

## On potentially $K_{r+1} - U$ -graphical Sequences Post-print

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### Abstract

Let  $K_m - H$  be the graph obtained from  $K_m$  by removing the edge set  $E(H)$  of the graph  $H$  ( $H$  is a subgraph of  $K_m$ ). We use the symbol  $Z_4$  to denote  $K_4 - P_2$ . A sequence  $S$  is potentially  $K_m - H$ -graphical if it has a realization containing a  $K_m - H$  as a subgraph. Let  $\sigma(K_m - H, n)$  denote the smallest degree sum such that every  $n$ -term graphical sequence  $S$  with  $\sigma(S) \geq \sigma(K_m - H, n)$  is potentially  $K_m - H$ -graphical. In this paper, we determine the values of  $\sigma(K_{r+1} - U, n)$  for  $n \geq 5r + 18, r + 1 \geq k \geq 7, j \geq 6$  where  $U$  is a graph on  $k$  vertices and  $j$  edges which contains a graph  $K_3 \cup P_3$  but does not contain a cycle on 4 vertices and does not contain  $Z_4$ . There are a number of graphs on  $k$  vertices and  $j$  edges which contain a graph  $(K_3 \cup P_3)$  but do not contain a cycle on 4 vertices and do not contain  $Z_4$ . (for example,  $C_3 \cup C_{i_1} \cup C_{i_2} \cup \dots \cup C_{i_p}$  ( $i_j \neq 4, j = 2, 3, \dots, p, i_1 \geq 5$ ),  $C_3 \cup P_{i_1} \cup P_{i_2} \cup \dots \cup P_{i_p}$  ( $i_1 \geq 3$ ),  $C_3 \cup P_{i_1} \cup C_{i_2} \cup \dots \cup C_{i_p}$  ( $i_j \neq 4, j = 2, 3, \dots, p, i_1 \geq 3$ ), etc)

### Full Text

## On Potentially $K_{r+1} - U$ -Graphic Sequences

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## Abstract

Let  $K_m - H$  denote the graph obtained from  $K_m$  by removing the edge set  $E(H)$  of a subgraph  $H$  of  $K_m$ . We use the symbol  $Z_4$  to denote  $K_4 - P_2$ . A sequence  $S$  is potentially  $K_m - H$ -graphic if it has a realization containing  $K_m - H$  as a subgraph. Let  $\sigma(K_m - H, n)$  denote the smallest degree sum such that every  $n$ -term graphic sequence  $S$  with  $\sigma(S) \geq \sigma(K_m - H, n)$  is potentially  $K_m - H$ -graphic. In this paper, we determine the values of  $\sigma(K_{r+1} - U, n)$  for  $n \geq 5r + 18$ ,  $r + 1 \geq k \geq 7$ , and  $j \geq 6$ , where  $U$  is a graph on  $k$  vertices and  $j$  edges that contains  $K_3 \cup P_3$  but contains neither a 4-cycle nor  $Z_4$ .

**Keywords:** graph; degree sequence; potentially  $K_{r+1} - U$ -graphic sequence; potentially  $K_{r+1} - K_3 \cup P_3$ -graphic sequence

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## 1 Introduction

Let  $NS_n$  denote the set of all non-increasing nonnegative integer sequences  $\pi = (d_1, d_2, \dots, d_n)$ . A sequence  $\pi \in NS_n$  is said to be graphic if it is the degree sequence of a simple graph  $G$  on  $n$  vertices, and such a graph  $G$  is called a realization of  $\pi$ . The set of all graphic sequences in  $NS_n$  is denoted by  $GS_n$ . A graphic sequence  $\pi$  is potentially  $H$ -graphic if there exists a realization of  $\pi$  containing  $H$  as a subgraph, while  $\pi$  is forcibly  $H$ -graphic if every realization of  $\pi$  contains  $H$  as a subgraph. If  $\pi$  has a realization in which the  $r + 1$  vertices of largest degree induce a clique, then  $\pi$  is said to be potentially  $A_{r+1}$ -graphic.

Let  $\sigma(\pi) = d_1 + d_2 + \dots + d_n$ , and let  $[x]$  denote the largest integer less than or equal to  $x$ . If  $G$  and  $G_1$  are graphs, then  $G \cup G_1$  denotes the disjoint union of  $G$  and  $G_1$ . If  $G = G_1$ , we abbreviate  $G \cup G_1$  as  $2G$ . We denote  $G + H$  as the graph with  $V(G + H) = V(G) \cup V(H)$  and  $E(G + H) = E(G) \cup \{xy : x \in V(G), y \in V(H)\}$ . Let  $K_k$ ,  $C_k$ ,  $T_k$ , and  $P_k$  denote a complete graph on  $k$  vertices, a cycle on  $k$  vertices, a tree on  $k + 1$  vertices, and a path on  $k + 1$  vertices, respectively. Let  $K_m - H$  be the graph obtained from  $K_m$  by removing the edge set  $E(H)$  of the graph  $H$  (where  $H$  is a subgraph of  $K_m$ ). Let  $F_k$  denote the friendship graph on  $2k + 1$  vertices, that is, the graph of  $k$  triangles intersecting in a single vertex. For  $0 \leq r \leq t$ , denote the generalized friendship graph on  $kt - kr + r$  vertices by  $F_{t,r,k}$ , where  $F_{t,r,k}$  is the graph of  $k$  copies of  $K_t$  meeting in a common  $r$ -set. We use the symbol  $Z_4$  to denote  $K_4 - P_2$ .

We use the symbol  $G[v_1, v_2, \dots, v_k]$  to denote the subgraph of  $G$  induced by the vertex set  $\{v_1, v_2, \dots, v_k\}$ . We use the symbol  $\epsilon(G)$  to denote the number of edges in graph  $G$ .

Given a graph  $H$ , what is the maximum number of edges of a graph with  $n$  vertices not containing  $H$  as a subgraph? This number is denoted  $\text{ex}(n, H)$  and is known as the Turán number. Mantel [?] proved that  $\text{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$ . This

was rediscovered by Turán [?] as a special case of his results on  $\text{ex}(n, K_k)$ .

In terms of graphic sequences, the number  $2\text{ex}(n, H) + 2$  is the minimum even integer  $l$  such that every  $n$ -term graphic sequence  $\pi$  with  $\sigma(\pi) \geq l$  is forcibly  $H$ -graphic. Here we consider the following variant: determine the minimum even integer  $l$  such that every  $n$ -term graphic sequence  $\pi$  with  $\sigma(\pi) \geq l$  is potentially  $H$ -graphic. We denote this minimum  $l$  by  $\sigma(H, n)$ . Erdős, Jacobson, and Lehel [?] showed that  $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$  and conjectured that equality holds. They proved that if  $\pi$  does not contain zero terms, this conjecture is true for  $k=3, n \geq 6$ . The conjecture was confirmed in [?, ?, ?, ?, ?].

Gould, Jacobson, and Lehel [?] also proved that  $\sigma(pK_2, n) = (p-1)(2n-2)+2$  for  $p \geq 2$  and  $\sigma(C_4, n) = 2 \lfloor \frac{3n-1}{2} \rfloor$  for  $n \geq 4$ . They also pointed out that it would be interesting to determine where in the range from  $3n-2$  to  $4n-4$  the value  $\sigma(K_4 - e, n)$  lies. Luo [?] characterized potentially  $C_k$ -graphic sequences for  $k=3, 4, 5$ . Luo and Warner [?] characterized potentially  $K_k$ -graphic sequences. Yin and Yin [?] characterized potentially  $(K_5 - e)$ -positive graphic sequences and gave two simple necessary and sufficient conditions for a positive graphic sequence  $\pi$  to be potentially  $K_5$ -graphic. Moreover, they also gave a simple necessary and sufficient condition for a positive graphic sequence  $\pi$  to be potentially  $K_6$ -graphic. Ferrara, Gould, and Schmitt [?] determined  $\sigma(F_k, n)$  for  $n$  sufficiently large. Ferrara [?] determined  $\sigma(F_{t,0,k}, n)$  for sufficiently large  $n$  and determined  $\sigma(F_{t,t-2,k}, n)$  for sufficiently large  $n$ . Yin and Chen [?] determined  $\sigma(F_{t,t-1,k}, n)$  for  $n \geq 3t + 2k^2 + 3k - 6$ . Yin, Chen, and Schmitt [?] determined  $\sigma(F_{t,r,k}, n)$  for  $k \geq 2, t \geq 3, 1 \leq r \leq t-2$ , and  $n$  sufficiently large. Gould et al. [?] determined  $\sigma(K_{2,2}, n)$  for  $n \geq 4$ . Yin et al. [?] determined  $\sigma(K_{r,s}, n)$  for  $s \geq r \geq 2$  and sufficiently large  $n$ .

Lai [?] determined  $\sigma(K_4 - e, n)$  for  $n \geq 4$ . Yin, Li, and Mao [?] determined  $\sigma(K_{r+1} - e, n)$  for  $r \geq 3, r+1 \leq n \leq 2r$ , and  $\sigma(K_5 - e, n)$  for  $n \geq 5$ . Yin and Li [?] gave a good method (the Yin-Li method) for determining the values  $\sigma(K_{r+1} - e, n)$  for  $r \geq 2$  and  $n \geq 3r^2 - r - 1$  (in fact, Yin and Li [?] also determined the values  $\sigma(K_{r+1} - ke, n)$  for  $r \geq 2$  and  $n \geq 3r^2 - r - 1$ ). After reading [?], using the Yin-Li method, Yin [?] determined  $\sigma(K_{r+1} - K_3, n)$  for  $n \geq 3r + 5, r \geq 3$ . Lai [?] determined  $\sigma(K_5 - K_3, n)$  for  $n \geq 5$ . Lai [?, ?] determined  $\sigma(K_5 - C_4, n), \sigma(K_5 - P_3, n)$ , and  $\sigma(K_5 - P_4, n)$  for  $n \geq 5$ .

Determining  $\sigma(K_{r+1} - H, n)$  where  $H$  is a tree on 4 vertices is more useful than when  $H$  is a cycle on 4 vertices (for example,  $C_4 \not\subset C_i$ , but  $P_3 \subset C_i$  for  $i \geq 5$ ). Therefore, after reading [?] and [?], using the Yin-Li method, Lai and Hu [?] determined  $\sigma(K_{r+1} - H, n)$  for  $n \geq 4r + 10, r \geq 3, r+1 \geq k \geq 4$ , and  $H$  a graph on  $k$  vertices containing a tree on 4 vertices but not containing a 3-cycle, and also determined  $\sigma(K_{r+1} - P_2, n)$  for  $n \geq 4r + 8, r \geq 3$ . Using the Yin-Li method, Lai and Sun [?] determined  $\sigma(K_{r+1} - (kP_2 \cup tK_2), n)$  for  $n \geq 4r + 10, r+1 \geq 3k + 2t, k+t \geq 2, k \geq 1, t \geq 0$ . To date, the problem of determining  $\sigma(K_{r+1} - H, n)$  for  $H$  not containing a 3-cycle and  $n$  sufficiently large has been solved. Using the Yin-Li method, Lai [?] determined the values of  $\sigma(K_{r+1} - Z, n)$

for  $n \geq 5r + 19$ ,  $r + 1 \geq k \geq 5$ ,  $j \geq 5$ , where  $Z$  is a graph on  $k$  vertices and  $j$  edges that contains  $Z_4$  but does not contain a 4-cycle. Using the Yin-Li method, Lai [?] also determined the values of  $\sigma(K_{r+1} - Z_4, n)$ ,  $\sigma(K_{r+1} - (K_4 - e), n)$ , and  $\sigma(K_{r+1} - K_4, n)$  for  $n \geq 5r + 16$ ,  $r \geq 4$ .

In this paper, using the Yin-Li method, we prove the following two theorems:

**Theorem 1.1.** If  $r \geq 6$  and  $n \geq 5r + 18$ , then

$$\sigma(K_{r+1} - (K_3 \cup P_3), n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

**Theorem 1.2.** If  $n \geq 5r + 18$ ,  $r + 1 \geq k \geq 7$ , and  $j \geq 6$ , then

$$\sigma(K_{r+1} - U, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

where  $U$  is a graph on  $k$  vertices and  $j$  edges that contains  $K_3 \cup P_3$  but contains neither a 4-cycle nor  $Z_4$ .

There are numerous graphs on  $k$  vertices and  $j$  edges that contain  $K_3 \cup P_3$  but contain neither a 4-cycle nor  $Z_4$  (for example,  $C_3 \cup C_{i_p}$  where  $i_j \neq 4$  for  $j = 2, 3, \dots, p$  and  $i_1 \geq 5$ ,  $C_3 \cup C_{i_p}$  where  $i_j \neq 4$  for  $j = 2, 3, \dots, p$  and  $i_1 \geq 3$ ,  $P_{i_1} \cup P_{i_2} \cup \dots, C_{i_2} \cup C_{i_1} \cup \dots \cup P_{i_p}$  where  $i_1 \geq 3$ ,  $C_3 \cup C_{i_2} \cup P_{i_1} \cup \dots$ , etc.).

## 2 Preparations

To prove our main results, we need the following notations and results.

Let  $\pi = (d_1, \dots, d_n) \in NS_n$  and  $1 \leq k \leq n$ . The residual sequence obtained by laying off  $d_k$  from  $\pi$  is denoted by  $\pi'_k$  and defined as:

$$\pi'_k = \begin{cases} (d_1 - 1, \dots, d_{k-1} - 1, d_{k+1} - 1, \dots, d_{d_k+1} - 1, d_{d_k+2}, \dots, d_n), & \text{if } d_k \geq k \\ (d_1 - 1, \dots, d_{d_k} - 1, d_{d_k+1}, \dots, d_{k-1}, d_{k+1}, \dots, d_n), & \text{if } d_k < k \end{cases}$$

Let  $\pi' = (d'_1, d'_2, \dots, d'_{n-1})$ , where  $d'_1 \geq d'_2 \geq \dots \geq d'_{n-1}$  is a rearrangement of the  $n - 1$  terms of  $\pi'_k$ . Then  $\pi'$  is called the residual sequence.

**Theorem 2.1** [?]. Let  $n \geq r + 1$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r+1} \geq r$ . If  $d_i \geq 2r - i$  for  $i = 1, 2, \dots, r - 1$ , then  $\pi$  is potentially  $A_{r+1}$ -graphic.

**Theorem 2.2** [?]. Let  $n \geq 2r + 2$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r+1} \geq r$ . If  $d_{2r+2} \geq r - 1$ , then  $\pi$  is potentially  $A_{r+1}$ -graphic.

**Theorem 2.3** [?]. Let  $n \geq r + 1$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r+1} \geq r - 1$ . If  $d_i \geq 2r - i$  for  $i = 1, 2, \dots, r - 1$ , then  $\pi$  is potentially  $K_{r+1} - e$ -graphic.

**Theorem 2.4** [?]. Let  $n \geq 2r + 2$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r-1} \geq r$ . If  $d_{2r+2} \geq r - 1$ , then  $\pi$  is potentially  $K_{r+1} - e$ -graphic.

**Theorem 2.5** [?]. Let  $\pi = (d_1, \dots, d_n) \in NS_n$  and  $1 \leq k \leq n$ . Then  $\pi \in GS_n$  if and only if  $\pi'_k \in GS_{n-1}$ .

**Theorem 2.6** [?]. Let  $\pi = (d_1, \dots, d_n) \in NS_n$  with even  $\sigma(\pi)$ . Then  $\pi \in GS_n$  if and only if for any  $t$ ,  $1 \leq t \leq n - 1$ ,

$$\sum_{i=1}^t d_i \leq t(t-1) + \sum_{j=t+1}^n \min\{t, d_j\}.$$

**Theorem 2.7** [?]. If  $\pi = (d_1, d_2, \dots, d_n)$  is a graphic sequence with a realization  $G$  containing  $H$  as a subgraph, then there exists a realization  $G'$  of  $\pi$  containing  $H$  as a subgraph such that the vertices of  $H$  have the largest degrees of  $\pi$ .

**Lemma 2.1** [?]. If  $\pi = (d_1, d_2, \dots, d_n) \in NS_n$  is potentially  $K_{r+1} - e$ -graphic, then there is a realization  $G$  of  $\pi$  containing  $K_{r+1} - e$  with vertices  $v_1, \dots, v_{r+1}$  such that  $d_G(v_i) = d_i$  for  $i = 1, 2, \dots, r + 1$  and  $e = v_r v_{r+1}$ .

**Lemma 2.2** [?]. Let  $n \geq 2r$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r-1} \geq r$  and  $d_{r+1} \geq r - 1$ . If  $d_i \geq 2r - i$  for  $i = 1, 2, \dots, r - 2$ , then  $\pi$  is potentially  $K_{r+1} - e$ -graphic.

**Lemma 2.3** [?]. Let  $\pi = (d_1, \dots, d_n) \in GS_n$  and  $G$  be a realization of  $\pi$ . If  $\epsilon(G[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 1$ , then there is a realization  $H$  of  $\pi$  such that  $d_H(v_i) = d_i$  for  $i = 1, 2, \dots, r + 1$  and  $v_r v_{r+1} \notin E(H)$ .

**Lemma 2.4** [?]. Let  $n \geq 2r + 2$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r-4} \geq r$ . If  $d_{2r+2} \geq r - 1$ , then  $\pi$  is potentially  $K_{r+1} - (P_2 \cup (r - 1)K_2)$ -graphic.

**Lemma 2.5** [?]. Let  $n \geq 2r$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r-2} \geq r + 1$ ,  $d_{r+1} \geq r$ , and  $d_{r-1} \geq d_{d_{r+1}+2}$ . If  $d_i \geq 2r - i$  for  $i = 1, 2, \dots, r - 3$ , then  $\pi$  is potentially  $A_{r+1}$ -graphic.

### 3 Proof of Main Results

**Lemma 3.1.** Let  $n \geq 2r + 2$ ,  $r \geq 4$ , and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r-2} \geq r - 1$  and  $d_{r+1} \geq r - 2$ . If

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

and  $d_i \geq 2r - i$  for  $i = 1, 2, \dots, r - 3$ , then  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic.

**Proof.** We consider two main cases.

**Case 1:**  $d_{r+1} \geq r - 1$ .

*Subcase 1.1:*  $d_{r-1} \geq r + 1$ . If  $d_{r-2} \geq r + 2$ , then  $\pi$  is potentially  $K_{r+1} - e$ -graphic by Theorem 2.3, and hence  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic. If  $d_{r-2} = r + 1$  and  $d_{r+1} = r + 1$ , then  $d_{r-2} = d_{r-1} = d_r = d_{r+1} = r + 1$ . Suppose  $\pi$  is not potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic. Let  $H$  be a realization of  $\pi$ ; then  $\epsilon(H[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$ . Let  $S = (d_1, d_2, \dots, d_{r-3}, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, d_{r+2}, \dots, d_n)$ . By Theorem 2.1,  $S$  is potentially  $A_{r+1}$ -graphic (denote  $S' = (d'_1, d'_2, \dots, d'_n)$ , where  $d'_1 \geq d'_2 \geq \dots \geq d'_n$  is a rearrangement of the  $n$  terms of  $S$ ; therefore  $S' \in GS_n$  by Lemma 2.3). Hence, there exists a realization  $G$  of  $S$  with vertices  $v_1, v_2, \dots, v_{r+1}$  ( $d(v_i) = d_i$  for  $i = 1, 2, \dots, r - 1$ ,  $d(v_r) = d_r + 1$ ,  $d(v_{r+1}) = d_{r+1} + 1$ ) where the  $r + 1$  vertices of highest degree contain a  $K_{r+1}$  by Theorem 2.7. Therefore,  $G - v_{r+1}v_r$  is a realization of  $\pi$  that is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic, which is a contradiction. Thus,  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic.

If  $d_{r+1} = r$  or  $d_{r+1} = r - 1$ , then  $d_{r-1} - 1 \geq r \geq d_{r+2}$ . The residual sequence  $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$  obtained by laying off  $d_{r+1}$  from  $\pi$  satisfies  $d'_{r-3} \geq d_{r-3} - 1 \geq 2(r - 1) - [(r - 1) - 2]$ ,  $d'_{r-2} \geq r - 1$ , and  $d'_{(r-1)+1} = d'_r \geq r - 2$ . By Lemma 2.2,  $\pi'_{r+1}$  is potentially  $K_{(r-1)+1} - e$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic by  $\{d_1 - 1, \dots, d_{r-2} - 1, d_{r-1} - 1\} \subseteq \{d'_1, \dots, d'_{(r-1)-2}\}$  and Lemma 2.1.

*Subcase 1.2:*  $d_{r-1} \leq r$ . If  $d_{r-2} \geq r + 1$ , then  $d_{r-2} - 1 \geq d_{r-1}$ . The residual sequence  $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$  obtained by laying off  $d_{r+1}$  from  $\pi$  satisfies  $d'_1 \geq d_1 - 1 \geq 2(r - 1) - [(r - 1) - 2]$ ,  $d'_{r-3} \geq d_{r-3} - 1 \geq 2(r - 1) - [(r - 1) - 2]$ ,  $d'_{r-2} \geq r - 1$ , and  $d'_{(r-1)+1} = d'_r \geq r - 2$ . By Lemma 2.2,  $\pi'_{r+1}$  is potentially  $K_{(r-1)+1} - e$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic by  $\{d_1 - 1, \dots, d_{r-2} - 1\} \subseteq \{d'_1, \dots, d'_{(r-1)-2}\}$  and Lemma 2.1.

If  $d_{r-2} = r$  and  $d_{r+1} = r$ , then  $d_{r-2} = d_{r-1} = d_r = d_{r+1} = r$ . Suppose  $\pi$  is not potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic. Let  $H$  be a realization of  $\pi$ ; then  $\epsilon(H[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$ . Let  $S = (d_1, d_2, \dots, d_{r-3}, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, d_{r+2}, \dots, d_n)$ . Denote  $S' = (d'_1, d'_2, \dots, d'_n)$ , where  $d'_1 \geq d'_2 \geq \dots \geq d'_n$  is a rearrangement of the  $n$  terms of  $S$ . Therefore  $S' \in GS_n$  by Lemma 2.3. The residual sequence  $S''_{r+1} = (d''_1, \dots, d''_{n-1})$  obtained by laying off  $d'_{r+1} = d_{r-1} = r$  from  $S'$  satisfies  $d''_{r-3} \geq d_{r-3} - 1 \geq 2(r - 1) - [(r - 1) - 2]$ ,  $d''_{r-2} \geq r$ , and  $d''_{(r-1)+1} = d''_r \geq r - 1$ . By Theorem 2.1,  $S''_{r+1}$  is potentially  $A_{(r-1)+1}$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic by  $\{d_1 - 1, \dots, d_{r-3} - 1, d_r, d_{r+1}\} \subseteq \{d''_1, \dots, d''_{(r-1)-2}\}$  and Theorem 2.7, which is a contradiction.

If  $d_{r+1} = r$  and  $d_r = r - 1$ , then  $d_{r-2} = d_{r-1} = d_r = r$ . The residual sequence  $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$  obtained by laying off  $d_{r+1}$  from  $\pi$  satisfies  $d'_{r-3} \geq d_{r-3} - 1 \geq 2(r - 1) - [(r - 1) - 2]$ ,  $d'_{r-2} \geq r - 1$ , and  $d'_{(r-1)+1} = d'_r \geq r - 2$ . By Lemma 2.2,  $\pi'_{r+1}$  is potentially  $K_{(r-1)+1} - e$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic by  $\{d_1 - 1, \dots, d_{r-2} - 1, d_{r-1}, d_r\} \subseteq \{d'_1, \dots, d'_{(r-1)-2}\}$  and Lemma 2.1.

If  $d_{r-2} = r - 1$ , then  $d_{r-2} = d_{r-1} = d_r = d_{r+1} = r - 1$ . Suppose  $\pi$  is not potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic. Let  $H$  be a realization of  $\pi$ ; then  $\epsilon(H[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$ . Let  $S = (d_1, d_2, \dots, d_{r-3}, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, d_{r+2}, \dots, d_n)$ . Denote  $S' = (d'_1, d'_2, \dots, d'_n)$ , where  $d'_1 \geq d'_2 \geq \dots \geq d'_n$  is a rearrangement of the  $n$  terms of  $S$ . Therefore  $S' \in GS_n$  by Lemma 2.3. The residual sequence  $S''_{r+1} = (d''_1, \dots, d''_{n-1})$  obtained by laying off  $d'_{r+1} = d_{r-1} = r - 1$  from  $S'$  satisfies  $d''_{r-3} \geq d_{r-3} - 1 \geq 2(r - 1) - [(r - 1) - 2]$ ,  $d''_{r-2} = r - 1$ , and  $d''_{(r-1)+1} = d''_r \geq r - 1$ . By Lemma 2.2,  $S''_{r+1}$  is potentially  $K_{(r-1)+1} - e$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic by  $\{d_1 - 1, \dots, d_{r-3} - 1, d_r, d_{r+1}, d_{r-2}\} = \{d''_1, \dots, d''_{(r-1)-2}\}$  and Lemma 2.1, which is a contradiction.

**Case 2:**  $d_{r+1} = r - 2$ .

*Subcase 2.1:*  $d_{r-1} < d_{r-2}$ . If  $d_{r-2} \geq r$ , then the residual sequence  $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$  obtained by laying off  $d_{r+1} = r - 2$  from  $\pi$  satisfies: (1)  $d'_i = d_i - 1$  for  $i = 1, 2, \dots, r - 2$ , (2)  $d'_{r-3} \geq d_{r-3} - 1 \geq 2(r - 1) - [(r - 1) - 2]$ ,  $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$ , and  $d'_{(r-1)+1} = d'_r \geq r - 2$ . By Lemma 2.2,  $\pi'_{r+1}$  is potentially  $K_{(r-1)+1} - e$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic by  $\{d_1 - 1, \dots, d_{r-2} - 1, d_{r-1}, d_r\} = \{d'_1, \dots, d'_{(r-1)-2}\}$  and Lemma 2.1.

If  $d_{r-2} = r - 1$ , then

$$\sigma(\pi) \leq (r-3)(n-1) + (r-1) + (r-2)(n-r+2) = (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+2) = (r-1)(2n-r)$$

which is a contradiction.

*Subcase 2.2:*  $d_{r-1} = d_{r-2}$ . Then  $\pi'_{r+1}$  satisfies  $d'_{r-3} \geq d_{r-3} - 1 \geq 2(r - 1) - [(r - 1) - 2]$ ,  $d'_{(r-1)-2} = d'_{r-3} \geq r - 1$ , and  $d'_{(r-1)+1} = d'_r \geq r - 2$ . By Lemma 2.2,  $\pi'_{r+1}$  is potentially  $K_{(r-1)+1} - e$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic by  $\{d_{r-1}, d_r, d_1 - 1, \dots, d_{r-2} - 1\} = \{d'_1, \dots, d'_{(r-1)-2}\}$  and Lemma 2.1.

**Lemma 3.2.** If  $n \geq r + 1$ ,  $r + 1 \geq k \geq 7$ , then

$$\sigma(K_{r+1} - U, n) \geq \begin{cases} (r - 1)(2n - r) - 3(n - r) - 1, & \text{if } n - r \text{ is odd} \\ (r - 1)(2n - r) - 3(n - r), & \text{if } n - r \text{ is even} \end{cases}$$

where  $U$  is a graph on  $k$  vertices and  $j$  edges that contains neither a 4-cycle nor  $Z_4$ .

**Proof.** Let

$$G = \begin{cases} K_{r-3} + \overline{K}_{n-r-1}, & \text{if } n - r \text{ is odd} \\ K_{r-3} + \overline{K}_{n-r}, & \text{if } n - r \text{ is even} \end{cases}$$

Then  $G$  is the unique realization of

$$\pi = \begin{cases} ((n - 1)^{r-3}, (r - 1)^1, (r - 2)^{n-r+1}, (r - 3)^1), & \text{if } n - r \text{ is odd} \\ ((n - 1)^{r-3}, (r - 1)^1, (r - 2)^{n-r+2}), & \text{if } n - r \text{ is even} \end{cases}$$

where the symbol  $x^y$  means  $x$  repeats  $y$  times in the sequence. Clearly,  $G$  does not contain  $K_{r+1} - U$ . Thus  $\sigma(K_{r+1} - U, n) \geq \sigma(\pi) + 2$ , which gives the desired lower bound.

**Proof of Theorem 1.1**

By Lemma 3.2, it suffices to verify that for  $n \geq 5r + 18$ ,

$$\sigma(K_{r+1} - (K_3 \cup P_3), n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

We now prove that if  $n \geq 5r + 18$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

then  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic.

If  $d_{r-4} \leq r - 1$ , then

$$\sigma(\pi) \leq (r-5)(n-1) + (r-1)(n-r+5) = (r-1)(n-1) - 4(n-1) + (r-1)(n-r+5) = (r-1)(2n-r) - 4(n-r) < (r-1)(2n-r) - 3(n-r)$$

which is a contradiction. Thus,  $d_{r-4} \geq r$ .

If  $d_{r-2} \leq r - 2$ , then

$$\sigma(\pi) \leq (r-3)(n-1) + (r-2)(n-r+3) = (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+3) = (r-1)(2n-r) - 3(n-r) < (r-1)(2n-r) - 3(n-r)$$

which is a contradiction. Thus,  $d_{r-2} \geq r - 1$ .

If  $d_{r+1} \leq r - 3$ , then

$$\sigma(\pi) = \sum_{i=1}^{r+1} d_i + \sum_{i=r+2}^n d_i \leq (r-1)r + \sum_{i=r+1}^n \min\{r, d_i\} + \sum_{i=r+1}^n d_i \leq (r-1)r + 2(n-r)(r-3) = (r-1)(2n-r) - 4(n-r) < (r-1)(2n-r) - 3(n-r)$$

which is a contradiction. Thus,  $d_{r+1} \geq r - 2$ .

If  $d_i \geq 2r - i$  for  $i = 1, 2, \dots, r - 3$  or  $d_{2r+2} \geq r - 1$ , then  $\pi$  is potentially  $K_{r+1} - (K_3 \cup P_3)$ -graphic by Lemma 3.1 or Lemma 2.4. If  $d_{2r+2} \leq r - 2$  and there exists an integer  $i$ ,  $1 \leq i \leq r - 3$ , such that  $d_i \leq 2r - i - 1$ , then

$$\sigma(\pi) \leq (i-1)(n-1) + (2r+1-i+1)(2r-i-1) + (r-2)(n+1-2r-2) = i^2 + i(n-4r-2) - (n-1) + (2r-1)(2r+2) + (r-2)(n-2r-1)$$

Since  $n \geq 5r + 18$ , it is easy to see that  $i^2 + i(n-4r-2)$ , considered as a function of  $i$ , attains its maximum value when  $i = r - 3$ . Therefore,

$$\sigma(\pi) \leq (r-3)^2 + (n-4r-2)(r-3) - (n-1) + (2r-1)(2r+2) + (r-2)(n-2r-1) = (r-1)(2n-r) - 3(n-r) - n + 5r + 16 < (r-1)(2n-r) - 3(n-r)$$

which is a contradiction.

Thus,

$$\sigma(K_{r+1} - (K_3 \cup P_3), n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

for  $n \geq 5r + 18$ .

### Proof of Theorem 1.2

By Lemma 3.2, for  $n \geq 5r + 18$ ,  $r + 1 \geq k \geq 7$ , and  $j \geq 6$ ,

$$\sigma(K_{r+1} - U, n) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

Obviously,  $\sigma(K_{r+1} - U, n) \leq \sigma(K_{r+1} - (K_3 \cup P_3), n)$ . By Theorem 1.1,

$$\sigma(K_{r+1} - (K_3 \cup P_3), n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

Therefore,

$$\sigma(K_{r+1} - U, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

for  $n \geq 5r + 18$ ,  $r + 1 \geq k \geq 7$ , and  $j \geq 6$ .

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