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# Besov Estimates for Sub-elliptic Equations in the Heisenberg Group

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## Abstract

This paper studies the regularity of weak solutions to non-degenerate divergence form subelliptic equations on the Heisenberg group. Based on more general assumptions on the coefficient matrix, this paper establishes, for both homogeneous and inhomogeneous cases, horizontal Calderón-Zygmund estimates for weak solutions in Besov spaces. The research in this paper will enrich and develop the nonlinear Calderón-Zygmund regularity theory on the Heisenberg group.

## Full Text

## Preamble

### Besov Estimates for Sub-elliptic Equations in the Heisenberg Group

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## Abstract

In this paper, we study weak solutions to non-degenerate sub-elliptic equations in the Heisenberg group and investigate the regularity of these solutions. We establish horizontal Calderón-Zygmund type estimates in Besov spaces under more general assumptions on the coefficients for both homogeneous and non-homogeneous equations. This work expands the Calderón-Zygmund theory in the Heisenberg group.

**Keywords:** Heisenberg group; sub-elliptic equations; regularity; Besov spaces.

**Mathematics Subject Classification (2020):** 35R03, 35H20, 35J70.

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## Introduction

The main purpose of this article is to study Besov regularity of weak solutions to a class of sub-elliptic equations of the type

$$\operatorname{div}_H A(x, Xu) = 0 \quad (1.1)$$

and

$$\operatorname{div}_H A(x, Xu) = \operatorname{div}_H \left(|F|^{p-2} F\right) \quad (1.2)$$

in $\Omega$, where $\Omega$ is an open and bounded subdomain in the Heisenberg group $\mathbb{H}^n = \mathbb{R}^{2n+1}$ ($n \geq 1$). We refer to (1.1) and (1.2) as the homogeneous and non-homogeneous equations, respectively. The unknown $u$ belongs to the local horizontal Sobolev space $HW^{1,p}_{\text{loc}}(\Omega)$, which will be defined in Section 2. In both equations, the horizontal divergence operator $\operatorname{div}_H$ and the horizontal gradient $X$ are defined by

$$\operatorname{div}_H F = \sum_{i=1}^{2n} X_i F_i, \quad Xu = (X_{1u}, X_{2u}, \dots, X_{2n-1}u, X_{2n}u)$$

in the distributional sense. Moreover, $A : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is assumed to be a Carathéodory vector field satisfying general growth and uniform ellipticity conditions, meaning there exist constants $\nu, L, k > 0$ and $0 < \mu < 1$ such that

$$[A(x,\xi) - A(x,\eta)] \cdot (\xi - \eta) \geq \nu \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2,$$

$$|A(x,\xi) - A(x,\eta)| \leq L \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|,$$

$$|A(x,\xi)| \leq k \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}$$

for every $\xi, \eta \in \mathbb{R}^{2n}$ and almost all $x \in \Omega$. In (1.2), $F : \Omega \to \mathbb{R}^{2n}$.

The regularity of solutions to elliptic equations in Euclidean spaces $\mathbb{R}^n$ has been well studied by Iwaniec [10], DiBenedetto and Manfredi [7]. This theory was subsequently extended to general elliptic problems; see the relevant papers [11, 12, 4, 3]. For nonlinear Calderón-Zygmund estimates in the Heisenberg group, Goldstein and Zatorska-Goldstein [8] treated the quadratic case $p = 2$. Later,

$HW^{1,p}$ estimates for sub-elliptic equations on $\mathbb{H}^n$ were proved by Mingione, Zatorska-Goldstein and Zhong [14], who considered equations of the form

$$\mathrm{div}_H[b(x)a(Xu)] = \mathrm{div}_H\left(|F|^{p-2}F\right)$$

with $b \in VMO_{\mathrm{loc}}(\Omega)$.

Currently, research has focused on regularity estimates for weak solutions in Besov spaces in both $\mathbb{R}^n$ and $\mathbb{H}^n$ ([2, 6, 9]). Besov spaces constitute a broader class of functions compared to classical Sobolev spaces. Baisón [1] treated nonlinear elliptic equations in divergence form and obtained Besov regularity estimates for weak solutions. Clop [5] and Lyaghfouri [13] extended these Besov space results by establishing higher integrability of weak solutions.

For the homogeneous case (1.1), we assume there exists a function $g \in L^\alpha(\Omega)$ $(0 < \alpha < 1)$ such that

$$|A(x,\xi) - A(y,\xi)| \leq d_{CC}(x,y)^\alpha (g(x) + g(y)) \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}$$

for almost every $x, y \in \Omega$ and all $\xi \in \mathbb{R}^{2n}$. Here $d_{CC}(x,y)$ denotes the CC-distance between points $x$ and $y$ in $\mathbb{H}^n$.

For the non-homogeneous case (1.2), we assume there exists a sequence of measurable non-negative functions $g_k \in L^\alpha(\Omega)$ $(k \in \mathbb{N},\ 0 < \alpha < 1)$ satisfying

$$\sum_{k=1}^{\infty} \|g_k\|^q_{L^\alpha(\Omega)} < \infty \quad (1 \leq q < \infty)$$

and

$$|A(x,\xi) - A(y,\xi)| \leq d_{CC}(x,y)^\alpha (g_k(x) + g_k(y)) \left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}$$

for $\xi \in \mathbb{R}^{2n}$ and almost all $x, y \in \Omega$ such that $2^{-k} \leq d_{CC}(x,y) < 2^{-k+1}$. Following (A5), we abbreviate this as $\{g_k\}_k \in \ell^q(L^\alpha(\Omega))$.

By introducing the auxiliary function

$$V(\xi) = \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{2}} \qquad (1.3)$$

with $\xi \in \mathbb{R}^{2n}$, we present the main results of this article.

**Theorem 1.1.** Let $0 < \alpha < 1$ and $2 \leq p < 4$. Assume that $A$ satisfies hypotheses (A1)-(A4) with $0 < \mu < 1$. If $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.1), then $V(Xu) \in B^\alpha_{2,\infty}(\Omega)$ locally.

**Theorem 1.2.** Let $0 < \alpha < 1$, $2 \leq p < 4$, and $1 \leq q < \frac{2Q}{Q-2\alpha}$. Assume that hypotheses (A1)-(A3) and (A5) hold. If $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.2) with $0 < \mu < 1$ and $|F|^{p-2}F \in B^\alpha_{2,q}(\Omega)$, then $V(Xu) \in B^\alpha_{2,q}(\Omega)$ locally.

See Section 2 for the definitions of $HW^{1,p}(\Omega)$ and $B^\alpha_{2,q}(\Omega)$.

The contribution of our main results is the study of a broad class of sub-elliptic equations in the Heisenberg group. Our aim is to obtain Besov regularity estimates for weak solutions. The hypotheses (A1)-(A4) (or (A5)) represent an extension of VMO conditions.

This article is organized as follows. In Section 2 we provide definitions and tools such as classical inequalities, and present two lemmas concerning reverse Hölder type inequalities for weak solutions. In Sections 3 and 4 we give the proofs of Theorem 1.1 and Theorem 1.2, respectively.

## 2.1 Heisenberg Group

In this section we collect basic notation and preliminaries for the Heisenberg group.

We denote by $(x,t) = (x_1, x_2, \dots, x_{2n}, t)$ the coordinates of points in the Heisenberg group $\mathbb{H}^n$. The group structure on $\mathbb{H}^n$ is given by

$$(x_1, x_2, \dots, x_{2n}, t)\circ(y_1, y_2, \dots, y_{2n}, s) = \left(x_1 + y_1, x_2 + y_2, \dots, x_{2n} + y_{2n}, t + s + \sum_{j=1}^{n}(x_j y_{n+j} - x_{n+j} y_j)\right).$$

An anisotropic dilation induces a homogeneous norm (gauge) on $(x,t)$ by

$$\|(x,t)\| = \left(|x|^4 + t^2\right)^{1/4}.$$

For $j = 1, \dots, n$, we define the left-invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} - \frac{x_{n+j}}{2}\frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} + \frac{x_j}{2}\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

which form a basis of the space of left-invariant vector fields on $\mathbb{H}^n$. The vector fields $X_1, X_2, \dots, X_{2n}$ are called horizontal vector fields. The length of the horizontal gradient is then given by

$$|Xu|^2 = \sum_{j=1}^{2n}(X_j u)^2.$$

## 2.2 CC-distance and CC-Balls

By considering the well-known Carnot-Carathéodory metric with CC-distance $d_{CC}$, we define CC-balls by

$$B_R(x_0) = \{y \in \mathbb{H}^n \mid d_{CC}(x_0, y) < R\}$$

with center $x_0$ and radius $R$. Introducing the homogeneous dimension $Q = 2n + 2$, we obtain the Lebesgue measure of a CC-ball $|B_R(x_0)| \approx R^Q$.

## 2.3 Horizontal Sobolev Spaces and Besov Spaces

Let $L^p(\mathbb{H}^n)$ denote the Lebesgue space on the Heisenberg group. The dual space of $L^p(\mathbb{H}^n)$ is $L^{p'}(\mathbb{H}^n)$ where $\frac{1}{p}+\frac{1}{p'}=1$. The horizontal Sobolev space with its norm is defined by

$$HW^{1,p}(\Omega)=\{u\in L^p(\Omega):Xu\in L^p(\Omega)\},\quad \|u\|_{HW^{1,p}(\Omega)}=\|u\|_{L^p(\Omega)}+\|Xu\|_{L^p(\Omega)}.$$

A function $u$ belongs to $HW^{1,p}_{\text{loc}}(\Omega)$ if $u\in HW^{1,p}(\Omega_0)$ for every $\Omega_0\Subset\Omega$.

Let the parameters satisfy $0<\alpha<1$, $1\le p<\infty$, and $1\le q\le\infty$. The Besov spaces $B^\alpha_{p,q}(\Omega)$ $(\Omega\subset\mathbb{H}^n)$ with their norm are defined via [16] as

$$\|u\|_{B^\alpha{}_{p,q}(\Omega)}=\|u\|_{L^p(\Omega)}+[u]_{B^\alpha{}_{p,q}(\Omega)},$$

where the seminorm $[u]_{B^\alpha{}_{p,q}(\Omega)}$ is given by

$$[u]_{B^\alpha{}_{p,q}(\Omega)}=\begin{cases}\left(\displaystyle\int_{\mathbb{H}^n}\frac{\|\Delta_h u\|^q_{L^p(\Omega)}}{|h|^{\alpha q}}\frac{dh}{|h|^Q}\right)^{1/q}, & 1\le q<\infty,\\ \displaystyle\sup_{h\neq 0}\frac{\|\Delta_h u\|_{L^p(\Omega)}}{|h|^\alpha}, & q=\infty.\end{cases}$$

In this article, we write $\Delta_h u=u(x+h)-u(x)$ for brevity.

## 2.4 Basic Tools

For every $\varepsilon>0$, there exists $C(\varepsilon)>0$ such that for all $s,t\ge 0$,

$$st\le\varepsilon s^p+C(\varepsilon)t^{p'},\quad (2.1)$$

which is the classical Young inequality, where $\frac{1}{p}+\frac{1}{p'}=1$. In particular,

$$ab\le\varepsilon a^2+C(\varepsilon)b^2.\quad (2.2)$$

Let $B_R\Subset\mathbb{H}^n$ be a CC-ball and $f$ an integrable function on $B_R$. We define the average of $f$ over the CC-ball $B_R$ as

$$(f)_{B_R}=\frac{1}{|B_R|}\int_{B_R}f(x)\,dx\approx R^{-Q}\int_{B_R}f(x)\,dx.\quad (2.3)$$

We present the definition of weak solutions. If for any $\phi\in C_0^\infty(\Omega)$,

$$\int_\Omega A(x,Xu)\cdot X\phi\,dx=\int_\Omega |F|^{p-2}F\cdot X\phi\,dx,\quad (2.4)$$

then $u\in HW^{1,p}_{\text{loc}}(\Omega)$ is a weak solution to (1.2). Here we call $\phi$ a test function.

## 2.5 Reverse Hölder Type Inequalities

The higher integrability estimates for Laplace and $p$-Laplace equations are well known (see [10] and [7]). In the Heisenberg group, we have the following two results for homogeneous and non-homogeneous situations (see [14]).

**Lemma 2.1.** Let $u \in HW^{1,p}(\Omega)$ with $2 < p < 4$ be a weak solution to (1.1) under hypotheses (A1)-(A4). There exists a constant $c(n, p, \nu, k, L)$, independent of $\mu$, the solution $u$, and the vector field $A(x, \nabla u)$, such that the following inequality holds for any CC-ball $B_R \Subset \Omega$:

$$\left(\frac{1}{|B_R|}\int_{B_R} |Xu|^p \, dx\right)^{1/p} \le c \left(\frac{1}{|B_{2R}|}\int_{B_{2R}} (\mu + |Xu|)^p \, dx\right)^{1/p}. \quad (2.5)$$

**Lemma 2.2.** Let $u \in HW^{1,p}(\Omega)$ with $2 < p < 4$ be a weak solution to equation (1.2). Assume that (A1)-(A3) and (A5) hold. If $F \in L^q_{\text{loc}}(\Omega)$, then $Xu \in L^q_{\text{loc}}(\Omega)$, where $q \in (p, \infty)$. Moreover, there exists a positive constant $C(n, p, \nu, L, q, a)$ such that

$$\left(\frac{1}{|B_R|}\int_{B_R} |Xu|^q \, dx\right)^{1/q} \le C \left(\frac{1}{|B_{2R}|}\int_{B_{2R}} (\mu + |Xu|)^p \, dx\right)^{1/p} + C \left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |F|^q \, dx\right)^{1/q} \quad (2.6)$$

for any CC-ball $B_R \Subset \Omega$.

# 3 Proof of Theorem 1.1

In this section we present the proof of Theorem 1.1. Inspired by [5], for the vector field $A(x, \xi)$ appearing in (1.2), we introduce for $\xi \in \mathbb{R}^{2n}$ and a CC-ball $B \subset \Omega$:

$$A_B(\xi) = \frac{1}{|B|}\int_B A(x, \xi) \, dx, \quad (3.1)$$

and define

$$V(x, B) = \sup_{\xi \in \mathbb{R}^{2n}} \frac{|A(x, \xi) - A_B(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}}, \quad (3.2)$$

where $B \subset \Omega$ is a CC-ball and $x \in \Omega$. It follows that if $A : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a Carathéodory vector field such that (A1)-(A4) hold, then $A$ is locally uniformly in VMO, that is,

$$\lim_{R \to 0} \sup_{c(B) \in K, r(B) < R} \frac{1}{|B|}\int_B V(x, B) \, dx = 0, \quad (3.3)$$

where $K \Subset \Omega$, and $c(B)$ and $r(B)$ denote the center and radius of the CC-ball $B$, respectively.

To prove Theorem 1.1, we note that there exists a constant $\widehat{C} > 0$ such that

$$\widehat{C}^{-1} \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2 \le |V(\xi) - V(\eta)|^2 \le \widehat{C} \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2, \quad (3.4)$$

for any $\xi, \eta \in \mathbb{R}^{2n}$ with $|\xi - \eta| \neq 0$.

We are now ready to present the proof.

**Proof of Theorem 1.1.** Let $B_{3R} \Subset \Omega$ and choose a test function $\phi = \Delta_{-h}(\eta^2 \Delta_h u)$ for (1.1), where $\eta \in C_0^\infty(B_{3R})$ is a cutoff function satisfying $0 \leq \eta(x) \leq 1$, $\eta(x) \equiv 1$ for $x \in B_R$, $\eta(x) \equiv 0$ for $x \in B_{3R}\ B_{2R}$, and $|X\eta| \leq C/R$.

Testing (1.1) with this $\phi$ yields

$$\int_\Omega [A(x+h, Xu(x+h)) - A(x+h, Xu)] \cdot \eta^2 \Delta_h Xu \, dx = G_1 + G_2 + G_3 + G_4,$$

where

$$G_1 = \int_\Omega [A(x+h, Xu(x+h)) - A(x+h, Xu)] \cdot \eta^2 \Delta_h Xu \, dx,$$

$$G_2 = \int_\Omega [A(x, Xu) - A(x+h, Xu)] \cdot \eta^2 \Delta_h Xu \, dx,$$

$$G_3 = \int_\Omega [A(x+h, Xu) - A(x+h, Xu(x+h))] \cdot 2\eta X\eta \Delta_h u \, dx,$$

$$G_4 = \int_\Omega [A(x, Xu) - A(x+h, Xu)] \cdot 2\eta X\eta \Delta_h u \, dx.$$

We estimate each $G_i$ $(1 \leq i \leq 4)$. By (A1), it is clear that

$$G_1 \geq \nu \int_\Omega \left(\mu^2 + |Xu(x+h)|^2 + |Xu|^2\right)^{\frac{p-2}{2}} |\Delta_h Xu|^2 \eta^2 \, dx. \quad (3.5)$$

For $G_2$, using (A4) and (2.2), we obtain

$$|G_2| \leq C \int_\Omega |h|^\alpha (g(x) + g(x+h)) \left(\mu^2 + |Xu|^2\right)^{\frac{p-1}{2}} |\Delta_h Xu| \eta^2 \, dx$$

$$\leq \varepsilon \int_\Omega \left(\mu^2 + |Xu|^2\right)^{\frac{p-2}{2}} |\Delta_h Xu|^2 \eta^2 \, dx + C(\varepsilon)|h|^{2\alpha} \int_\Omega (g(x)+g(x+h))^2 \left(\mu^2 + |Xu|^2\right)^{\frac{p}{2}} \, dx, \quad (3.6)$$

where $\varepsilon > 0$ will be chosen later. By (A2) and (2.2), we deduce that

$$|G_3| \leq C \int_\Omega \left(\mu^2 + |Xu|^2 + |Xu(x+h)|^2\right)^{\frac{p-2}{2}} |\Delta_h Xu| \eta |X\eta| |\Delta_h u| \, dx$$

$$\leq \varepsilon \int_\Omega \left(\mu^2 + |Xu|^2 + |Xu(x+h)|^2\right)^{\frac{p-2}{2}} |\Delta_h Xu|^2 \eta^2 \, dx + C(\varepsilon) \int_\Omega \left(\mu^2 + |Xu|^2 + |Xu(x+h)|^2\right)^{\frac{p-2}{2}} |X\eta|^2 |\Delta_h u|^2 \, dx.$$

Applying the Lagrange Mean Value Theorem, we obtain

$$\int_\Omega \left(\mu^2 + |Xu|^2 + |Xu(x+h)|^2\right)^{\frac{p-2}{2}} |X\eta|^2 |\Delta_h u|^2 \, dx \leq C|h|^2 \int_{B_{2R+|h|}} \left(\mu^2 + 2|Xu|^2\right)^{\frac{p-2}{2}} |Xu|^2 \, dx \leq C|h|^2 \int_{B_{2R+|h|}} (\mu$$

To estimate $G_4$, hypothesis (A4) and (2.2) give

$$|G_4| \le C \int_\Omega |h|^\alpha (g(x) + g(x+h)) \left(\mu^2 + |Xu|^2\right)^{\frac{p-1}{2}} \eta |X\eta| |\Delta_h u| \, dx$$

$$\le \varepsilon \int_\Omega \left(\mu^2 + |Xu|^2\right)^{\frac{p-2}{2}} \eta^2 |\Delta_h u|^2 \, dx + C(\varepsilon)|h|^{2\alpha} \int_\Omega (g(x)+g(x+h))^2 \left(\mu^2 + |Xu|^2\right)^{\frac{p}{2}} \, dx. \quad (3.9)$$

Here we note that

$$\int_\Omega \left(\mu^2 + |Xu|^2\right)^{\frac{p-2}{2}} \eta^2 |\Delta_h u|^2 \, dx \le C|h|^2 \int_{B_{2R+|h|}} (\mu + |Xu|)^p \, dx. \quad (3.10)$$

Combining the estimates for $G_i$ and choosing $\varepsilon$ sufficiently small, we obtain

$$\int_\Omega \left(\mu^2 + |Xu(x+h)|^2 + |Xu|^2\right)^{\frac{p-2}{2}} |\Delta_h Xu|^2 \eta^2 \, dx \le C|h|^{2\alpha} \int_\Omega (g(x)+g(x+h))^2 \left(\mu^2 + |Xu|^2\right)^{\frac{p}{2}} \, dx + C|h|^2 \int_{B_{2R+|h|}}$$

By the definition of $V$ and (3.4), we have

$$|\Delta_h V|^2 \le C \left(\mu^2 + |Xu(x+h)|^2 + |Xu|^2\right)^{\frac{p-2}{2}} |\Delta_h Xu|^2. \quad (3.12)$$

Integrating both sides of (3.12) over $B_R$ and applying the properties of $\eta$, we get

$$\int_{B_R} |\Delta_h V|^2 \, dx \le C \int_\Omega \left(\mu^2 + |Xu(x+h)|^2 + |Xu|^2\right)^{\frac{p-2}{2}} |\Delta_h Xu|^2 \eta^2 \, dx$$

$$\le C|h|^{2\alpha} \int_\Omega (g(x)+g(x+h))^2 \left(\mu^2 + |Xu|^2\right)^{\frac{p}{2}} \, dx + C|h|^2 \int_{B_{2R+|h|}} (\mu+|Xu|)^p \, dx. \quad (3.13)$$

Dividing both sides of (3.13) by $|h|^{2\alpha}$, we obtain

$$\int_{B_R} \frac{|\Delta_h V|^2}{|h|^{2\alpha}} \, dx \le C \int_\Omega (g(x)+g(x+h))^2 \left(\mu^2 + |Xu|^2\right)^{\frac{p}{2}} \, dx + C|h|^{2-2\alpha} \int_{B_{2R+|h|}} (\mu+|Xu|)^p \, dx =: P_1 + P_2. \quad (3.14)$$

Finally, we show that $P_i$ is bounded for each $i$. By Lemma 2.1, we have $|Xu|^p \in L^t(\Omega)$ for some $t > 1$. In particular, $|Xu|^p \in L^{\frac{Q}{Q-2\alpha}}(\Omega)$. Choosing $0 < |h| < \delta < R$ and using (A4), we obtain

$$P_1 \le C\|g\|^2_{L^\alpha(\Omega)} \left\| \left(\mu^2 + |Xu|^2\right)^{\frac{p}{2}} \right\|_{L^{\frac{Q}{Q-2\alpha}}(\Omega)} < \infty.$$

Since $u \in HW^{1,p}_{\text{loc}}(\Omega)$, we have $P_2 < \infty$. It follows that

$$\sup_{|h|<\delta} \int_{B_R} \frac{|\Delta_h V|^2}{|h|^{2\alpha}} \, dx < \infty,$$

that is, $V(Xu) \in B^\alpha_{2,\infty}(\Omega)$ locally.

## 4 Proof of Theorem 1.2

For the non-homogeneous case, we need the following lemma.

**Lemma 4.1.** Let $A : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a Carathéodory vector field such that (A1)-(A3) and (A5) hold. Then $A$ is locally uniformly in VMO, that is,

$$\lim_{R\to 0} \sup_{c(B)\in K, r(B)<R} \frac{1}{|B|}\int_B V(x,B)\,dx = 0, \quad (4.1)$$

where $V(x,B)$ is given in (3.2), $K \Subset \Omega$, and $c(B)$ and $r(B)$ denote the center and radius of the CC-ball $B$, respectively.

**Proof.** Given a point $x \in \Omega$, let $A_k(x) = \{y \in \Omega : 2^{-k} \le d_{CC}(x,y) < 2^{-k+1}\}$. We obtain

$$\frac{1}{|B|}\int_B V(x,B)\,dx \le \frac{1}{|B|}\int_B \sup_{\xi\in\mathbb{R}^{2n}} \frac{|A(x,\xi)-A(y,\xi)|}{(\mu^2+|\xi|^2)^{\frac{p-1}{2}}}\,dy\,dx$$

$$\le \frac{C(Q,\alpha)}{|B|}\sum_k \int_{B\cap A_k(x)} d_{CC}(x,y)^\alpha (g_k(x)+g_k(y))\,dy\,dx.$$

By Hölder's inequality, we acquire

$$\frac{1}{|B|}\int_{B\cap A_k(x)} (g_k(x)+g_k(y))^\alpha\,dy\,dx \le C(Q,\alpha,q)|B|^{-\frac{\alpha}{Q}}\|g_k\|_{L^\alpha(B)}.$$

Choosing $r > 0$ small enough and observing that $x \mapsto \|g_k\|_{L^\alpha(B_r(x))}$ is continuous on the set $\{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > r\}$, we find that for each $x_r \in K$ with $r$ sufficiently small,

$$\|g_k\|_{L^\alpha(B_r(x_r))} \to 0 \quad \text{as } r \to 0.$$

Each of the limits on the right-hand side equals zero, which completes the proof.

With the help of the preceding lemma, we can now prove Theorem 1.2.

**Proof of Theorem 1.2.** Assume that $B_{3R} \Subset \Omega$ and choose a test function $\phi = \Delta_{-h}(\eta^2\Delta_h u)$ for (1.2), where $\eta \in C_0^\infty(\Omega)$ is a cutoff function satisfying $0 \le \eta(x) \le 1$, $\eta(x) \equiv 1$ for $x \in B_R$, $\eta(x) \equiv 0$ for $x \in B_{3R}$ $B_{2R}$, and $|X\eta| \le C/R$.

According to the definition of weak solution and our choice of test function, we obtain

$$\int_\Omega [A(x+h, Xu(x+h)) - A(x+h, Xu)]\cdot \eta^2\Delta_h Xu\,dx$$

$$+\int_\Omega [A(x,Xu) - A(x+h,Xu)]\cdot\eta^2\Delta_h Xu\,dx$$

$$+\int_\Omega [A(x+h,Xu) - A(x+h,Xu(x+h))]\cdot 2\eta X\eta\Delta_h u\,dx$$

$$+\int_\Omega [A(x,Xu)-A(x+h,Xu)]\cdot 2\eta X\eta\Delta_h u\,dx$$

$$=\int_\Omega |F|^{p-2}F\cdot 2\eta X\eta\Delta_h u\,dx+\int_\Omega |F|^{p-2}F\cdot \eta^2\Delta_h Xu\,dx,$$

which we denote as $G_1+G_2+G_3+G_4=G_5+G_6$. (4.2)

We have already estimated the terms $G_1$ through $G_4$ in the proof of Theorem 1.1. Thus it remains to estimate $G_5$ and $G_6$.

Applying (2.2), we get

$$|G_5|\le C\int_\Omega |\Delta_h(|F|^{p-2}F)||\Delta_h u|\eta\,dx$$

$$\le C|h|^{2\alpha}\int_\Omega |F|^{2(p-1)}\,dx+C\int_\Omega |\Delta_h u|^2\eta^2\,dx.$$

By the Lagrange Mean Value Theorem, the second term can be controlled by

$$\int_\Omega |\Delta_h u|^2\eta^2\,dx\le C|h|^2\int_{B_{2R+|h|}}(\mu+|Xu|)^p\,dx. \quad (4.3)$$

For the estimate of $G_6$, we have

$$|G_6|\le C\int_\Omega |\Delta_h(|F|^{p-2}F)||\Delta_h Xu|\eta^2\,dx$$

$$\le C|h|^{2\alpha}\int_\Omega |F|^{2(p-1)}\,dx+\varepsilon\int_\Omega |\Delta_h Xu|^2\eta^2\,dx. \quad (4.4)$$

Similarly, one obtains

$$\int_\Omega |\Delta_h Xu|^2\eta^2\,dx\le \mu^{p-2}\int_\Omega |\Delta_h Xu|^2\eta^2\,dx\le \int_\Omega \left(\mu^2+|Xu(x+h)|^2+|Xu|^2\right)^{\frac{p-2}{2}}|\Delta_h Xu|^2\eta^2\,dx. \quad (4.5)$$

Combining the estimates for $G_i$, we have

$$(\nu-2\varepsilon)\int_\Omega \left(\mu^2+|Xu(x+h)|^2+|Xu|^2\right)^{\frac{p-2}{2}}|\Delta_h Xu|^2\eta^2\,dx$$

$$\le C|h|^{2\alpha}\int_\Omega (g_k(x)+g_k(x+h))^2\left(\mu^2+|Xu|^2\right)^{\frac{p}{2}}\,dx+C|h|^{2\alpha}\int_\Omega |F|^{2(p-1)}\,dx+C|h|^2\int_{B_{2R+|h|}}(\mu+|Xu|)^p\,dx. \quad (4.6)$$

Choosing $\varepsilon=\nu/4$, we obtain

$$\int_\Omega \left(\mu^2+|Xu(x+h)|^2+|Xu|^2\right)^{\frac{p-2}{2}}|\Delta_h Xu|^2\eta^2\,dx$$

$$\leq C|h|^{2\alpha}\int_{\Omega}(g_k(x)+g_k(x+h))^2\left(\mu^2+|Xu|^2\right)^{\frac{p}{2}}\,dx+C|h|^{2\alpha}\int_{\Omega}|F|^{2(p-1)}\,dx+C|h|^2\int_{B_{2R+|h|}}(\mu+|Xu|)^p\,dx. \quad (4.7)$$

Using (1.3) and (3.4), we have

$$|\Delta_h V|^2 \leq C\left(\mu^2+|Xu(x+h)|^2+|Xu|^2\right)^{\frac{p-2}{2}}|\Delta_h Xu|^2. \quad (4.8)$$

From (4.7), it follows that

$$\int_{B_R}|\Delta_h V|^2\,dx \leq C|h|^2\int_{B_{2R+|h|}}(\mu+|Xu|)^p\,dx+C|h|^{2\alpha}\int_{\Omega}(g_k(x)+g_k(x+h))^2\left(\mu^2+|Xu|^2\right)^{\frac{p}{2}}\,dx+C|h|^{2\alpha}\int_{\Omega}|F|^{2(p-1}$$

Dividing both sides of (4.9) by $|h|^{2\alpha}$ and applying the properties of $\eta$, we derive

$$\int_{B_R}\frac{|\Delta_h V|^2}{|h|^{2\alpha}}\,dx \leq C|h|^{2-2\alpha}\int_{B_{2R+|h|}}(\mu+|Xu|)^p\,dx+C\int_{\Omega}(g_k(x)+g_k(x+h))^2\left(\mu^2+|Xu|^2\right)^{\frac{p}{2}}\,dx+C\int_{\Omega}|F|^{2(p-1)}\,dx.$$

Taking the 1/2 power, we obtain

$$\left(\int_{B_R}\frac{|\Delta_h V|^2}{|h|^{2\alpha}}\,dx\right)^{1/2} \leq C|h|^{1-\alpha}\left(\int_{B_{2R+|h|}}(\mu+|Xu|)^p\,dx\right)^{1/2}$$

$$+C\left(\int_{\Omega}(g_k(x)+g_k(x+h))^2\left(\mu^2+|Xu|^2\right)^{\frac{p}{2}}\,dx\right)^{1/2}+C\left(\int_{\Omega}|F|^{2(p-1)}\,dx\right)^{1/2}. \quad (4.11)$$

Restricting to $B_\delta$ with $0<|h|<\delta$ and taking the $L^q$ norm with respect to the measure $\frac{dh}{|h|^Q}$, we obtain

$$\left(\int_{B_\delta}\left(\int_{B_R}\frac{|\Delta_h V|^2}{|h|^{2\alpha}}\,dx\right)^{q/2}\frac{dh}{|h|^Q}\right)^{1/q}$$

$$\leq C\left(\int_{B_\delta}|h|^{(1-\alpha)q}\left(\int_{B_{2R+|h|}}(\mu+|Xu|)^p\,dx\right)^{q/2}\frac{dh}{|h|^Q}\right)^{1/q}$$

$$+C\left(\int_{B_\delta}\left(\int_{\Omega}(g_k(x)+g_k(x+h))^2\left(\mu^2+|Xu|^2\right)^{\frac{p}{2}}\,dx\right)^{q/2}\frac{dh}{|h|^Q}\right)^{1/q}$$

$$+C\left(\int_{B_\delta}\left(\int_{\Omega}|F|^{2(p-1)}\,dx\right)^{q/2}\frac{dh}{|h|^Q}\right)^{1/q} =: P_1+P_2+P_3. \quad (4.12)$$

We shall show that each $P_i$ $(1 \le i \le 3)$ is bounded. Since $|F|^{p-2}F \in B^{\alpha}_{2,q}(\Omega)$ and $1 \le q < \frac{2Q}{Q-2\alpha}$, we have $|F|^{p-2}F \in L^{\frac{Q}{Q-2\alpha}}(\Omega)$. By Lemma 2.2, we get $|Xu|^{p-2}Xu \in L^{\frac{2Q}{Q-2\alpha}}(\Omega)$, which implies $|Xu|^p \in L^{\frac{Q}{Q-2\alpha}}(\Omega)$.

To estimate $P_1$, we write the $L^q$ norm in polar coordinates. There is no harm in assuming $\delta = 1$, so $h \in B_1 \cap \mathbb{R}^{2n}$ is equivalent to $h = r\xi$ with $0 \le r < 1$ and $\xi$ in the unit sphere $S^{2n-1}$. Let $d\sigma(\xi)$ be the surface measure on $S^{2n-1}$. Setting $r_k = 2^{-k}$, we estimate $P_1$ as

$$P_1 \le C \sum_{k=0}^{\infty} \int_{r_{k+1}}^{r_k} \int_{S^{2n-1}} \|(g_k(\cdot + r\xi) + g_k(\cdot))(\mu^2 + |Xu|^2)^{\frac{p}{2}}\|^{q/2}_{L^1(B_{2R})} r^{Q-1}\, d\sigma(\xi)\, dr.$$

Since $|Xu|^p \in L^{\frac{Q}{Q-2\alpha}}(\Omega)$ and $g_k \in L^{\alpha}(\Omega)$, we have

$$\|(g_k(\cdot+r\xi)+g_k(\cdot))(\mu^2+|Xu|^2)^{\frac{p}{2}}\|_{L^1(B_{2R})} \le \|g_k(\cdot+r\xi)+g_k(\cdot)\|_{L^{\alpha}(B_{2R})}\|(\mu^2+|Xu|^2)^{\frac{p}{2}}\|_{L^{\frac{Q}{Q-2\alpha}}(B_{2R})}.$$

Moreover,

$$\|g_k(\cdot + r\xi) + g_k(\cdot)\|_{L^{\alpha}(B_{2R})} \le \|g_k\|_{L^{\alpha}(B_{2R+r_k\xi})} + \|g_k\|_{L^{\alpha}(B_{2R})} \le 2\|g_k\|_{L^{\alpha}(\widetilde{B}_R)}$$

for each $\xi \in S^{2n-1}$ and $r_{k+1} \le r \le r_k$, where $\widetilde{B}_R = B_{3R}$. Therefore,

$$P_1 \le C\|(\mu^2 + |Xu|^2)^{\frac{p}{2}}\|_{L^{\frac{Q}{Q-2\alpha}}(B_{2R})} \|\{g_k\}_k\|_{\ell^q(L^{\alpha}(\widetilde{B}_R))} < \infty.$$

In the Heisenberg group, a direct calculation gives

$$\int_{B_\delta \cap \mathbb{H}^n} |h|^{(1-\alpha)q-Q}\, dx = C(\alpha, q, Q) \int_0^{\delta} \rho^{(1-\alpha)q-1}\, d\rho < \infty,$$

provided $(1-\alpha)q > 0$, which holds by our assumptions.

Since $u \in HW^{1,p}(\Omega)$, we deduce that

$$P_2 \le C \left( \int_{B_\delta} |h|^{(1-\alpha)q-Q}\, dh \right)^{1/q} \left( \int_{B_{2R+|h|}} (\mu + |Xu|)^p\, dx \right)^{1/2} < \infty.$$

Finally, because $|F|^{p-2}F \in B^{\alpha}_{2,q}(\Omega)$, we have

$$P_3 = C \left\| |F|^{p-2}F \right\|_{L^q(\frac{dh}{|h|^Q}; L^2(B_{2R}))} < \infty.$$

Therefore, we complete the proof of Theorem 1.2.

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## References

[1] Baisón A. L., Clop A., Giova R., Orobitg J., Passarelli di Napoli A., Fractional Differentiability for Solutions of Nonlinear Elliptic Equations, Potential Analysis, 2017, 46: 1-23.

[2] Balci A. K., Diening L., Weimar M., Higher Order Calderón-Zygmund Estimates for the p-Laplace Equation, Journal of Differential Equations, 2020, 268 (2): 590-635.

[3] Byun S. S., Wang L., $L^p$-estimates for General Nonlinear Elliptic Equations, Indiana University Mathematics Journal, 2007, 56 (6), 3193-3222.

[4] Caffarelli L. A., Peral I., On $W^{1,p}$ Estimates for Elliptic Equations in Divergence Form, Communications on Pure and Applied Mathematics, 1998, 51 (1): 1-21.

[5] Clop A., Giova R., Passarelli di Napoli A., Besov Regularity for Solutions of p-harmonic Equations, Advances in Nonlinear Analysis, 2017, 8 (1): 762-778.

[6] Dahlke S., Diening L., Hartmann C., Scharf B., Weimar M., Besov Regularity of Solutions to the p-Poisson Equation, Nonlinear Analysis, 2016, 130: 298-329.

[7] DiBenedetto E., Manfredi J., On the Higher Integrability of the Gradient of Weak Solutions of Certain Degenerate Elliptic Systems, American Journal of Mathematics, 1993, 115 (5): 1107-1134.

[8] Goldstein P., Zatorska-Goldstein A., Calderón-Zygmund type estimates for non-linear systems with quadratic growth on the Heisenberg group, Forum Math, 2008, 20: 679-710.

[9] Hartmann C., Weimar M., Besov Regularity of Solutions to the p-Poisson Equation in the Vicinity of a Vertex of a Polygonal Domain, Results in Mathematics, 2018, 73 (41): 1-15.

[10] Iwaniec T., Projections onto Gradient Fields and $L^p$-estimates for Degenerated Elliptic Operators, Studia Mathematica, 1983, 75 (3): 293-312.

[11] Kinnunen J., Zhou S., A Local Estimate for Nonlinear Equations with Discontinuous Coefficients, Communications in Partial Differential Equations, 1999, 24 (11-12): 2043-2068.

[12] Kinnunen J., Zhou S., A Boundary Estimate for Nonlinear Equations with Discontinuous Coefficients, Differential and Integral Equations, 2001, 14 (4): 475-492.

[13] Lyaghfouri A., Global Higher Integrability of the Gradient of the A-Laplace Equation Solution, arXiv:2104.08831, 2021.

[14] Mingione G, Zatorska-Goldstein A, Zhong X., Gradient Regularity for Elliptic Equations in the Heisenberg Group, Advances in Mathematics, 2009, 222 (1): 62-129.

[15] Musielak J., Orlicz Spaces and Modular Spaces, Lecture Notes in Mathematics, Volume 1034, Springer, 2006.

[16] Triebel H., Theory of Function Spaces, Birkhäuser Verlag, Basel, 1983.

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