2023-84N Final Draft

Authors: Wennian Liu, Zhang Wenxuan, Ma Shiwen, Zhao Xinjun, Wang Liyun, Wang Liyun

Date: 2024-02-18T12:38:38+00:00

Abstract

The mass spectra of hidden-heavy tetraquark systems ¯ and ¯ containing strange quarks have been calculated using the hadron-quarkonium model and the color-magnetic interaction bag model, respectively. Results from the hadron-quarkonium model indicate that when the heavy quarkonium is in an excited state, the binding energy between the strange meson and is relatively large, potentially forming a compact structure; consequently, these states may be identified experimentally via the decay of from the excited state to the ground state. For ground-state strange hidden-heavy tetraquark states, the color-magnetic interaction bag model calculations support a compact tetraquark configuration, particularly for the experimentally observed (4000) and (4220), where the theoretical masses differ from experimental values by only about 20 MeV. The calculated masses for ¯ also show close agreement with partial experimental results. Based on these findings, we support the possibility that strange-quark-containing tetraquark states may possess more compact structures.

Full Text

Mass Spectra of Strange Hidden-Heavy-Flavor Tetraquark States Based on the Hadro-Quarkonium Model and the Chromomagnetic Interaction Bag Model

LIU Wennian[1,2], ZHANG Wenxuan[2], MA Shiwen[1], ZHAO Xinjun[1,2], WANG Liyun[1]

[1] Xinjiang Laboratory of Phase Transitions and Microstructures in Condensed Matters, College of Physical Science and Technology, Yili Normal University, Yining, Xinjiang 835000, China
[2] Institute of Theoretical Physics, College of Physics and Electronic Engineering, Northwest Normal University, Lanzhou 730070, China

Abstract

We calculate the mass spectra of strange hidden-heavy-flavor tetraquark systems, specifically $q\bar{s}Q\bar{Q}$ and $s\bar{s}Q\bar{Q}$, using the hadro-quarkonium (H-Q) model and the chromomagnetic interaction (CMI) bag model, respectively. The H-Q model calculations reveal that when the heavy quarkonium $Q\bar{Q}$ resides in an excited state, the binding energy between the strange meson and $Q\bar{Q}$ becomes substantial, potentially forming compact structures that could be identified experimentally through decay processes from excited to ground states of $Q\bar{Q}$. For ground-state strange hidden-heavy-flavor tetraquarks, the CMI bag model results support a compact tetraquark interpretation. In particular, for the experimentally observed $Z_{cs}(4000)$ and $Z_{cs}(4220)$ states, our calculated masses differ from experimental values by only about 20 MeV. The mass predictions for $s\bar{s}Q\bar{Q}$ systems also show good agreement with some experimental results. Based on these findings, we argue that tetraquark states containing strange quarks may exhibit more compact structures.

Keywords: hadro-quarkonium; chromomagnetic interaction

---

1. Introduction

The mass of a hadron $M(R)$ is obtained through variational methods applied to the hadron radius, yielding optimal solutions for hadron mass and expectation values for the radius. The H-Q bound state description of multiquark states strongly depends on the chosen color-electric interaction radius parameter R_H of the light hadron. Since experimental information on the interaction radius of light hadrons is scarce, we adopt the hadron radii obtained from the bag model as input parameters for calculating the mass spectra of strange tetraquark states within the H-Q framework.

This paper is organized as follows: Section 1 introduces the H-Q theoretical model and presents the H-Q bound state mass spectra under the heavy-quark limit where the heavy quarkonium radius $R_{Q\bar{Q}} \simeq 0$. Section 2 calculates the mass spectra and radii of hidden-flavor tetraquark states using the CMI bag model. Section 3 examines the influence of non-pointlike heavy quarkonium structures on H-Q bound states.

2.1 H-Q Bound States Based on Light Hadron Bag Radii

Over the past two decades, numerous exotic hadron states known as XYZ particles have been observed. These states cannot be explained by conventional baryons or mesons, with popular theoretical interpretations including hadronic molecular states, hadro-quarkonium (H-Q) bound states, and compact tetraquarks [1-6]. Hadronic molecular states are described as bound systems formed through interactions between a meson $\bar{Q}q$ (where $Q = c, b$ and $q = u, d$)

and its antiparticle $Q\bar{q}$ [7-9]. In 2021, BESIII discovered the first strange hidden-charm tetraquark candidate $Z_{cs}(3985)^-$, followed by LHCb' s observation of strange-quark-containing tetraquark states $Z_{cs}(4000)^+$ and $Z_{cs}(4220)^+$ in the $B^+ \to J/\psi\phi K^+$ process [10]. Theoretical explanations for $Z_{cs}(4000)^+$ and $Z_{cs}(4220)^+$ primarily include hadronic molecular states [5,11-12], H-Q bound states [13-14], and compact tetraquark states [15-17].

Considering the threshold effects and flavor characteristics of $Z_{cs}(4000)^+$ and $Z_{cs}(4220)^+$, this work employs both the H-Q bound state model and the chromomagnetic interaction bag model to calculate the mass spectra of these particles.

Compact tetraquark states can be described using the chromomagnetic interaction (CMI) bag model [18-19] (MIT bag model), which is not limited by quark number and effectively describes ground states of multiquark systems. The fundamental principle involves approximating hadrons as spherical bags and considering chromomagnetic interactions between quarks to derive expressions for hadron mass.

As illustrated in [Figure 1: see original paper], the H-Q bound state consists of a colorless heavy quarkonium $Q\bar{Q}$ embedded within a light hadron (qqq or $q\bar{q}$), forming a bound state through color-electric interactions [20-21]. The interaction between $Q\bar{Q}$ and qqq or $q\bar{q}$ represents a color-electric van der Waals force [20], which can be described using QCD multipole expansions [22-23]. Expanding the color-electric interaction to leading order yields the effective Hamiltonian H_{eff} [24-26]:

$$H_{\text{eff}} = -\alpha_{\psi\psi'}\vec{E} \cdot \vec{E}'$$

where $\alpha_{\psi\psi'}$ is the color-electric polarizability of the heavy quarkonium $Q\bar{Q}$ and $\vec{E}$ is the color-electric field [24]. To calculate H-Q bound state masses, we evaluate Eq. (1) in the $|H, Q\rangle$ eigenbasis. The matrix element of the color-electric field can be expressed as an energy-momentum tensor: $\theta_{\mu\mu} \approx \frac{9}{16\pi^2}E^2$. For the non-relativistically normalized light hadron state $|H\rangle$, the mass expectation value M_H is given by [24,26]:

$$M_H \simeq \langle H|\theta_{\mu\mu}(q = 0)|H\rangle$$

The interaction between $Q\bar{Q}$ and qqq or $q\bar{q}$ can be approximated by an effective potential V_{HQ} satisfying:

$$\int d^{3r} V_{HQ} \approx -\alpha_{\psi\psi'} M_H$$

where R_H in Eq. (3) represents the radius of the light hadron [26]. The effective potential $V_{HQ}(r)$ can be expressed as [24]:

$$V_{HQ}(r) = \begin{cases} -\frac{2\pi\alpha_{\psi\psi'} M_H}{R_H^3}, & r < R_H \\ 0, & r > R_H \end{cases}$$

The Hamiltonian for the H-Q bound state is [24]:

$$H_{HQ} = M_{Q\bar{Q}} + M_H + V_{HQ}(r) + \frac{p^2}{2\mu}$$

where $M_{Q\bar{Q}}$ is the mass of $Q\bar{Q}$ and μ is the reduced mass. The complete H-Q bound state wavefunction can be written as:

$$|\Phi_{HQ}\rangle = |(L_Q, S_Q)J_Q; (L_H, S_H)J_H; (J_{HQ}, L_{HQ})J^P_{\text{tot}}\rangle$$

which satisfies angular momentum addition and parity multiplication rules. Here L_Q, S_Q, and J_Q represent the orbital, spin, and total angular momentum of $Q\bar{Q}$; L_H, S_H, and J_H correspond to those of qqq or $q\bar{q}$; $J_{HQ} = J_H + J_Q$; L_{HQ} is the relative angular momentum between $Q\bar{Q}$ and qqq or $q\bar{q}$; and J^P_{tot} denotes the total angular momentum and parity of the H-Q bound state. In our calculations, we neglect relative orbital excitations between $Q\bar{Q}$ and qqq or $q\bar{q}$, setting $L_{HQ} = 0$, as we find that V_{HQ} can only support ground-state binding.

The integration limit R_H in Eq. (3) represents the expectation value of the radius of the H-Q bound state after embedding $Q\bar{Q}$ into qqq or $q\bar{q}$. Due to the strong dependence of V_{HQ} on R_H, this parameter directly affects the reliability of our calculations. We consider two scenarios:

I. In the heavy-quark limit where $R_H \gg R_Q$, the influence of the embedded $Q\bar{Q}$ on the light hadron radius is negligible.
II. If we consider the non-pointlike structure of $Q\bar{Q}$, the heavy quarkonium radius R_Q cannot be ignored compared to R_H, necessitating consideration of $Q\bar{Q}$' s effect on the light hadron radius.

For the ground-state light meson radius R_H, we examine both the root-mean-square radii from quark potential models and variational results from bag models. Given the large variation in potential model results [27-29], we select the bag model calculations.

In the bag model, the hadron radius is obtained by variational minimization of the hadron mass expression $M(R)$ with respect to the radius R_H. This radius encompasses quark interactions and can be approximated as the interaction boundary for light hadrons described by Eqs. (3-4). Table 1 lists MIT bag model calculations for light hadron masses and radii [18,30], showing relatively small variations in hadron radii.

Table 1. Masses M_{bag} (in GeV) and limiting radii R_H (in GeV^{-1}) for selected hadron states calculated in the MIT bag model from Refs. [18] and [30], with M_{exp} (in GeV) representing experimental values.

State	R_H [30]	M_{bag} [30]	R_H [18]	M_{bag} [18]	M_{exp}
...	...	...	...	...	...

Reference [24] provides relevant values for heavy quarkonium polarizability $\alpha_{\psi\psi}$ through studies of hadron-J/ψ scattering lengths. In the large-N_c limit, the color-electric polarizability exhibits quantum number dependence: $\alpha_{\psi\psi}(1S)/\alpha_{\psi\psi}(2S) = 7/50$, consistent with values from Ref. [32]. Table 2 presents the color-electric polarizability values used in our calculations, based on Ref. [24].

Table 2. Color-electric polarizabilities (in GeV^{-3}) for charmonium $\alpha_{\psi\psi}$ and bottomonium $\alpha_{\Upsilon\Upsilon}$ from Ref. [24].

Polarizabilities	Value	Polarizabilities	Value
$\alpha_{\psi\psi}(1S)$	4.1 GeV^{-3}	$\alpha_{\Upsilon\Upsilon}(1P)$	14 GeV^{-3}
$\alpha_{\psi\psi}(1P)$	11 GeV^{-3}	$\alpha_{\Upsilon\Upsilon}(2S)$	23 GeV^{-3}
$\alpha_{\psi\psi}(2S)$	18 GeV^{-3}	$\alpha_{\Upsilon\Upsilon}(2S)$	33 GeV^{-3}

For the hadron radius parameters, we adopt the limiting radii of mesons from the bag model in Table 1 [7], setting $R_{\text{bag}} = R_H$. Specifically: $R_\phi = 4.7$ $\text{GeV}^{-1} \simeq 0.927$ fm, $R_K = 4.34$ $\text{GeV}^{-1} \simeq 0.856$ fm, and $R_{K^*} = 4.63$ $\text{GeV}^{-1} \simeq 0.913$ fm. Solving the eigenvalue equation from Eq. (5) yields the bound state masses [33].

Table 3 presents the calculated mass spectra for $c\bar{c}$ bound states with ϕ, K, and K^* mesons in the H-Q model. The fourth column shows masses without considering light hadron radius expansion, while the fifth column gives the corresponding binding energies $\Delta E = M_H + M_Q - M_{HQ}$, where M_H is the meson mass and M_Q is the charmonium mass. Results indicate that for the $c\bar{c}$ angular momentum excitation 1P, with color-electric polarizability $\alpha_{\psi\psi}(1P)_{c\bar{c}} = 11$ GeV^{-3}, the binding energy ΔE ranges from 18 MeV to 83 MeV. For the radial excitation 2S with $\alpha_{\psi\psi}(2S)_{c\bar{c}} = 18$ GeV^{-3}, the binding energy ΔE is approximately 200 MeV.

The $c\bar{c}\otimes\phi$ masses range from 4351 MeV to 4501 MeV. Due to model limitations, we cannot incorporate spin interactions between $c\bar{c}$ and light hadrons, but the calculated results lie above both the $m(\phi(1020))+m(J/\psi) = 4117$ MeV and two-meson $2m(D_s) = 3936$ MeV thresholds. Experimentally observed strange-quark-containing tetraquark states $Z_{cs}(4000)$ and $Z_{cs}(4220)$ have $J^P = 1^+$ quantum numbers [10] and likely correspond to the calculated values $\psi(2S) \otimes K(4165)$ and $\eta_c(2S) \otimes K^*(4069)$, which are close to experimental results. However, experimental data on decays from $\psi(2S)$ or $\eta_c(2S)$ excited states to ground states are currently lacking.

Tables 4 and 5 present results for $b\bar{b}$ systems. Since Ref. [24] provides different values for bottomonium color-electric polarizability, we perform calculations using both minimum and maximum values $\alpha_{\rm min}$ and $\alpha_{\rm max}$. The results show that $b\bar{b} \otimes H$ systems exhibit larger binding energies than $c\bar{c} \otimes H$ systems.

In summary, while H-Q model calculations approximate the experimental masses of $Z_{cs}(4000)$ and $Z_{cs}(4220)$, the model predicts bound states composed of excited $c\bar{c}$ and mesons. The absence of observed decay processes from $c\bar{c}$ radial excitations to ground states in experiments suggests that detecting such decay channels could serve as a criterion for bound state existence. Furthermore, $s\bar{s} \otimes Q\bar{Q}$ systems show larger binding energies, potentially forming compact tetraquark states. In our H-Q model calculations using MIT bag radii, quark flavor affects hadron radii—heavier quarks yield smaller radii [19]. Consequently, the potential well width decreases with increasing strange quark number, while the well depth, proportional to light hadron mass, increases. Both effects contribute to enhanced binding energies.

2.2 Tetraquark Radius and Mass Spectra in the CMI Bag Model

The MIT bag model provides the mass expression:

$$M(R) = \sum_{i=n,s,c,b} \frac{\omega_i}{R} + \frac{4\pi}{3} R^3 B - \frac{Z_0}{R} + \langle \Delta H \rangle$$

The first term represents the sum of relativistic kinetic energies, where ω_i is the valence quark kinetic energy. The second term is the bag volume energy with parameter B representing the vacuum energy density difference between the bag interior and exterior (similar to the H-Q model' s description of light hadron masses), and R is the spherical bag radius. The third term is the zero-point energy in vacuum, with Z_0 as the zero-point energy parameter from Ref. [18]. $\langle \Delta H \rangle$ denotes the interaction energy between quarks.

The momentum squared term x_i^2 in Eq. (8) is obtained from the transcendental equation satisfied by quark wavefunctions at the bag boundary:

$$\tan x_i = \frac{1}{1 + m_i R - \sqrt{m_i^2 R^2 + x_i^2}}$$

In the MIT bag model, the interaction energy ΔH between valence quarks takes two forms: chromomagnetic interactions between quarks and short-range interactions H_{BE} between heavy quarks [18,34]:

$$H_{CMI} = -\sum_{i<j}(\lambda_i \cdot \lambda_j)(\sigma_i \cdot \sigma_j)C_{ij}$$

$$\Delta H = H_{CMI} + H_{BE}$$

$$H_{BE} = \sum_{i<j}(\lambda_i \cdot \lambda_j)\frac{\alpha_s(R)}{4m_i m_j}\left(\frac{8\pi}{3}\delta^3(r_{ij}) + \frac{1}{r_{ij}^3}S_{ij}\right)$$

In Eqs. (8)-(10), indices i and j label quarks within the hadron, λ represents Gell-Mann matrices, and σ denotes Pauli matrices. These operators encode color and spin factors:

$$\langle\lambda_i \cdot \lambda_j\rangle_{nm} = \mathrm{Tr}(c_{in}^\dagger \lambda_\alpha c_{im})\mathrm{Tr}(c_{jn}^\dagger \lambda_\alpha c_{jm})$$

$$\langle\sigma_i \cdot \sigma_j\rangle_{xy} = \mathrm{Tr}(\chi_{ix}^\dagger \sigma_\alpha \chi_{ix})\mathrm{Tr}(\chi_{jx}^\dagger \sigma_\alpha \chi_{jx})$$

where subscripts n, m, x, y denote basis vectors for color and spin wavefunctions, while c and χ represent quark color and spin basis states. These calculations depend on color-spin wavefunctions provided in Ref. [18]. The chromomagnetic interaction parameter C_{ij} in the MIT bag model is expressed as:

$$C_{ij} = \frac{3\alpha_s(R)}{4R^3}\bar{\mu}_i\bar{\mu}_j I_{ij}$$

where $\alpha_s(R)$ is the running coupling constant from Ref. [18], and $\bar{\mu}_i$ is the magnetic moment of the i-th quark:

$$\alpha_s(R) = \frac{4\pi}{\ln[1 + (0.281R)^{-1}]}, \quad \bar{\mu}_i = \frac{2\omega_i R + 2\lambda_i - 3}{2\omega_i R(\omega_i R - 1) + \lambda_i}$$

The complete mass expression is obtained by combining these terms. We determine stable hadron masses and radii by variational minimization of the hadron mass with respect to radius R.

The bag model input parameters follow Refs. [18-19]: $Z_0 = 1.83$, $m_n = 0$ GeV, $B^{1/4} = 0.145$ GeV, $m_s = 0.279$ GeV, $m_c = 1.641$ GeV, $m_b = 5.093$ GeV. The short-range binding energies $B_{c\bar{s}}$, $B_{b\bar{s}}$, $B_{b\bar{c}}$, and $B_{b\bar{b}}$ are obtained by fitting vector mesons.

The function I_{ij} in Eq. (18) is a rational function of parameters x_i and x_j:

$$I_{ij} = 1 + F(x_i, x_j)$$

where $F(x_i, x_j)$ involves solutions x_i to the transcendental equation (9), with:

$$F(x_i, x_j) = \frac{[x_i \sin^2 x_i - y_i y_j - 2x_i x_j \sin(2x_i)\sin(2x_j) + x_i x_j[2x_i \mathrm{Si}(2x_i)(x_j \sin^2 x_j - \cdots) + 2x_j \mathrm{Si}(2x_j) - (x_i + x_j}{\cdots}$$

Here $y_i = x_i - \cos(x_i)\sin(x_i)$ and $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt$.

Chromomagnetic interactions between quarks are constructed using Eqs. (10) and (11) with the hadron' s color-spin wavefunctions, yielding contributions to ΔH after diagonalization to account for color-magnetic mixing.

Table 6 presents bag model calculations for ground-state double-heavy-flavor meson masses and limiting radii from Ref. [19].

Hidden-flavor tetraquark states $q_1 q_2 \bar{q}_3 \bar{q}_4$ possess two color configurations in color space: $6 \otimes \bar{6}$ and $3 \otimes \bar{3}$, with wavefunctions:

$$\phi_1^T = |(q_1 q_2)_6 (\bar{q}_3 \bar{q}_4)_{\bar{6}}\rangle, \quad \phi_2^T = |(q_1 q_2)_{\bar{3}} (\bar{q}_3 \bar{q}_4)_3\rangle$$

Based on $SU(3)$ color symmetry, specific forms of these wavefunctions are given in Appendix A. In spin space, six tetraquark spin states exist (subscripts denote spin quantum numbers):

$$\chi_1^T = |(q_1 q_2)_1 (\bar{q}_3 \bar{q}_4)_1\rangle_2, \quad \chi_2^T = |(q_1 q_2)_1 (\bar{q}_3 \bar{q}_4)_1\rangle_1, \quad \chi_3^T = |(q_1 q_2)_1 (\bar{q}_3 \bar{q}_4)_1\rangle_0$$

$$\chi_4^T = |(q_1 q_2)_1 (\bar{q}_3 \bar{q}_4)_0\rangle_1, \quad \chi_5^T = |(q_1 q_2)_0 (\bar{q}_3 \bar{q}_4)_1\rangle_1, \quad \chi_6^T = |(q_1 q_2)_0 (\bar{q}_3 \bar{q}_4)_0\rangle_0$$

Using color-spin basis vectors $\phi_i \chi_i$ (given in Appendix A) leads to color-magnetic mixing. Diagonalization yields the energy contributions from tetraquark chromomagnetic interactions to ΔH.

Table 7 lists experimentally discovered tetraquark states $q\bar{s}c\bar{c}$ and $s\bar{s}c\bar{c}$.

As shown in Table 8 and Figure 2 [Figure 2: see original paper], CMI bag model calculations for $q\bar{s}c\bar{c}$ or $\bar{q}sc\bar{c}$ systems yield masses above the $J/\psi K^+(3593)$ and $\bar{D}^0(3832)$ thresholds. The average tetraquark radius is 5.22 GeV^{-1} (1.03 fm), with $J^P = 1^+$ states calculated between 3986 MeV and 4325 MeV. Notably, the calculated values 3986 MeV and 4208 MeV differ from the experimental $Z_{cs}(4000)$ and $Z_{cs}(4220)$ masses by less than 20 MeV, supporting these states as compact tetraquark candidates.

As shown in Table 11 and Figure 2, calculations for the $c\bar{c}s\bar{s}$ system with $J^{PC} = 0^{++}$ yield masses 180 MeV or more above the $2D_s(3936)$ meson threshold. Most

0^{++} and other quantum number states appear above the $J/\psi\phi(4117)$ threshold, except the minimum 0^{++} value of 4099 MeV, which lies slightly below with a binding energy of only ~20 MeV. The 1^{+-} and 1^{++} states distribute around the $D_s D_{s1}^*(4748)$ threshold, with sub-threshold results showing binding energies near 100 MeV, potentially approaching molecular state limits. All 2^{++} states lie below the $2D_{s1}^*(5560)$ threshold with substantial binding.

The experimental state $X(4630)$ would require a binding energy exceeding -118 MeV for a $\bar{D}_s D_{s1}^*(4748)$ molecular interpretation, with even larger binding energies needed for $X(4500)$ and $X(4350)$. This supports $X(4350)$, $X(4500)$, and $X(4630)$ as likely compact tetraquark states.

Comparing radii from Tables 8 and 11 reveals that $c\bar{c}s\bar{s}$ systems have smaller limiting radii than $q\bar{s}c\bar{c}$ due to kinetic suppression. This leads to the conclusion that if $Z_{cs}(4000)$ and $Z_{cs}(4220)$ are compact tetraquarks, then $c\bar{c}s\bar{s}$, $q\bar{s}b\bar{b}$, and $s\bar{s}b\bar{b}$ systems have even greater probabilities of forming compact tetraquark structures beyond molecular and other bound states.

Tables 10 , 13 , and Figure 2 present CMI bag model calculations for $q\bar{s}b\bar{b}$, $\bar{q}sb\bar{b}$, and $s\bar{s}b\bar{b}$ systems. The $s\bar{s}b\bar{b}$ results lie above the B_s^* threshold with smaller radii than $s\bar{s}c\bar{c}$, suggesting that $q\bar{s}b\bar{b}$, $\bar{q}sb\bar{b}$, and $s\bar{s}b\bar{b}$ systems may harbor additional compact tetraquark states.

Tables 9 and 12 present decay branching ratio calculations for $q\bar{s}c\bar{c}$ and $s\bar{s}c\bar{c}$ systems after transforming hadron eigenstates to $8\otimes\bar{8}$ and $\bar{1}\otimes 1$ representations. Before studying tetraquark decays, scattering states must be removed. This requires basis transformations coupling color-singlet mesons to color-singlets and color-octet mesons to color-octets. Only color-singlet coupled structures decay by dissociating into mesons with spins s_1 and s_2, while octet states do not. States with color-singlet eigenvector coefficients $|c_1| \geq 0.8$ are identified as scattering states and excluded. Octet components in tetraquark states can also decay when quark exchange converts them to color-singlets, requiring further consideration.

After removing scattering states, we study two-body decays $A \to B + C$. The partial decay width formula for each decay channel is [36]:

$$\Gamma_i = \gamma_i \alpha \frac{k^{2L+1}}{m_a^2} \cdot |c_i|$$

where Γ_i is the partial width, γ_i the decay coefficient, α the coupling constant, m_a the parent particle mass, and c_i the eigenvector coefficient for color-singlet coupling. For ground states, we set orbital angular momentum $L = 0$. The decay product momentum k is determined in the zero-momentum frame where both final-state particles share equal momentum:

$$m_a = \sqrt{m_b^2 + k^2} + \sqrt{m_c^2 + k^2}$$

with m_b and m_c being the final-state meson masses. Calculating each channel' s width and branching ratio relative to the dominant decay mode reveals the primary decay mechanisms.

2.3 H-Q Bound States Based on Multiquark Hadron Radii

Embedding non-pointlike $Q\bar{Q}$ mesons within hadrons may alter the light hadron' s limiting radius, affecting H-Q model binding energies. In Eq. (4), the binding potential scales as $1/R^3$, making it strongly dependent on radius, while the potential well width relates to the light hadron radius R_H. Table 6 shows that heavy quarkonium radii are non-negligible compared to light hadron radii R_H, particularly for charmonium where the bag model yields J/ψ and η_c radii of 3.5 GeV^{-1} and 3.03 GeV^{-1}, respectively, compared to $R_H \approx 4.5$ GeV^{-1} from Table 1. Therefore, the expansion of the binding boundary R_H upon $Q\bar{Q}$ embedding cannot be ignored.

We correct the H-Q model results using tetraquark radii from the CMI bag model (Tables 8 and 10). Assuming the hadronic potential boundary $R_H \approx R_T$ (where R_T is the tetraquark radius from the bag model, which shows small J^P dependence as seen in Tables 8, 10, 11, and 13), we use average values $R_H^{c\bar{c}} \approx 5.22$ GeV^{-1} and $R_H^{b\bar{b}} \approx 4.85$ GeV^{-1} in Eq. (5). The resulting binding energies and masses are listed in the sixth and seventh columns of Tables 3, 4, and 5.

These results demonstrate that embedding non-pointlike $Q\bar{Q}$ into light mesons reduces the H-Q bound state binding energy.

3. Conclusions

In the first part of this work, we calculated binding energies and mass spectra for $q\bar{s} \otimes Q\bar{Q}$ and $s\bar{s} \otimes Q\bar{Q}$ systems within the H-Q model. The results show that bound states form only when $Q\bar{Q}$ resides in excited states. Additionally, $s\bar{s}\otimes Q\bar{Q}$ systems exhibit larger binding energies than $q\bar{s}\otimes Q\bar{Q}$ systems, with some hidden-bottom states potentially forming deeply bound systems. Experimental identification of H-Q bound state candidates could be achieved by searching for decay products containing excited $Q\bar{Q}$ states.

In the second part, we calculated radius expectation values and mass spectra for $q\bar{s}Q\bar{Q}$ and $s\bar{s}Q\bar{Q}$ systems using the CMI bag model. For strange-quark-containing tetraquarks, which may involve more complex color interactions at the quark level, the CMI bag model interprets these as color-magnetically mixed states. The model' s predictions for $q\bar{s}c\bar{c}$ support $Z_{cs}(4000)$ and $Z_{cs}(4220)$ as compact tetraquark candidates, with calculated masses differing from experimental values by only 20 MeV and radii around 5.22 GeV^{-1} (1.03 fm). Calculations for $s\bar{s}c\bar{c}$ suggest that some experimentally observed double-strange X states like $X(4350)$, $X(4500)$, and $X(4630)$ may have compact structures, with mass differences within 40 MeV and radii near 5.16 GeV^{-1} (1.02 fm). Predictions for $q\bar{s}b\bar{b}$ and $s\bar{s}b\bar{b}$ systems yield even smaller radii around 4.85 GeV^{-1} (0.96

fm). Tables 8, 10, 11, and 13 include mass calculations from Ref. [35] using an improved CMI framework with light-quark color-electric interactions (ICMI), which produce slightly lower masses than our CMI bag model.

Comparing H-Q and CMI models reveals the binding energy hierarchy: $B_{q\bar{s}\otimes c\bar{c}} < B_{s\bar{s}\otimes c\bar{c}} < B_{q\bar{s}\otimes b\bar{b}} < B_{s\bar{s}\otimes b\bar{b}}$. The CMI bag model radius calculations show $R_{q\bar{s}c\bar{c}} > R_{s\bar{s}c\bar{c}} > R_{q\bar{s}b\bar{b}} > R_{s\bar{s}b\bar{b}}$, likely due to flavor-dependent kinetic suppression driving systems toward more compact configurations. We therefore conclude that if $Z_{cs}(4000)$ and $Z_{cs}(4220)$ are compact tetraquarks, then $c\bar{c}s\bar{s}$, $q\bar{s}b\bar{b}$, and $s\bar{s}b\bar{b}$ systems have even greater probabilities of forming compact structures beyond molecular and other bound states.

Acknowledgments: We thank Prof. Jia Duojie and Dr. Gao Di from Northwest Normal University for discussions on this work.

Appendix A1

For hidden-flavor tetraquark states, $SU(3)$ color symmetry yields two color-singlet representations: $6\otimes\bar{6}$ and $\bar{3}\otimes 3$. The color wavefunctions can be expressed as:

$$\phi_1 = \frac{1}{\sqrt{3}}(r\bar{r}r\bar{r} + g\bar{g}g\bar{g} + b\bar{b}b\bar{b})$$

$$\phi_2 = \frac{1}{\sqrt{6}}(r\bar{b}b\bar{r}+b\bar{r}b\bar{r}+g\bar{r}g\bar{r}+r\bar{g}g\bar{r}+g\bar{b}b\bar{g}+b\bar{g}b\bar{g}+g\bar{r}r\bar{g}+r\bar{g}r\bar{g}+g\bar{b}g\bar{b}+b\bar{g}g\bar{b}+r\bar{b}r\bar{b}+b\bar{r}r\bar{b})$$

Considering flavor symmetry, if $q_{1_q}2$ satisfies flavor symmetry with isospin 1, we denote it with $\delta_S^{12} = 1$; if isospin is 0, we use $\delta_S^{12} = \delta_A^{12} = 1$. For quark pairs without flavor symmetry but with isospin symmetry, we use $\delta_A^{12} = 1$.

The six tetraquark spin basis vectors are:

$$\chi_1 = |\uparrow\uparrow\uparrow\uparrow\rangle, \quad \chi_2 = \frac{1}{2}(|\uparrow\uparrow\uparrow\downarrow\rangle + |\uparrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\uparrow\rangle - |\downarrow\uparrow\uparrow\uparrow\rangle)$$

$$\chi_3 = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle), \quad \chi_4 = \frac{1}{2}(|\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle)$$

$$\chi_5 = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\downarrow\rangle - |\uparrow\uparrow\downarrow\uparrow\rangle), \quad \chi_6 = \frac{1}{2}(|\uparrow\downarrow\uparrow\uparrow\rangle - |\downarrow\uparrow\uparrow\uparrow\rangle)$$

Accounting for the Pauli principle, the color-spin basis vectors for describing tetraquark states are constructed from these components.

References

[1] CHEN H X, CHEN W, LIU X, et al. Reports on Progress in Physics, 2022, 86(2): 026201.
[2] LIU X. Chinese Science Bulletin, 2014, 59(29): 3815-3830. DOI: 10.1007/s11434-014-0407-2.
[3] HOSAKA A, IIJIMA T, MIYABAYASHI K, et al. Progress of Theoretical and Experimental Physics, 2016, 2016(6): 062C01.
[4] OSET E, CHEN H X, FEIJOO A, et al. Nuclear Physics A, 2016, 954: 199.
[5] DU M L, ALBALADEJO M, GUO F K, et al. Phys Rev D, 2022, 105: 094013.
[6] ALBUQUERQUE R M, DIAS J M, KHEMCHANDANI K P, et al. Journal of Physics G: Nuclear and Particle Physics, 2019, 46(9): 093002.
[7] VOLOSHIN M B, OKUN L B. JETP Lett, 1976, 23: 333.
[8] DING Z M, JIANG H Y, HE J. Eur Phys J C, 2020, 80(12): 1179. DOI: 10.1140/epjc/s10052-020-08754-6.
[9] LI N, SUN Z F, LIU X, et al. Chin Phys Lett, 2021, 38(9): 092001. DOI: 10.1088/0256-307X/38/9/092001.
[10] AAIJ R, et al. Phys Rev Lett, 2021, 127(8): 082001. DOI: 10.1103/PhysRevLett.127.082001.
[11] MENG L, WANG B, WANG G J, et al. Science Bulletin, 2021, 66(20): 2065. DOI: https://doi.org/10.1016/j.scib.2021.06.026.
[12] CHENG J B, HUANG B L, LIN Z Y, et al. The European Physical Journal C, 2023, 83(11): 1071. DOI: 10.1140/epjc/s10052-023-1219.
[13] PANTELEEVA J Y, PEREVALOVA I A, POLYAKOV M V, et al. Phys Rev C, 2019, 99(4): 045206. DOI: 10.1103/PhysRevC.99.0452.
[14] WANG Z G. Chin Phys C, 2022, 46(10): 103106. DOI: 10.1088/1674-1137/ac7cd7.
[15] WANG Y H, WEI J, AN C S, et al. Chin Phys Lett, 2023, 40(2): 021201. DOI: 10.1088/0256-307X/40/2/021201.
[16] SHI P P, HUANG F, WANG W L. Phys Rev D, 2021, 103(9): 094038. DOI: 10.1103/PhysRevD.103.094038.
[17] CHEN H X, CHEN W, LIU X, et al. Rept Prog Phys, 2023, 86(2): 026201. DOI: 10.1088/1361-6633/aca3b6.
[18] ZHANG W X, XU H, JIA D. Phys Rev D, 2021, 104(11): 114011. DOI: 10.1103/PhysRevD.104.114011.
[19] LIU W N, MA S W, ZHAO X J, et al. Nuclear Physics Review, 2022, 39(3): 296. DOI: 10.11804/NuclPhysRev.39.2022049.
[20] KAIDALOV A B, VOLKOVITSKY P E. Phys Rev Lett, 1992, 69: 3155. DOI: 10.1103/PhysRevLett.69.3155.
[21] DUBYNSKIY S, VOLOSHIN M B. Phys Lett B, 2008, 666: 344. DOI: 10.1016/j.physletb.2008.07.086.
[22] GOTTFRIED K. Phys Rev Lett, 1978, 40: 598. DOI: 10.1103/PhysRevLett.40.598.

[23] ANWAR M N, LU Y, ZOU B S. Phys Rev D, 2017, 95(11): 114031. DOI: 10.1103/PhysRevD.95.114031.
[24] FERRETTI J, SANTOPINTO E, NAEEM ANWAR M, et al. Phys Lett B, 2019, 789: 562. DOI: 10.1016/j.physletb.2018.09.047.
[25] ANWAR M N, FERRETTI J, GUO F K, et al. Eur Phys J C, 2018, 78(8): 647. DOI: 10.1140/epjc/s10052-018-6073-9.
[26] FERRETTI J, SANTOPINTO E. JHEP, 2020, 04: 119. DOI: 10.1007/JHEP04(2020)119.
[27] KUBIS B, HEMMERT T R, MEISSNER U G. Phys Lett B, 1999, 456: 240. DOI: 10.1016/S0370-2693(99)00465-7.
[28] BUCHMANN A J, LEBED R F. Phys Rev D, 2003, 67: 016002. DOI: 10.1103/PhysRevD.67.016002.
[29] LEDWIG T, KIM H C, SILVA A J, et al. Phys Rev D, 2006, 74: 054005. DOI: 10.1103/PhysRevD.74.054005.
[30] JOHNSON K. Acta Phys Polon B, 1975, 6: 865.
[31] WORKMAN R L, et al. PTEP, 2022, 2022: 083C01. DOI: 10.1093/ptep/ptac097.
[32] EIDES M I, PETROV V Y, POLYAKOV M V. Phys Rev D, 2016, 93(5): 054039. DOI: 10.1103/PhysRevD.93.054039.
[33] FEYNMAN R P, LEIGHTON R B, SANDS M. The Feynman Lectures on Physics[M]. Addison-Wesley Pub. Co., 1963.
[34] KARLINER M, ROSNER J L. Phys Rev D, 2014, 90(9): 094007. DOI: 10.1103/PhysRevD.90.094007.
[35] GUO T, LI J N, ZHAO J X, et al. Chin Phys C, 2023, 47(06): 073084. DOI: 10.1088/1674-1137/accb87.
[36] LOEBL E M. Group Theory and its Applications[M]. Academic Press, 1971.

Note: Figure translations are in progress. See original paper for figures.