A note on potentially $K_4 - e$ graphical sequences postprint

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Date: 2024-02-10T15:28:22+00:00

Abstract

“A sequence $S$is potentially $K_4 - e$graphical if it has a realization containing a $K_4 - e$as a subgraph. Let $\sigma(K_4 - e, n)$denote the smallest degree sum such that every n-term graphical sequence $S$with $\sigma(S) \geq \sigma(K_4 - e, n)$is potentially $K_4 - e$graphical. Gould, Jacobson, Lehel raised the problem of determining the value of $\sigma(K_4 - e, n)$. In this paper, we prove that $\sigma(K_4 - e, n) = 2[(3n-1)/2]$for $n \geq 7$and $n = 4, 5$, and $\sigma(K_4 - e, 6) = 20$.’’

Full Text

Preamble

On Potentially $\mathbf{K_4 - e}$ Graphical Sequences

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Abstract

A sequence S is *potentially* $K_4 - e$ *graphical* if it has a realization containing a $K_4 - e$ as a subgraph. Let $\sigma(K_4 - e, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_4 - e, n)$ is potentially $K_4 - e$ graphical. Gould, Jacobson, and Lehel raised the problem of determining the value of $\sigma(K_4 - e, n)$. In this paper, we prove that $\sigma(K_4 - e, n) = 2\lfloor(3n-1)/2\rfloor$ for $n \geq 7$, as well as for $n = 4, 5$, and that $\sigma(K_4 - e, 6) = 20$.

1. Introduction

Let $S = (d_1, d_2, ..., d_n)$ be a sequence of non-negative integers. We call S *graphical* if there exists a simple graph G of order n whose degree sequence

$(d(v_1), d(v_2), ..., d(v_n))$ is precisely S. If G is such a graph, then G is said to *realize* S, or be a *realization* of S. A graphical sequence S is *potentially* H *graphical* if there exists a realization of S containing H as a subgraph, while S is *forcibly* H *graphical* if every realization of S contains H as a subgraph.

Let $\sigma(S) = d_1 + d_2 + ... + d_n$, and let $[x]$ denote the largest integer less than or equal to x. If G and G_1 are graphs, then $G \cup G_1$ denotes the disjoint union of G and G_1. If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. Let K_k denote a complete graph on k vertices, and C_k a cycle of length k.

Given a graph H, what is $\mathrm{ex}(n, H)$, the maximum number of edges in a graph with n vertices not containing H as a subgraph? This problem was proposed for $H = C_4$ by Erdős [2] in 1938 and in general by Turán [9]. In terms of graphic sequences, the number $2\,\mathrm{ex}(n, H) + 2$ is the minimum even integer m such that every n-term graphical sequence S with $\sigma(S) \geq m$ is forcibly H graphical. Here we consider the following variant: determine the minimum even integer m such that every n-term graphical sequence S with $\sigma(S) \geq m$ is potentially H graphical. We denote this minimum m by $\sigma(H, n)$.

Erdős, Jacobson, and Lehel [1] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that $\sigma(K_k, n) = (k-2)(2n-k+1)+2$. They proved that if S contains no zero terms, this conjecture holds for $k = 3, n \geq 6$. Li and Song [6,7,8] proved that if S contains no zero terms, the conjecture holds for $k = 4, n \geq 8$ and $k = 5, n \geq 10$, and that $\sigma(K_k, n) \leq 2n(k-2)+2$ for $n \geq 2k-1$. Gould, Jacobson, and Lehel [3] proved the conjecture for $k = 4, n \geq 9$; they also showed that if $n = 8$ and $\sigma(S) \geq 28$, then either S has a realization containing K_4 or $S = (4^7, 0^1)$ (i.e., S consists of seven 4' s and one 0). Additionally, they established that $\sigma(pK_2, n) = (p-1)(2n-2)+2$ for $p \geq 2$, that $\sigma(C_4, n) = 2\lfloor(3n-1)/2\rfloor$ for $n \geq 4$, that $\sigma(C_4, n) \leq \sigma(K_4 - e, n) \leq \sigma(K_4, n)$, and raised the problem of determining $\sigma(K_4 - e, n)$. Lai [4,5] proved that $\sigma(C_{2m+1}, n) = m(2n-m-1)+2$ for $m \geq 2, n \geq 3m$ and $\sigma(C_{2m+2}, n) = m(2n-m-1)+4$ for $m \geq 2, n \geq 5m-2$. In this paper, we determine the values of $\sigma(K_4 - e, n)$.

2. $\sigma(K_4 - e, n)$

Theorem 1. For $n = 4, 5$ and $n \geq 7$,

$$\sigma(K_4 - e, n) = \begin{cases} 3n-1 & \text{if } n \text{ is odd,} \\ 3n-2 & \text{if } n \text{ is even.} \end{cases}$$

For $n = 6$, if S is a 6-term graphical sequence with $\sigma(S) \geq 16$, then either S has a realization containing $K_4 - e$ or $S = (3^6)$. (Thus $\sigma(K_4 - e, 6) = 20$.)

Proof. By [3], we have $\sigma(K_4 - e, n) \geq \sigma(C_4, n) = 2\lfloor(3n-1)/2\rfloor$, which equals $3n-1$ when n is odd and $3n-2$ when n is even, for $n \geq 4$.

Assume $d_1 \geq d_2 \geq ... \geq d_n \geq 0$. For $n = 4$, if a graph has size $q \geq 5$, then it clearly contains a $K_4 - e$, so $\sigma(K_4 - e, n) \leq 3n-2$. For $n = 5$, we have $q \geq 7$.

There are exactly four graphs of order 5 and size 7, and each contains a $K_4 - e$. Thus $\sigma(K_4 - e, n) \leq 3n - 1$.

Suppose for $5 \leq t < n$ that S_1 is a t-term graphical sequence such that

$$\sigma(S_1) \geq \begin{cases} 3t-1 & \text{if } t \text{ is odd,} \\ 3t-2 & \text{if } t \text{ is even.} \end{cases}$$

Then either S_1 has a realization containing a $K_4 - e$ or $S_1 = (3^6)$.

We now consider two main cases based on the parity of n.

Case 1: n **is even.** Let S be an n-term graphical sequence with $\sigma(S) \geq 3n-2$, and let G be a realization of S with degree sequence $d_1 \geq d_2 \geq ... \geq d_n \geq 0$.

Subcase 1.1: $\sigma(S) = 3n - 2$. If $d_n \leq 1$, let S' be the degree sequence of $G - v_n$. Then $\sigma(S') \geq 3n - 2 - 2 = 3(n-1) - 1$. By induction, S' has a realization containing a $K_4 - e$, and therefore S has a realization containing a $K_4 - e$. Hence, we may assume $d_n \geq 2$. Since $\sigma(S) = 3n - 2$, we must have $d_n = d_{n-1} = 2$. Let v_n be adjacent to vertices x and y.

If $x = v_{n-1}$ or $y = v_{n-1}$, let S'' be the degree sequence of $G - v_n - v_{n-1}$. Then $\sigma(S'') = 3n - 2 - 6 = 3(n-2) - 2$. Clearly $S'' \neq (3^6)$, so by induction, S'' has a realization containing a $K_4 - e$, and thus S has a realization containing a $K_4 - e$.

If $x \neq v_{n-1}$ and $y \neq v_{n-1}$, and v_{n-1} is adjacent to x and y, we first assume x is adjacent to y. Then G contains a $K_4 - e$. Therefore, we may assume x is not adjacent to y. The edge interchange that removes edges xv_{n-1} and yv_n and inserts edges xy and $v_{nv_{n-1}}$ produces a realization G' of S containing $v_{n-1}v_n$, reducing to the previous case.

If $x \neq v_{n-1}$ and $y \neq v_{n-1}$, and v_{n-1} is not adjacent to x, let v_{n-1} be adjacent to z_1 and z_2. First consider when x is not adjacent to z_1. The edge interchange that removes edges $v_{n-1}z_1$ and v_{nx} and inserts edges xz_1 and $v_{n-1}v_n$ produces a realization G' of S containing $v_{n-1}v_n$, reducing to an earlier case. Next, if x is not adjacent to z_2, a similar argument shows S has a realization containing a $K_4 - e$. Finally, if x is adjacent to both z_1 and z_2, we assume z_1 is adjacent to z_2. Then G contains a $K_4 - e$. Otherwise, if z_1 is not adjacent to z_2, the edge interchange that removes edges $v_{n-1}z_1$, $v_{n-1}z_2$, and v_{nx} and inserts edges $v_{n-1}v_n$, $z_{1z}2$, and $v_{n-1}x$ produces a realization G' of S containing $v_{n-1}v_n$, reducing to a previous case.

If $x \neq v_{n-1}$ and $y \neq v_{n-1}$, and v_{n-1} is not adjacent to y, a similar argument shows S has a realization containing a $K_4 - e$.

Subcase 1.2: $\sigma(S) = 3n$. If $d_n \leq 2$, let S' be the degree sequence of $G - v_n$. Then $\sigma(S') \geq 3n - 4 = 3(n-1) - 1$. By induction, S' has a realization containing a $K_4 - e$, and thus S has a realization containing a $K_4 - e$. Therefore, we may assume $d_n \geq 3$, which implies $S = (3^n)$. If $n = 6$, let G_1 be a realization of

(3^6); clearly G_1 does not contain a $K_4 - e$. If $n = 4p$ with $p \geq 2$, then pK_4 is a realization of $S = (3^n)$ containing a $K_4 - e$. Finally, if $n = 4p + 2$ with $p \geq 2$, then $G_1 \cup (p-1)K_4$ is a realization of $S = (3^n)$ containing a $K_4 - e$.

Subcase 1.3: $3n + 2 \leq \sigma(S) \leq 4n - 2$. Then $d_n \leq 3$. Let S' be the degree sequence of $G - v_n$, so $\sigma(S') \geq 3n + 2 - 6 = 3(n-1) - 1$. By induction, S' has a realization containing a $K_4 - e$, and hence S has a realization containing a $K_4 - e$.

Subcase 1.4: $\sigma(S) \geq 4n$. If $n \geq 8$, then by Proposition 2 and Theorem 4 of [3], S has a realization containing a K_4. If $n = 6$ and $4n \leq \sigma(S) \leq 5n-2$, then $d_n \leq 4$. Let S' be the degree sequence of $G - v_n$, giving $\sigma(S') \geq 4n-8 = 16 = 3(n-1)+1$. By induction, S' has a realization containing a $K_4 - e$, so S has a realization containing a $K_4 - e$. Finally, if $\sigma(S) \geq 5n = 30$, then $\sigma(S) = 30$ and the realization of S is K_6, which contains a $K_4 - e$.

Case 2: n **is odd.** Let S be an n-term graphical sequence with $\sigma(S) \geq 3n-1$, and let G be a realization of S.

Subcase 2.1: $\sigma(S) = 3n - 1$. Then $d_n \leq 2$. Let S' be the degree sequence of $G - v_n$, so $\sigma(S') \geq 3n - 1 - 4 = 3(n-1) - 2$. By induction, either S' has a realization containing a $K_4 - e$ or $S' = (3^6)$. Therefore, either S has a realization containing a $K_4 - e$ or $S = (4^1, 3^5, 1^1)$. Clearly, $(4^1, 3^5, 1^1)$ has a realization containing a $K_4 - e$ (see Appendix Figure 1 [Figure 1: see original paper]). Hence, S has a realization containing a $K_4 - e$.

Subcase 2.2: $3n + 1 \leq \sigma(S) \leq 4n - 2$. Then $d_n \leq 3$. Let S' be the degree sequence of $G - v_n$, giving $\sigma(S') \geq 3n + 1 - 6 = 3(n-1) - 2$. By induction, either S' has a realization containing a $K_4 - e$ or $S' = (3^6)$. Therefore, either S has a realization containing a $K_4 - e$ or $S = (4^2, 3^4, 2^1)$ or $S = (4^3, 3^4)$. Both $(4^2, 3^4, 2^1)$ and $(4^3, 3^4)$ have realizations containing a $K_4 - e$ (see Appendix Figure 2 [Figure 2: see original paper]). Hence, S has a realization containing a $K_4 - e$.

Subcase 2.3: $\sigma(S) \geq 4n$. If $n \geq 9$, then by Theorem 4 of [3], S has a realization containing a K_4. If $n = 7$ and $4n \leq \sigma(S) \leq 5n - 1$, then $d_n \leq 4$. Let S' be the degree sequence of $G - v_n$, so $\sigma(S') \geq 4n - 8 = 3n - 1 = 3(n-1) + 2$. Clearly $S' \neq (3^6)$, so by induction, S' has a realization containing a $K_4 - e$, and thus S has a realization containing a $K_4 - e$. Finally, if $\sigma(S) \geq 5n + 1 = 36$, then since $(6^6, 0^1)$ is not graphical, we have $d_7 \geq 1$, and by Theorem 2.2 of [6], S has a realization containing a K_4.

Acknowledgment

This paper was written while I was a visiting scholar at the University of Science and Technology of China. I thank Professor Li Jiong-Sheng for his valuable advice and the referee for many helpful comments.

References

[1] P. Erdős, M. S. Jacobson and J. Lehel, Graphs realizing the same degree sequences and their respective clique numbers, in *Graph Theory, Combinatorics & Applications*, Vol. 1 (Y. Alavi et al., eds.), John Wiley & Sons, Inc., New York, 1991, 439–449.

[2] P. Erdős, On sequences of integers no one of which divides the product of two others and some related problems, *Izv. Nauchno-Issl. Mat. i Meh. Tomsk* 2 (1938).

[3] R. J. Gould, M. S. Jacobson and J. Lehel, Potentially G-graphical degree sequences, a lecture presented at Kalamazoo Meeting in 1996.

[4] Lai Chunhui, Potentially C_k-graphical degree sequences, *J. Zhangzhou Teachers College* 11(4) (1997), 27–31.

[5] Lai Chunhui, The smallest degree sum that yields potentially C_k-graphical sequences, a lecture presented at Hefei Meeting in 1997.

[6] Li Jiong-Sheng and Song Zi-Xia, An extremal problem on the potentially P_k-graphical sequences, in *The International Symposium on Combinatorics and Applications*, June 28–30, 1996 (W. Y. C. Chen et al., eds.), Nankai University, 269–276.

[7] Li Jiong-Sheng and Song Zi-Xia, The smallest degree sum that yields potentially P_k-graphical sequences, *J. Graph Theory* 29 (1998), 63–72.

[8] Li Jiong-Sheng and Song Zi-Xia, On the potentially P_k-graphic sequences, *Discrete Math.* 195 (1999), 255–262.

[9] P. Turán, On an extremal problem in graph theory, *Mat. Fiz. Lapok* 48 (1941).

Appendix

[Figure 1: see original paper] Realization of $(4^1, 3^5, 1^1)$ containing $K_4 - e$

[Figure 2: see original paper] Realizations of $(4^2, 3^4, 2^1)$ and $(4^3, 3^4)$ containing $K_4 - e$

Note: Figure translations are in progress. See original paper for figures.