is positive (the case $k_0 \leq 0$ is similar). Using Minkowski’ s inequality:

$$\|I_\alpha^p f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{Q}_p^n)} = \sup_{\epsilon>0} \epsilon^{\theta/u(1+\epsilon)} \left\{ \sum_{k=-\infty}^{k_0} p^{-k_0\lambda u(1+\epsilon)} p^{\eta(\cdot)ku(1+\epsilon)} \|I_\alpha^p f\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} .$$

We decompose the sum into three parts:

$$\|I_\alpha^p f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{Q}_p^n)} \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=-\infty}^{k-2} \|I_\alpha^p(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} + C \sup_{\epsilon>0} p^{-k_0\lambda} \{$$

Estimation of E_2. On one hand, if $k > 0, l > 0$, since $\eta(\cdot) \in C^{\log}(\mathbb{Q}_p^n)$, Definition 2.3 gives for $z_1 \in S_k$:

$$|\eta(z_1) - \eta(\infty)| \leq \frac{C}{\log_p(p + p^k)},$$

implying $p^{k\eta(z_1)} \approx p^{k\eta(\infty)}$. Thus $\eta(\infty)$ can be replaced by $\eta(z_1)$. On the other hand, if $k < 0, l < 0$, for $z_1 \in S_k$:

$$|\eta(z_1) - \eta(0)| \leq \frac{C}{\log_p p^k} = \frac{C}{k},$$

implying $p^{k\eta(z_1)} \approx p^{k\eta(0)}$. Thus $\eta(0)$ can be replaced by $\eta(z_1)$. By Lemma 2.3:

$$E_2 \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=k-1}^{k+1} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} \leq C\|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)}.$$

Estimation of E_1. For any $k \in \mathbb{Z}$, $l \leq k-2$, and a.e. $z_1 \in S_k$, $z_2 \in S_l$, we have $|z_1 - z_2|_p \approx p^k$. By Lemma 2.1:

$$|I_\alpha^p(f\chi_l)(z_1)| \leq \int_{S_l} \frac{|f(z_2)|}{|z_1 - z_2|_p^{n-\alpha}} \, dz_2 \leq Cp^{k(\alpha-n)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)}. \tag{3.1}$$

Applying Minkowski’ s inequality:

$$E_1 \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=-\infty}^{k-2} \|I_\alpha^p(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \{ \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1$$

Using Lemma 2.8 and (3.1), for $k, l < 0$ and the condition $\alpha - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, let $\omega = n/q_1'(\cdot) - \eta(\cdot)$. Then:

$$E_{11} \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l=-\infty}^{k-2} p^{l\eta(\cdot)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} p^{\omega(l-k)} \right)^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} .$$

By Fubini' s theorem, Lemma 2.1, and the estimate $p^{-u(1+\epsilon)} < p^{-u}$:

$$E_{11} \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{l=-\infty}^{k_0-2} p^{l\eta(\cdot)u(1+\epsilon)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}^{u(1+\epsilon)} \left(\sum_{k=l+2}^{k_0} p^{\omega(l-k)u(1+\epsilon)} \right) \right\}^{1/u(1+\epsilon)} \leq C\|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)}.$$

For E_{12}, applying Minkowski' s inequality yields:

$$E_{12} \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=-\infty}^{-1} \|I_\alpha^p(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} + C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1}$$

The estimate for N_2 is similar to E_{11}, substituting $q_1'(\infty)$ and using $\eta(\cdot) < n/q_1'(\infty)$. For N_1, since $\eta(\cdot) < n/q_1'(\infty)$, Lemma 2.8 and (3.1) give:

$$N_1 \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l=-\infty}^{-1} p^{l\eta(\cdot)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} p^{(\eta(\cdot)-n/q_1'(\infty))k} \right)^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} \leq C\|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)}.$$

Estimation of E_3. For $k \in \mathbb{Z}$, $l \geq k+2$, and a.e. $z_1 \in S_k$, $z_2 \in S_l$, we have $|z_1 - z_2|_p \approx p^l$. By Lemma 2.1:

$$|I_\alpha^p(f\chi_l)(z_1)| \leq Cp^{l(\alpha-n)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)}. \tag{3.2}$$

Splitting E_3 using Minkowski' s inequality:

$$E_3 \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=k+2}^{\infty} \|I_\alpha^p(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} =: E_{31} + E_{32}.$$

For E_{32}, let $d = n/q_1(\infty) + \eta(\cdot) > 0$. By Lemma 2.8 and (3.2):

$$E_{32} \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l=k+2}^{\infty} p^{l(\alpha+\eta(\cdot))} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} p^{d(k-l)} \right)^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} \leq C\|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)},$$

where the last step uses $\lambda + \alpha < d/2$.

For E_{31}, applying Minkowski’ s inequality yields:

$$E_{31} \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=k+2}^{-1} \|I_\alpha^p(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} + C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1} \right.$$

The estimate for W_1 is similar to E_{32}, substituting $q_1(\cdot)$ and using $n/q_1(\cdot) + \eta(\cdot) > 0$. For W_2, using $\lambda + \alpha - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, Definition 2.5, and Lemmas 2.7-2.8:

$$W_2 \leq C \sup_{\epsilon>0} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l=0}^{\infty} p^{l(\alpha-\eta(\cdot)-n/q_1(\infty)+\lambda)} \right)^{u(1+\epsilon)} \right\}^{1/u(1+\epsilon)} \|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)} \leq C\|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)},$$

where the last step uses $\lambda + \alpha - n/q_1(\infty) < \eta(\cdot)$.

Combining the estimates for E_1, E_2, and E_3 yields:

$$\|I_\alpha^p f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{Q}_p^n)} \leq C\|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)},$$

which completes the proof of Theorem 1.1.

Proof of Theorem 1.2. For all $x \in \mathbb{Q}_p^n$, Lemma 2.4 gives $|M_\alpha^p(f)(x)| \leq C|I_\alpha^p(|f|)(x)|$. The result follows directly from Theorem 1.1.

Proof of Theorem 1.3. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, then from (1.1) we have:

$$|[b, I_\alpha^p]f(x)| \leq \int_{\mathbb{Q}p^n} \frac{|b(x)-b(y)||f(y)|}{|x-y|_p^{n-\alpha}}\, dy \leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} |I_{\alpha+\beta}^p f(x)|.$$

The result follows from Theorem 1.1.

Proof of Theorem 1.4. For any fixed $x \in \mathbb{Q}_p^n$, if $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, Lemma 2.5 gives $M_{\alpha,b}^p \leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M_{\alpha+\beta}^p(f)(x)$. The result follows from Theorem 1.2.

Proof of Theorem 1.5. For any fixed $x \in \mathbb{Q}_p^n$, if $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$, Lemma 2.6 yields $|[b, M_\alpha^p](f)(x)| \leq M_{\alpha,b}^p(f)(x)$. The result follows from Theorem 1.4.

Proof of Theorem 1.6. For any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, the triangle inequality gives:

$$|[|b|, M_p^\sharp]f(x)| \leq 2M_{|b|}^p f(x).$$

Since $M_p^\sharp(f) \leq 2M^p(f)$, for $x \in \mathbb{Q}_p^n$:

$$|[b, M_p^\sharp](f)(x)| \leq 4\left(b^-(x)M^p(f)(x) + M^p(b^-f)(x)\right) + 2M_{|b|}^p f(x).$$

Given $b \geq 0$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, we apply Corollary 1.17 to complete the proof.

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Conflict of Interest

The authors declare no conflict of interest.

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All data generated or analyzed during this study are included in this published article.

Author Contributions

All authors contributed equally to the writing of this article. All authors read and approved the final manuscript.

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Note: Figure translations are in progress. See original paper for figures.