The smallest degree sum that yields potentially C_k -graphical sequences postprint

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Abstract

Summary: "In this paper we consider a variation of the classical Turán-type extremal problems. Let S be an n-term graphical sequence, and $\sigma(S)$ be the sum of the terms in S. Let H be a graph. The problem is to determine the smallest even l such that any n-term graphical sequence S having $\sigma(S) \geq l$ has a realization containing H as a subgraph. Denote this value l by $\sigma(H, n)$. We show $\sigma(C_{2m+1}, n) = m(2n - m - 1) + 2$, for $m \geq 3$, $n \geq 3m$; $\sigma(C_{2m+2}, n) = m(2n - m - 1) + 4$, for $m \geq 3$, $n \geq 5m - 2$.' "'

Full Text

Preamble

The Smallest Degree Sum that Yields Potentially Ck-graphical Sequences

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Abstract

In this paper we consider a variation of the classical Turán-type extremal problems. Let S be an n-term graphical sequence, and $\sigma(S)$ be the sum of the terms in S. Let H be a graph. The problem is to determine the smallest even l such that any n-term graphical sequence S having $\sigma(S) \geq l$ has a realization containing H as a subgraph. Denote this value l by $\sigma(H, n)$. We show $\sigma(C_{2m+1}, n) =$

$m(2n-m-1)+2$, for $m \geq 3, n \geq 3m$; $\sigma(C_{2m+2}, n) = m(2n-m-1)+4$, for $m \geq 3, n \geq 5m-2$.

Key words: graph; degree sequence; potentially H-graphic sequence
AMS Subject Classifications: 05C07, 05C35

Introduction

If $S = (d_1, d_2, ..., d_n)$ is a sequence of non-negative integers, then it is called *graphical* if there is a simple graph G of order n, whose degree sequence $(d(v_1), d(v_2), ..., d(v_n))$ is precisely S. If G is such a graph then G is said to *realize* S or be a *realization* of S. A graphical sequence S is *potentially H-graphical* if there is a realization of S containing H as a subgraph, while S is *forcibly H-graphical* if every realization of S contains H as a subgraph.

Let $\sigma(S) = d(v_1) + d(v_2) + ... + d(v_n)$, and let $[x]$ denote the largest integer less than or equal to x. If G and G_1 are graphs, then $G \cup G_1$ is the disjoint union of G and G_1. If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. Let K_k and C_k denote a complete graph on k vertices and a cycle on k vertices, respectively.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $\mathrm{ex}(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdős [2] in 1938 and in general by Turán [11]. In terms of graphic sequences, the number $2\,\mathrm{ex}(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence S with $\sigma(S) \geq l$ is forcibly H-graphical. Here we consider the following variant: determine the minimum even integer l such that every n-term graphical sequence S with $\sigma(S) \geq l$ is potentially H-graphical. We denote this minimum l by $\sigma(H, n)$.

Erdős, Jacobson and Lehel [3] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that equality holds. They proved that if S does not contain zero terms, this conjecture is true for $k = 3, n \geq 6$. The conjecture is confirmed in [4], [7], [8], [9] and [10].

Gould, Jacobson and Lehel [4] also proved that $\sigma(pK_2, n) = (p-1)(2n-2)+2$ for $p \geq 2$; and $\sigma(C_4, n) = 2\left[\frac{3n-1}{2}\right]$ for $n \geq 4$. Lai [5, 6] proved that $\sigma(C_5, n) = 4n-4$ for $n \geq 5$, and $\sigma(C_6, n) = 4n-2$ for $n \geq 7$, along with the general lower bounds $\sigma(C_{2m+1}, n) \geq m(2n-m-1)+2$ for $n \geq 2m+1, m \geq 2$, and $\sigma(C_{2m+2}, n) \geq m(2n-m-1)+4$ for $n \geq 2m+2, m \geq 2$, and $\sigma(K_4 - e, n) = 2\left[\frac{3n-1}{2}\right]$ for $n \geq 7$. In this paper we prove that $\sigma(C_{2m+1}, n) = m(2n-m-1)+2$ for $n \geq 3m, m \geq 3$; and $\sigma(C_{2m+2}, n) = m(2n-m-1)+4$ for $n \geq 5m-2, m \geq 3$.

2 Main Results

Theorem 1

Let $k \geq 4$. Let S be a potentially C_k-graphical n-term sequence. If there exists $x \notin C_k$, $w \in C_k$ such that $d(x) \geq \left[\frac{k}{2}\right]+1$, $d(w) \geq 3$, then S has a realization containing a C_{k+1}.

Assume C_k is $w_1 w_2 \cdots w_{kw} 1$. Let $w_{k+i} = w_i$. We first establish three lemmas.

Lemma (a). For any $x \notin C_k$, if there exist w_r, w_{r+1} such that $w_{rx}, w_{r+1}x \in E(G)$, then G contains a C_{k+1}: $w_1 w_2 \cdots w_{rxw_{r+1}} \cdots w_{kw} 1$.

Lemma (b). For any $x, y \notin C_k$, $xy \in E(G)$, if there exists w_r such that $w_{rx} \in E(G)$, $w_{ry} \notin E(G)$, then S has a realization containing a C_{k+1}. (We note that the edge $w_{r+1}x$ is not in G, otherwise a C_{k+1} would already exist. However, the edge interchange that removes edges $w_{rw_{r+1}}$ and xy and inserts edges $w_{r+1}x$ and w_{ry} produces a realization containing a C_{k+1}: $w_1 w_2 \cdots w_{rxw_{r+1}} \cdots w_{kw} 1$.)

Lemma (c). For any $x, y \notin C_k$, $xy \in E(G)$, if there exist w_r, w_{r+2} such that $w_{rx}, w_{r+2}x \in E(G)$, then S has a realization containing a C_{k+1}. (If $w_{r+2}y \notin E(G)$, then by Lemma (b), S has a realization containing a C_{k+1}. Otherwise, $w_{r+2}y \in E(G)$ and so G contains a C_{k+1}: $w_1 w_2 \cdots w_{rxyw_{r+2}} w_{r+3} \cdots w_{kw} 1$.)

Proof of Theorem 1. Assume every realization of S does not contain a C_{k+1}. Since $d(x) \geq \left[\frac{k}{2}\right] + 1$, there exists $x_1 \notin C_k$ such that $xx_1 \in E(G)$. Thus, by Lemma (a), x is adjacent to at most $\left[\frac{k}{2}\right] - 1$ vertices of C_k. Since $d(x) \geq \left[\frac{k}{2}\right] + 1$, Lemma (c) implies that x is adjacent to at most $\left[\frac{k}{2}\right] - 2$ vertices of C_k when $k \geq 4$. Hence there exists $x_2 \notin C_k$, $x_2 \neq x_1$, such that $xx_2 \in E(G)$.

Case 1. Suppose there exists $w_i \in C_k$ such that $w_{ix} \in E(G)$. By Lemma (b), $w_{ix}1, w_{ix}2 \in E(G)$. By Lemma (a), $w_{i+1}x, w_{i+1}x_1, w_{i+1}x_2 \notin E(G)$. By Lemma (c), $w_{i+2}x, w_{i+2}x_1, w_{i+2}x_2 \notin E(G)$. Then the edge interchange that removes edges $w_{i+1}w_{i+2}$ and xx_2 and inserts edges $w_{i+2}x$ and $w_{i+1}x_2$ produces a realization containing a C_{k+1}: $w_1 w_2 \cdots w_{ix} 1 x w_{i+2} w_{i+3} \cdots w_{kw} 1$. This is a contradiction.

Case 2. Suppose for any $w_i \in C_k$, $w_{ix} \notin E(G)$. Since $d(x) \geq \left[\frac{k}{2}\right] + 1 \geq 2 + 1 = 3$, there exists $x_3 \notin C_k$, $x_3 \neq x_1, x_3 \neq x_2$ such that $xx_3 \in E(G)$. By Lemma (b), $w_{ix}1, w_{ix}2, w_{ix}3 \notin E(G)$ for all $w_i \in C_k$. Since there exists $w \in C_k$ such that $d(w) \geq 3$, there is x_4 such that $wx_4 \notin E(C_k)$ but $wx_4 \in E(G)$. By Lemma (b), x_4 is not one of x_1, x_2, x_3. If $x_3x_4 \in E(G)$, then by Lemma (b), $wx_3 \in E(G)$ and thus, by Lemma (b) as well, $wx \in E(G)$. This is a contradiction. Thus $x_3x_4 \notin E(G)$. Then the edge interchange that removes edges wx_4 and xx_3 and inserts edges wx and x_3x_4 produces a realization containing the edge wx. By Case 1, S has a realization containing a C_{k+1}. This is a contradiction.

Theorem 2

Let $m \geq 3$. Let S be an n-term graphical sequence. Suppose S satisfies the following two conditions: (i) there is a realization G of S containing a C_{2m+1}, such that for all $x, y \notin C_{2m+1}$, $d(x) = d(y) = m$ and $xy \notin E(G)$; (ii) there is no realization of S containing a C_{2m+2}. Then $\sigma(S) \leq m(2n - m - 1) + 2$.

Proof. Let C_{2m+1} be $w_1 w_2 \cdots w_{2m+1} w_1$, and let $w_{2m+1+i} = w_i$. Since every realization of S does not contain a C_{2m+2}, Lemma (a) implies that for any

$v \notin C_{2m+1}$, there are no w_r, w_{r+1} such that $w_{rv}, w_{r+1}v \in E(G)$. Since for any $x, y \notin C_{2m+1}$, $xy \notin E(G)$ and $d(x) = d(y) = m$, each such vertex is adjacent to exactly m vertices of C_{2m+1}. Assume without loss of generality that $w_1x, w_4x, w_6x, \cdots, w_{2m}x \in E(G)$.

Case 1. Suppose there exists $y \notin C_{2m+1}$, $y \neq x$ such that there is a $w_i \in C_{2m+1}$ with $w_{ix} \in E(G)$ but $w_{iy} \notin E(G)$.

Subcase 1. Suppose $w_2y \in E(G)$. By Lemma (a), $w_3y, w_1y \notin E(G)$ and at most one vertex of $\{w_4, w_5\}$ is adjacent to y. If $w_6y \in E(G)$, then G contains a C_{2m+2}: $w_6w_7 \cdots w_{2m+1}w_1xw_4w_3w_2yw_6$. This is a contradiction, thus $w_6y \notin E(G)$. Next, if $w_7y \in E(G)$, then by Lemma (a), $w_8y, w_6y \notin E(G)$. Since y is adjacent to m vertices of C_{2m+1}, Lemma (a) forces $w_9y, w_{11}y, \cdots, w_{2m+1}y \in E(G)$. Then G contains a C_{2m+2}: $w_{2m+1}yw_2w_1xw_4w_5 \cdots w_{2m}w_{2m+1}$. This is a contradiction, thus $w_7y \notin E(G)$. Finally, suppose $w_6y, w_7y \notin E(G)$. Then, by Lemma (a), y is adjacent to at most $m-1$ vertices of C_{2m+1}—a contradiction.

Subcase 2. Suppose $w_3y \in E(G)$. By a similar method as Subcase 1 we obtain a contradiction.

Subcase 3. Suppose $w_2y, w_3y \notin E(G)$. Lemma (a) forces y to be adjacent to the following m vertices of C_{2m+1}: $w_1, w_4, w_6, \cdots, w_{2m}$. This contradicts the supposition of Case 1.

Case 2. Suppose for any $y \notin C_{2m+1}$, $y \neq x$, and any $w_i \in C_{2m+1}$, if $w_{ix} \in E(G)$ then $w_{iy} \in E(G)$. Then $w_1y, w_4y, w_6y, \cdots, w_{2m}y \in E(G)$.

Subcase 1. Suppose $w_2w_5 \in E(G)$. Then G contains a C_{2m+2}: $w_5w_2w_3w_4xw_1w_{2m+1}w_{2m} \cdots w_5$. This is a contradiction.

Subcase 2. Suppose $w_{2m+1}w_2 \in E(G)$. Then G contains a C_{2m+2}: $w_2w_{2m+1}w_1xw_{2m}w_{2m-1} \cdots w_2$. This is a contradiction.

Subcase 3. Suppose there exists i $(3 \leq i \leq m-1)$ such that $w_2w_{2i+1} \in E(G)$. Then G contains a C_{2m+2}: $w_{2i+1}w_2w_3w_4 \cdots w_{2i-2}xw_1w_{2m+1}w_{2m} \cdots w_{2i+2}yw_{2i}w_{2i+1}$. This is a contradiction.

Subcase 4. Suppose $w_3w_5 \in E(G)$. Then G contains a C_{2m+2}: $w_3w_5w_4xw_6w_7 \cdots w_{2m+1}w_1w_2w_3$. This is a contradiction.

Subcase 5. Suppose $w_3w_{2m+1} \in E(G)$. Then G contains a C_{2m+2}: $w_{2m+1}w_3w_2w_1xw_4w_5 \cdots w_{2m}w_{2m+1}$. This is a contradiction.

Subcase 6. Suppose there exists i $(3 \leq i \leq m-1)$ such that $w_3w_{2i+1} \in E(G)$. Then G contains a C_{2m+2}: $w_{2i+1}w_3w_2w_1xw_4w_5 \cdots w_{2i}yw_{2m}w_{2m-1} \cdots w_{2i+1}$. This is a contradiction.

Subcase 7. Suppose there exists j $(2 \leq j \leq m-1)$ such that $w_{2j+1}w_{2m+1} \in E(G)$. Then G contains a C_{2m+2}: $w_{2j+1}w_{2m+1}w_{2m} \cdots w_{2j+2}xw_1w_2 \cdots w_{2j}w_{2j+1}$. This is a contradiction.

Subcase 8. Suppose there exist j and i $(2 \leq j < i \leq m-1)$ such that $w_{2j+1}w_{2i+1} \in E(G)$. Then G contains a C_{2m+2}: $w_{2j+1}w_{2i+1}w_{2i} \cdots w_{2j+2}xw_{2i+2}w_{2i+3} \cdots w_{2m+1}w_1w_2 \cdots w_{2j}w_{2j+1}$. This is a contradiction.

Subcase 9. Suppose for any i $(2 \leq i \leq m)$, $w_2w_{2i+1}, w_3w_{2i+1} \notin E(G)$, and for any i, j $(2 \leq j < i \leq m)$, $w_{2j+1}w_{2i+1} \notin E(G)$. Then $d(w_2), d(w_3) \leq m+1$ and $d(w_5), d(w_7), \cdots, d(w_{2m+1}) \leq m$. Since for any $y \notin C_{2m+1}$, $d(y) = m$, we have:

$$\sigma(S) \leq m(n-2m-1)+d(w_2)+d(w_3)+d(w_5)+d(w_7)+\cdots+d(w_{2m+1})+d(w_1)+d(w_4)+d(w_6)+\cdots+d(w_{2m})$$

$$\leq m(n-2m-1)+2(m+1)+m(m-1)+(n-1)m = (n-2m-1+2+m-1+n-1)m+2 = m(2n-m-1)+2.$$

Theorem 3

Let $m \geq 2$. If $k = 2m+1$ and $n \geq 3m$, then $\sigma(C_k, n) = m(2n-m-1)+2$; if $k = 2m+2$ and $n \geq 3m$, then $\sigma(C_k, n) \leq m(2n-m-1)+2m+2$.

Proof. By [5] Theorems 2 and 3, $\sigma(C_5, n) = 4n-4$ for $n \geq 5$, and $\sigma(C_6, n) = 4n-2$ for $n \geq 7$. Clearly $\sigma(C_6, 6) = 24$. Hence for $m = 2$, if $k = 2m+1$ and $n \geq 3m$, then $\sigma(C_k, n) \leq m(2n-m-1)+2$; if $k = 2m+2$ and $n \geq 3m$, then $\sigma(C_k, n) \leq m(2n-m-1)+2m+2$.

Suppose for some t with $2 \leq t < m$, if $k = 2t+1$ and $n \geq 3t$, then $\sigma(C_k, n) \leq t(2n-t-1)+2$, and if $k = 2t+2$ and $n \geq 3t$, then $\sigma(C_k, n) \leq t(2n-t-1)+2t+2$.

Case 1. Let S be an n-term graphical sequence with $k = 2m+1$, $n \geq 3m$, and $\sigma(S) \geq m(2n-m-1)+2$. For $n = 3m$, $\sigma(S) \geq m(6m-m-1)+2 = 5m^2-m+2 = 2\left[\binom{k-1}{2}+1\right]$, which by [1] (Chapter III, Theorem 5.9) implies that all realizations of S contain a C_k. Now assume that S_1 is a p-term graphical sequence with $3m \leq p < n$, $\sigma(S_1) \geq m(2p-m-1)+2$, and that there is a realization of S_1 containing a C_k. We will show that if $S = (d_1, d_2, ..., d_{p+1})$ is a $(p+1)$-term graphical sequence with realization G and $\sigma(S) \geq m(2(p+1)-m-1)+2$, then S has a realization containing a C_{2m+1}. Assume $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$. Let S' be the degree sequence of $G - v_{p+1}$ and suppose $d_{p+1} \leq m$. Then $\sigma(S') \geq m(2(p+1)-m-1)+2-2m = m(2p-m-1)+2$. Therefore, by our assumption, S' has a realization containing a C_k. Hence S has a realization containing a C_k. Thus, we may assume that $d_{p+1} \geq m+1$. Since $\sigma(S) \geq m(2(p+1)-m-1)+2 \geq (m-1)(2(p+1)-(m-1)-1)+2(m-1)+2$, by our assumption there is a realization of S containing a C_{2m}. Theorem 1 then implies that S has a realization containing a C_{2m+1}.

Case 2. If $k = 2m+2$, $n \geq 3m$, and S is an n-term graphical sequence with $\sigma(S) \geq m(2n-m-1)+2m+2$, then we can prove via a similar method as Case 1 that S has a realization containing a C_{2m+2}.

Hence $\sigma(C_{2m+1}, n) \leq m(2n-m-1)+2$ and $\sigma(C_{2m+2}, n) \leq m(2n-m-1)+2m+2$.

By [5], Theorem 1 (Theorem A below), for $m \geq 2$, $k = 2m+1$, and $n \geq 2m+1$, we have $\sigma(C_k, n) \geq m(2n-m-1)+2$. Therefore, for $m \geq 2$, if $k = 2m+1$ and $n \geq 3m$, then $\sigma(C_k, n) = m(2n-m-1)+2$.

Lemma 4

If $m \geq 3$ and $n = 3m + t$ $(t = 0, 1, 2, \cdots, 2m-2)$, then

$$\sigma(C_{2m+2}, n) \leq m(2n - m - 1) + 2m + 2 - 2\left[\frac{t}{2}\right].$$

Proof. By Theorem 3, Lemma 4 holds for $t = 0, 1$. Now assume that Lemma 4 holds for all $t - 1$ where $1 \leq t \leq 2m - 2$. We prove that it holds for t.

Let $S = (d_1, d_2, ..., d_n)$ be an n-term graphical sequence with $n = 3m + t$, and let G be a realization of S with $\sigma(S) \geq m(2n-m-1) + 2m + 2 - 2\left[\frac{t}{2}\right]$. Assume $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$.

Let S' be the degree sequence of $G - v_n$. If $d_n \leq m - 1$, then

$$\sigma(S') \geq m(2n-m-1)+2m+2-2\left[\frac{t}{2}\right]-2(m-1) \geq m(2(n-1)-m-1)+2m+2-2\left[\frac{t-1}{2}\right].$$

By the induction hypothesis, S' has a realization containing a C_{2m+2}. Hence S has a realization containing a C_{2m+2}. Thus, we may assume that $d_n \geq m$.

Since $t \leq 2m - 2$, we have $\sigma(S) \geq m(2n - m - 1) + 2m + 2 - 2\left[\frac{t}{2}\right] > m(2n - m - 1) + 2$. This implies, by Theorem 3, that S has a realization containing a C_{2m+1}.

Let $w \in C_{2m+1}$ and $x, y \notin C_{2m+1}$, and assume that every realization of S does not contain a C_{2m+2}. If $d(x) \geq m + 1$, then since $d(w) \geq d_n \geq 3$, Theorem 1 implies that S has a realization containing a C_{2m+2}—a contradiction. Hence for any $x \notin C_{2m+1}$, $d(x) = m$.

If for any $x, y \notin C_{2m+1}$, $xy \notin E(G)$, then by Theorem 2, $\sigma(S) \leq m(2n - m - 1) + 2 \leq \sigma(S)$. This is a contradiction. Thus, we may assume that there exist $x, y \notin C_{2m+1}$ such that $xy \in E(G)$. Let S' be the degree sequence of $G - \{x, y\}$. Since $d(x) = d(y) = m$, we have:

$$\sigma(S') \geq m(2n-m-1)+2m+2-2\left[\frac{t}{2}\right]-4m+2 = m(2(n-2)-m-1)+2m+2-2\left[\frac{t-2}{2}\right].$$

By the induction hypothesis, S' has a realization containing a C_{2m+2}. Hence S has a realization containing a C_{2m+2}—a contradiction.

Therefore, $\sigma(C_{2m+2}, n) \leq m(2n - m - 1) + 2m + 2 - 2\left[\frac{t}{2}\right]$.

Theorem 5

$\sigma(C_{2m+2}, n) = m(2n - m - 1) + 4$, for $m \geq 3$ and $n \geq 5m - 2$.

Proof. By Lemma 4, for $m \geq 3$ and $n = 5m - 2$, we have $\sigma(C_{2m+2}, n) \leq m(2n - m - 1) + 2m + 2 - 2\left[\frac{2m-2}{2}\right] = m(2n - m - 1) + 4$.

Suppose for some p with $5m - 2 \leq p < n$ that $\sigma(C_{2m+2}, p) \leq m(2p - m - 1) + 4$.

Let $S = (d_1, d_2, ..., d_n)$ be an n-term graphical sequence with realization G and $\sigma(S) \geq m(2n - m - 1) + 4$. Assume $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$.

If $d_n \leq m$, then consider the degree sequence S' formed by $G - v_n$. We have $\sigma(S') \geq m(2n - m - 1) + 4 - 2m = m(2(n - 1) - m - 1) + 4$. By the induction hypothesis, S' has a realization containing a C_{2m+2}. Hence S has a realization containing a C_{2m+2}. Thus, we may assume that $d_n \geq m + 1$.

Since $\sigma(S) \geq m(2n - m - 1) + 4 \geq m(2n - m - 1) + 2$, Theorem 3 implies that S has a realization containing a C_{2m+1}. Therefore, by Theorem 1, S has a realization containing a C_{2m+2}.

Thus $\sigma(C_{2m+2}, n) \leq m(2n - m - 1) + 4$. By [5] Theorem 1 (Theorem A below), for $m \geq 2$ and $n \geq 2m + 2$, we have $\sigma(C_{2m+2}, n) \geq m(2n - m - 1) + 4$. Hence $\sigma(C_{2m+2}, n) = m(2n - m - 1) + 4$ for $m \geq 3$ and $n \geq 5m - 2$.

For completeness, we give short proofs of the lower bounds for $\sigma(C_{2m+1}, n)$ and $\sigma(C_{2m+2}, n)$:

Theorem A

$\sigma(C_{2m+1}, n) \geq m(2n - m - 1) + 2$, for $n \geq 2m + 1$, $m \geq 2$; and $\sigma(C_{2m+2}, n) \geq m(2n - m - 1) + 4$, for $n \geq 2m + 2$, $m \geq 2$.

Proof. This result follows easily by noting that $G = K_m + K_{n-m}$ gives a uniquely realizable degree sequence that clearly does not contain C_{2m+1}, and $H = K_m + (K_{n-m-2} \cup K_2)$ gives a uniquely realizable degree sequence that clearly does not contain C_{2m+2}.

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Note: Figure translations are in progress. See original paper for figures.