The Minimum Degree Sum That Yields Potentially $K_{r+1} - Z$-Graphical Sequences (Postprint)

Authors: Chunhui LAI, Chunhui LAI

Date: 2024-02-13T20:00:00+00:00

Abstract

Let $K_m - H$ be the graph obtained from K_m by removing the edge set $E(H)$ of the graph H (H is a subgraph of K_m). We use the symbol Z_4 to denote $K_4 - P_2$.. A sequence S is potentially $K_m - H$-graphical if it has a realization containing a $K_m - H$ as a subgraph. Let $\sigma(K_m - H, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_m - H, n)$ is potentially $K_m - H$-graphical. In this paper, we determine the values of $\sigma(K_{r+1} - Z, n)$ for $n \geq 5r + 19$, $r + 1 \geq k \geq 5$, $j \geq 5$ where Z is a graph on k vertices and j edges which contains a graph Z_4 but does not contain a cycle on 4 vertices. We also determine the values of $\sigma(K_{r+1} - Z_4, n)$, $\sigma(K_{r+1} - (K_4 - e), n)$, $\sigma(K_{r+1} - K_4, n)$ for $n \geq 5r + 16, r \geq 4$.

Full Text

The Smallest Degree Sum That Yields Potentially $K_{r+1} - Z$-Graphical Sequences

Chunhui Lai
Department of Mathematics, Zhangzhou Teachers College, Zhangzhou, Fujian 363000, P. R. China
Email: zjlaichu@public.zzptt.fj.cn

Abstract

Let $K_m - H$ be the graph obtained from K_m by removing the edge set $E(H)$ of graph H (where H is a subgraph of K_m). We use the symbol Z_4 to denote $K_4 - P_2$. A sequence S is potentially $K_m - H$-graphical if it has a realization containing $K_m - H$ as a subgraph. Let $\sigma(K_m - H, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_m - H, n)$

is potentially $K_m - H$-graphical. In this paper, we determine the values of $\sigma(K_{r+1} - Z, n)$ for $n \geq 5r + 19$, $r + 1 \geq k \geq 5$, $j \geq 5$, where Z is a graph on k vertices and j edges that contains a graph Z_4 but does not contain a cycle on 4 vertices. We also determine the values of $\sigma(K_{r+1} - Z_4, n)$, $\sigma(K_{r+1} - (K_4 - e), n)$, and $\sigma(K_{r+1} - K_4, n)$ for $n \geq 5r + 16$, $r \geq 4$.

Keywords: subgraph; degree sequence; potentially $K_{r+1} - Z$-graphic; potentially $K_{r+1} - Z_4$-graphic sequence
AMS Subject Classifications: 05C07, 05C35

1 Introduction

Let NS_n denote the set of all non-increasing nonnegative integer sequences $\pi = (d_1, d_2, \dots, d_n)$. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. A graphical sequence π is potentially H-graphical if there exists a realization of π containing H as a subgraph, while π is forcibly H-graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r + 1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic.

Let $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$, and let $[x]$ denote the largest integer less than or equal to x. If G and G_1 are graphs, then $G \cup G_1$ is the disjoint union of G and G_1. If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. We denote $G + H$ as the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. Let K_k, C_k, T_k, and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k+1$ vertices, and a path on $k+1$ vertices, respectively. Let $K_m - H$ be the graph obtained from K_m by removing the edge set $E(H)$ of graph H (where H is a subgraph of K_m). We use the symbol Z_4 to denote $K_4 - P_2$.

We use the symbol $G[v_1, v_2, \dots, v_k]$ to denote the subgraph of G induced by the vertex set $\{v_1, v_2, \dots, v_k\}$. We use the symbol $\epsilon(G)$ to denote the number of edges in graph G.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $\mathrm{ex}(n, H)$ and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdős [2] in 1938 and in general by Turán [19]. In terms of graphic sequences, the number $2\,\mathrm{ex}(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is forcibly H-graphical. Here we consider the following variant: determine the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is potentially H-graphical. We denote this minimum l by $\sigma(H, n)$.

Erdős, Jacobson and Lehel [4] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that equality holds. They proved that if π does not contain zero terms, this conjecture is true for $k = 3$, $n \geq 6$. The conjecture was confirmed in [5], [14], [15], [16] and [17].

Gould, Jacobson and Lehel [5] also proved that $\sigma(pK_2, n) = (p-1)(2n-2)+2$ for $p \geq 2$ and $\sigma(C_4, n) = 2\left[\frac{3n-1}{2}\right]$ for $n \geq 4$. They also pointed out that it would be nice to determine where in the range $3n-2$ to $4n-4$ the value $\sigma(K_4 - e, n)$ lies. Luo [18] characterized potentially C_k-graphic sequences for $k = 3, 4, 5$. Lai [7] determined $\sigma(K_4 - e, n)$ for $n \geq 4$. Yin, Li and Mao [21] determined $\sigma(K_{r+1} - e, n)$ for $r \geq 3$, $r+1 \leq n \leq 2r$ and $\sigma(K_5 - e, n)$ for $n \geq 5$. Yin and Li [20] gave a good method (the Yin-Li method) for determining the values $\sigma(K_{r+1} - e, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$ (in fact, Yin and Li [20] also determined the values $\sigma(K_{r+1} - ke, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$). After reading [20], using the Yin-Li method, Yin [22] determined $\sigma(K_{r+1} - K_3, n)$ for $n \geq 3r+5$, $r \geq 3$. Lai [8] determined $\sigma(K_5 - K_3, n)$ for $n \geq 5$. Lai [9] gave a lower bound for $\sigma(K_{t+p} - K_p, n)$.

Lai [10, 11] determined $\sigma(K_5 - C_4, n)$, $\sigma(K_5 - P_3, n)$, and $\sigma(K_5 - P_4, n)$ for $n \geq 5$. Determining $\sigma(K_{r+1} - H, n)$, where H is a tree on 4 vertices, is more useful than for a cycle on 4 vertices (for example, $C_4 \not\subset C_i$, but $P_3 \subset C_i$ for $i \geq 5$). So, after reading [20] and [22], using the Yin-Li method, Lai and Hu [12] determined $\sigma(K_{r+1} - H, n)$ for $n \geq 4r+10$, $r \geq 3$, $r+1 \geq k \geq 4$, where H is a graph on k vertices that contains a tree on 4 vertices but does not contain a cycle on 3 vertices, and also determined $\sigma(K_{r+1} - P_2, n)$ for $n \geq 4r+8$, $r \geq 3$. Using the Yin-Li method, Lai and Sun [13] determined $\sigma(K_{r+1} - (kP_2 \cup tK_2), n)$ for $n \geq 4r+10$, $r+1 \geq 3k+2t$, $k+t \geq 2$, $k \geq 1$, $t \geq 0$.

To date, the problem of determining $\sigma(K_{r+1} - H, n)$ for H not containing a cycle on 3 vertices and sufficiently large n has been solved. In this paper, using the Yin-Li method we prove the following two theorems.

Theorem 1.1. If $r \geq 4$ and $n \geq 5r+16$, then

$$\sigma(K_{r+1} - K_4, n) = \sigma(K_{r+1} - (K_4 - e), n) = \sigma(K_{r+1} - Z_4, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, & \text{if } n-r \text{ is even} \end{cases}$$

Theorem 1.2. If $n \geq 5r+19$, $r+1 \geq k \geq 5$, and $j \geq 5$, then

$$\sigma(K_{r+1} - Z, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

where Z is a graph on k vertices and j edges that contains a graph Z_4 but does not contain a cycle on 4 vertices.

There are a number of graphs on k vertices and j edges that contain a graph Z_4 but do not contain a cycle on 4 vertices.

2 Preparations

In order to prove our main results, we need the following notation and results.

Let $\pi = (d_1, \dots, d_n) \in NS_n$ and $1 \le k \le n$. Denote $\pi'_k = (d'_1, \dots, d'_{n-1})$, where

$$(d'_1, \dots, d'_{n-1}) = \begin{cases} (d_1 - 1, \dots, d_{k-1} - 1, d_{k+1} - 1, \dots, d_{d_k+1} - 1, d_{d_k+2}, \dots, d_n), & \text{if } d_k \ge k \\ (d_1 - 1, \dots, d_{d_k} - 1, d_{d_k+1}, \dots, d_{k-1}, d_{k+1}, \dots, d_n), & \text{if } d_k < k \end{cases}$$

Then π'_k is a rearrangement of the $n-1$ terms obtained by laying off d_k from π. The sequence π'_k is called the residual sequence.

Theorem 2.1 [20]. Let $n \ge r+1$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r+1} \ge r$. If $d_i \ge 2r - i$ for $i = 1, 2, \dots, r-1$, then π is potentially A_{r+1}-graphic.

Theorem 2.2 [20]. Let $n \ge 2r+2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r+1} \ge r$. If $d_{2r+2} \ge r-1$, then π is potentially A_{r+1}-graphic.

Theorem 2.3 [20]. Let $n \ge r+1$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r+1} \ge r-1$. If $d_i \ge 2r - i$ for $i = 1, 2, \dots, r-1$, then π is potentially $K_{r+1} - e$-graphic.

Theorem 2.4 [20]. Let $n \ge 2r+2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-1} \ge r$. If $d_{2r+2} \ge r-1$, then π is potentially $K_{r+1} - e$-graphic.

Theorem 2.5 [6]. Let $\pi = (d_1, \dots, d_n) \in NS_n$ and $1 \le k \le n$. Then $\pi \in GS_n$ if and only if $\pi'_k \in GS_{n-1}$.

Theorem 2.6 [3]. Let $\pi = (d_1, \dots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for any t, $1 \le t \le n-1$,

$$\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{j=t+1}^{n} \min\{t, d_j\}.$$

Theorem 2.7 [5]. If $\pi = (d_1, d_2, \dots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph such that the vertices of H have the largest degrees of π.

Theorem 2.8 [9]. If $n \ge p+t$, then

$$\sigma(K_{p+t} - K_p, n) \ge 2\left[\frac{(p+2t-3)n + p + 2t + 1 - pt - t^2}{2}\right].$$

Lemma 2.1 [22]. If $\pi = (d_1, d_2, \dots, d_n) \in NS_n$ is potentially $K_{r+1} - e$-graphic, then there is a realization G of π containing $K_{r+1} - e$ with the $r+1$ vertices $v_1, \dots, v_{r+1}$ such that $d_G(v_i) = d_i$ for $i = 1, 2, \dots, r+1$ and $e = v_r v_{r+1}$.

Lemma 2.2 [12]. Let $n \ge 2r+2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-2} \ge r$. If $d_{2r+2} \ge r-1$, then π is potentially $K_{r+1} - P_2$-graphic.

Lemma 2.3. Let $\pi = (d_1, \dots, d_n) \in GS_n$ and G be a realization of π. If $\epsilon(G[v_1, v_2, \dots, v_{r+1}]) \le \epsilon(K_{r+1}) - 1$, then there is a realization H of π such that $d_H(v_i) = d_i$ for $i = 1, 2, \dots, r+1$ and $v_r v_{r+1} \notin E(H)$.

The proof is similar to the proof of Lemma 2.1.

---

3 Proof of Main Results

Lemma 3.1. Let $n \geq 2r$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-1} \geq r$ and $d_{r+1} \geq r-1$. If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-2$, then π is potentially $K_{r+1} - e$-graphic.

Proof. We consider the following two cases.

Case 1: $d_{r+1} \geq r$.

If $d_{r-1} \geq r+1$, then π is potentially $K_{r+1} - e$-graphic by Theorem 2.3.

If $d_{r-1} = r$, then $d_{r-1} = d_r = d_{r+1} = r$. Suppose π is not potentially $K_{r+1} - e$-graphic. Let H be a realization of π; then $\epsilon(H[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 2$. Let $S = (d_1, d_2, \dots, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, \dots, d_n)$. By Theorem 2.1, S is potentially A_{r+1}-graphic (denote $S' = (d'_1, \dots, d'_n)$ as a rearrangement of the n terms of S; therefore $S' \in GS_n$ by Lemma 2.3, and S' satisfies the conditions of Theorem 2.1). Hence, there is a realization G of S with vertices $v_1, v_2, \dots, v_{r+1}$ ($d(v_i) = d_i$ for $i = 1, 2, \dots, r-1$, $d(v_r) = d_r + 1$, $d(v_{r+1}) = d_{r+1} + 1$) where the $r+1$ vertices of highest degree contain a K_{r+1}. Therefore, $G - v_{r+1}v_r$ is a realization of π. Thus, π is potentially $K_{r+1} - e$-graphic, which is a contradiction.

Case 2: $d_{r+1} = r-1$.

Then the residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off $d_{r+1} = r-1$ from π satisfies: $d'_1 \geq 2(r-1)-1$, ..., $d'_{r-2} \geq 2(r-1)-(r-2)$, $d'_{r-1} \geq r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - e$-graphic by $\{d_1 - 1, \dots, d_{r-1} - 1\} \subseteq \{d'_1, \dots, d'_{r-1}\}$ and Theorem 2.7.

Lemma 3.2. Let $n \geq 2r$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-2} \geq r+1$, $d_{r+1} \geq r$, and $d_{r-1} \geq d_{2r+2}$. If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$, then π is potentially A_{r+1}-graphic.

Proof. The residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 \geq 2(r-1)-1$, ..., $d'_{r-3} \geq 2(r-1)-(r-3)$, $d'_{r-2} \geq r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially A_{r+1}-graphic by $\{d_1 - 1, \dots, d_r - 1\} = \{d'_1, \dots, d'_r\}$ and Theorem 2.7.

Lemma 3.3. Let $n \geq 2r+2$, $r \geq 4$, and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-2} \geq r-1$ and $d_{r+1} \geq r-2$. If

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

and $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$, then π is potentially $K_{r+1} - Z_4$-graphic.

Proof. We consider the following two cases.

Case 1: $d_{r+1} \geq r-1$.

Subcase 1.1: $d_{r-1} \geq r+1$.

If $d_{r-2} \geq r+2$, then π is potentially $K_{r+1}-e$-graphic by Theorem 2.3. Hence, π is potentially $K_{r+1}-Z_4$-graphic.

If $d_{r-2} = r+1$, then $d_{r-3}-1 \geq d_{r-2}$. The residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 \geq 2(r-1)-1$, ..., $d'_{r-3} \geq 2(r-1)-(r-3)$, $d'_{r-2} \geq r-1$, $d'_r \geq (r-1)-1$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1}-e$-graphic. Therefore, π is potentially $K_{r+1}-Z_4$-graphic by $\{d_1-1, \dots, d_{r-3}-1\} \subseteq \{d'_1, \dots, d'_{r-1}\}$ and Lemma 2.1.

Subcase 1.2: $d_{r-1} \leq r$.

Then $d_{r-3}-1 \geq d_{r-1}$. The residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 \geq 2(r-1)-1$, ..., $d'_{r-3} \geq 2(r-1)-(r-3)$, $d'_{r-2} \geq r-1$, $d'_r \geq (r-1)-1$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1}-e$-graphic. Therefore, π is potentially $K_{r+1}-Z_4$-graphic by $\{d_1-1, \dots, d_{r-3}-1\} \subseteq \{d'_1, \dots, d'_{r-1}\}$ and Lemma 2.1.

Case 2: $d_{r+1} = r-2$.

If $d_{r-1} < d_{r-2}$:

When $d_{r-2} \geq r$, the residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off $d_{r+1} = r-2$ from π satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \dots, r-2$, (2) $d'_{r-3} \geq d_{r-3}-1 \geq 2(r-1)-[(r-1)-2]$, $d'_{r-2} \geq r-1$, and $d'_r = d_r \geq r-2$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1}-e$-graphic. Therefore, π is potentially $K_{r+1}-Z_4$-graphic by $\{d_1-1, \dots, d_{r-2}-1, d_{r-1}, d_r\} = \{d'_1, \dots, d'_r\}$ and Lemma 2.1.

When $d_{r-2} = r-1$, then $d_{r-1} = d_r = r-2$ and

$$\sigma(\pi) \leq (r-3)(n-1)+r-1+(r-2)(n-r+2) = (r-1)(n-1)-2(n-1)+(r-1)(n-r+3)-(n-r+2) = (r-1)(2n-r)-$$

Hence, $\pi = ((n-1)^{r-3}, (r-1)^1, (r-2)^{n-r+2})$ and $n-r$ is even. Clearly, π is potentially $K_{r+1}-Z_4$-graphic.

If $d_{r-1} = d_{r-2}$ and $d_{r-3} \geq d_r$, then π'_{r+1} satisfies: $d'_1 \geq 2(r-1)-1$, ..., $d'_{r-3} \geq d_{r-3}-1 \geq 2(r-1)-[(r-1)-2]$, $d'_{r-2} \geq r-1$, and $d'_r \geq r-2$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1}-e$-graphic. Therefore, π is potentially $K_{r+1}-Z_4$-graphic by $\{d_{r-1}, d_r, d_1-1, \dots, d_{r-2}-1\} = \{d'_1, \dots, d'_r\}$ and Lemma 2.1.

If $d_{r-1} = d_{r-2}$ and $d_{r-3} = d_r$, then $d_{r-3} = d_{r-2} = d_{r-1} = d_r \geq r+3$. Let H be a realization of π. Since $d_{r+1} = r-2$, there exist $i, j \leq r$ such that $v_{r+1}v_i, v_{r+1}v_j \notin E(H)$. Let $S = (d_1, d_2, \dots, d_i+1, \dots, d_j+1, \dots, d_r, d_{r+1}+2, \dots, d_n)$. By Theorem 2.1, S is potentially A_{r+1}-graphic (denote $S' = (d'_1, \dots, d'_n)$ as a rearrangement of the n terms of S; therefore $S' \in GS_n$ and S' satisfies the conditions of Theorem 2.1). Hence, there is a realization G of S with vertices $v_1, v_2, \dots, v_{r+1}$ ($d(v_t) = d_t$ for $t \neq i, j, r+1$, $d(v_i) = d_i+1$, $d(v_j) = d_j+1$, $d(v_{r+1}) = d_{r+1}+2$) where the

$r+1$ vertices of highest degree contain a K_{r+1}. Therefore, $G-\{v_{r+1}v_i, v_{r+1}v_j\}$ is a realization of π. Thus, π is potentially $K_{r+1}-Z_4$-graphic.

Lemma 3.4. Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-t} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1}-K_{1,t}$-graphic.

Proof. We consider the following two cases.

Case 1: If $d_{r-1} \geq r$, then π is potentially $K_{r+1}-e$-graphic by Theorem 2.4. Hence, π is potentially $K_{r+1}-K_{1,t}$-graphic.

Case 2: $d_{r-1} \leq r-1$, that is, $d_{r-1} = r-1$. Then $d_{r-1} = d_r = d_{r+1} = \dots = d_{2r+2} = r-1$ and π'_{r+1} satisfies $d'_{2(r-1)+2} = d'_r \geq r-1$. By Theorem 2.2, π'_{r+1} is potentially A_r-graphic. Therefore, π is potentially $K_{r+1}-K_{1,t}$-graphic by $\{d_1-1, \dots, d_{r-t}-1\} \subseteq \{d'_1, \dots, d'_{r-1}\}$ and Theorem 2.7.

Lemma 3.5. Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-4} \geq r$ and

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r)-3(n-r)-1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r)-3(n-r)-2, & \text{if } n-r \text{ is even} \end{cases}$$

If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1}-(P_2 \cup K_2)$-graphic.

Proof. We consider the following two cases.

Case 1: If $d_{r-2} \geq r$, then π is potentially $K_{r+1}-P_2$-graphic by Lemma 2.2. Hence, π is potentially $K_{r+1}-(P_2 \cup K_2)$-graphic.

Case 2: $d_{r-2} = r-1$.

Subcase 2.1: $d_{r-3} \geq r$. Then $d_{r-3} \geq d_r + 1 = d_{r+1} + 1 = r > r-1 = d_{r-2} = d_{r-1}$. Suppose π is not potentially $K_{r+1}-(P_2 \cup K_2)$-graphic. Let H be a realization of π; then $\epsilon(H[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$. Let $S = (d_1, d_2, \dots, d_{r-2}, d_{r-1}, d_r+1, d_{r+1}+1, \dots, d_n)$. By Theorem 2.4, S is potentially $K_{r+1}-e$-graphic (denote $S' = (d'_1, \dots, d'_n)$ as a rearrangement of the n terms of S; therefore $S' \in GS_n$ by Lemma 2.3, and S' satisfies the conditions of Theorem 2.4). Hence, there is a realization G of S with vertices $v_1, v_2, \dots, v_{r+1}$ ($d(v_i) = d_i$ for $i = 1, 2, \dots, r-1$, $d(v_r) = d_r+1$, $d(v_{r+1}) = d_{r+1}+1$) where the $r+1$ vertices of highest degree contain a $K_{r+1}-e$ and $e = v_{r-1}v_{r-2}$ by Lemma 2.1. Therefore, $G - v_{r+1}v_r$ is a realization of π. Thus, π is potentially $K_{r+1}-(P_2 \cup K_2)$-graphic, which is a contradiction.

Subcase 2.2: $d_{r-3} = r-1$. Then

$$\sigma(\pi) \leq (r-4)(n-1)+(r-1)(n-r+4) = (r-1)(n-1)-3(n-1)+(r-1)(n-r+1)+3(r-1) = (r-1)(2n-r)-3(n-r)$$

Since $\sigma(\pi) \geq (r-1)(2n-r)-3(n-r)-1$ when $n-r$ is odd and $\sigma(\pi) \geq (r-1)(2n-r)-3(n-r)-2$ when $n-r$ is even, we find that π must be one of the following sequences:

If $n-r$ is odd: - $((n-1)^{r-5}, (n-2)^1, (r-1)^{n-r+4})$ - $((n-1)^{r-4}, (r-1)^{n-r+3}, (r-2)^1)$

If $n-r$ is even: - $((n-1)^{r-4},(r-1)^{n-r+4})$ - $((n-1)^{r-6},(n-2)^2,(r-1)^{n-r+4})$ - $((n-1)^{r-5},(n-3)^1,(r-1)^{n-r+4})$ - $((n-1)^{r-5},(n-2)^1,(r-1)^{n-r+3},(r-2)^1)$ - $((n-1)^{r-4},(r-1)^{n-r+3},(r-3)^1)$ - $((n-1)^{r-4},(r-1)^{n-r+2},(r-2)^2)$

Clearly, π is potentially $K_{r+1}-(P_2\cup K_2)$-graphic.

Lemma 3.6. If $r\geq 4$ and $n\geq r+1$, then $\sigma(K_{r+1}-Z_4,n)\geq\sigma(K_{r+1}-K_4,n)$. Moreover,

$$\sigma(K_{r+1}-K_4,n)\geq\begin{cases}(r-1)(2n-r)-3(n-r)+1, & \text{if } n-r \text{ is odd}\\(r-1)(2n-r)-3(n-r)+2, & \text{if } n-r \text{ is even}\end{cases}$$

Proof. Obviously, for $r\geq 4$ and $n\geq r+1$, we have $\sigma(K_{r+1}-Z_4,n)\geq\sigma(K_{r+1}-K_4,n)$. By Theorem 2.8, for $r\geq 4$ and $n\geq r+1$,

$$\sigma(K_{r+1}-K_4,n)=\sigma(K_{4+(r-3)}-K_4,n)\geq 2\left[\frac{(4+2(r-3)-3)n+4+2(r-3)+1-4(r-3)-(r-3)^2}{2}\right].$$

Hence,

$$\sigma(K_{r+1}-K_4,n)\geq\begin{cases}(r-1)(2n-r)-3(n-r)+1, & \text{if } n-r \text{ is odd}\\(r-1)(2n-r)-3(n-r)+2, & \text{if } n-r \text{ is even}\end{cases}$$

Lemma 3.7. If $n\geq r+1$ and $r+1\geq k\geq 4$, then

$$\sigma(K_{r+1}-H,n)\geq\begin{cases}(r-1)(2n-r)-3(n-r)-1, & \text{if } n-r \text{ is odd}\\(r-1)(2n-r)-3(n-r)-2, & \text{if } n-r \text{ is even}\end{cases}$$

where H is a graph on k vertices that does not contain a cycle on 4 vertices.

Proof. Let

$$G=\begin{cases}K_{r-3}+\left(\frac{n-r+1}{2}+1\right)K_2, & \text{if } n-r \text{ is odd}\\K_{r-3}+\left(\frac{n-r+2}{2}+1\right)K_2, & \text{if } n-r \text{ is even}\end{cases}$$

Then G is a unique realization of

$$\begin{cases}((n-1)^{r-3},(r-2)^{n-r+3}), & \text{if } n-r \text{ is odd}\\((n-1)^{r-3},(r-2)^{n-r+2},(r-3)^1), & \text{if } n-r \text{ is even}\end{cases}$$

and G clearly does not contain $K_{r+1}-H$, where the symbol x^y means x repeats y times in the sequence. Thus $\sigma(K_{r+1}-H,n)\geq\sigma(\pi)+2$. Therefore,

$$\sigma(K_{r+1}-H,n)\geq\begin{cases}(r-1)(2n-r)-3(n-r)-1, & \text{if } n-r \text{ is odd}\\(r-1)(2n-r)-3(n-r)-2, & \text{if } n-r \text{ is even}\end{cases}$$

Proof of Theorem 1.1

According to Lemma 3.6 and the inequality $\sigma(K_{r+1}-K_4,n) \leq \sigma(K_{r+1}-(K_4-e),n) \leq \sigma(K_{r+1}-Z_4,n)$, it suffices to verify that for $n \geq 5r+16$,

$$\sigma(K_{r+1}-Z_4,n) \leq \begin{cases} (r-1)(2n-r)-3(n-r)+1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r)-3(n-r)+2, & \text{if } n-r \text{ is even} \end{cases}$$

We now prove that if $n \geq 5r+16$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r)-3(n-r)+1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r)-3(n-r)+2, & \text{if } n-r \text{ is even} \end{cases}$$

then π is potentially $K_{r+1}-Z_4$-graphic.

If $d_{r-3} \leq r-1$, then

$$\sigma(\pi) \leq (r-4)(n-1)+(r-1)(n-r+4) = (r-1)(n-1)-3(n-1)+(r-1)(n-r+4) = (r-1)(2n-r)-3(n-r) < (r-1)$$

which is a contradiction. Thus, $d_{r-3} \geq r$.

If $d_{r-2} \leq r-2$, then

$$\sigma(\pi) \leq (r-3)(n-1)+(r-2)(n-r+3) = (r-1)(n-1)-2(n-1)+(r-1)(n-r+3)-(n-r+3) = (r-1)(2n-r)-3(n-$$

which is a contradiction. Thus, $d_{r-2} \geq r-1$.

If $d_{r+1} \leq r-3$, then

$$\sigma(\pi) = \sum_{i=1}^{r+1} d_i + \sum_{i=r+2}^{n} d_i \leq \sum_{i=1}^{r+1} \min\{r, d_i\} + \sum_{i=r+2}^{n} d_i \leq (r-1)r+2(n-r)(r-3) = (r-1)(2n-r)-4(n-r) < (r-1)(2n$$

which is a contradiction. Thus, $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1}-Z_4$-graphic by Lemma 3.3 or Lemma 3.4. If $d_{2r+2} \leq r-2$ and there exists an integer i, $1 \leq i \leq r-3$, such that $d_i \leq 2r-i-1$, then

$$\sigma(\pi) \leq (i-1)(n-1)+(2r+1-i+1)(2r-i-1)+(r-2)(n+1-2r-2) = i^2+i(n-4r-2)-(n-1)+(2r-1)(2r+2)+(r-$$

Since $n \geq 5r+16$, it is easy to see that $i^2+i(n-4r-2)$, considered as a function of i, attains its maximum value when $i = r-3$. Therefore,

$$\sigma(\pi) \leq (r-3)^2+(n-4r-2)(r-3)-(n-1)+(2r-1)(2r+2)+(r-2)(n-2r-1) = (r-1)(2n-r)-3(n-r)-n+5r+16$$

which is a contradiction.

Thus,

$$\sigma(K_{r+1}-Z_4,n) \leq \begin{cases} (r-1)(2n-r)-3(n-r)+1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r)-3(n-r)+2, & \text{if } n-r \text{ is even} \end{cases}$$

for $n \geq 5r+16$.

Proof of Theorem 1.2

According to Lemma 3.7, it suffices to verify that for $n \geq 5r + 19$,

$$\sigma(K_{r+1} - Z, n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

We now prove that if $n \geq 5r + 19$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

then π is potentially $K_{r+1} - Z$-graphic.

If $d_{r-4} \leq r-1$, then

$$\sigma(\pi) \leq (r-5)(n-1)+(r-1)(n-r+5) = (r-1)(n-1)-4(n-1)+(r-1)(n-r+5) = (r-1)(2n-r)-4(n-r) < (r-1)$$

which is a contradiction. Thus, $d_{r-4} \geq r$.

If $d_{r-2} \leq r-2$, then

$$\sigma(\pi) \leq (r-3)(n-1)+(r-2)(n-r+3) = (r-1)(n-1)-2(n-1)+(r-1)(n-r+3)-(n-r+3) = (r-1)(2n-r)-3(n-$$

which is a contradiction. Thus, $d_{r-2} \geq r-1$.

If $d_{r+1} \leq r-3$, then

$$\sigma(\pi) = \sum_{i=1}^{r+1} d_i + \sum_{i=r+2}^{n} d_i \leq \sum_{i=1}^{r+1} \min\{r, d_i\} + \sum_{i=r+2}^{n} d_i \leq (r-1)r+2(n-r)(r-3) = (r-1)(2n-r)-4(n-r) < (r-1)(2n$$

which is a contradiction. Thus, $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - Z$-graphic by Lemma 3.3 or Lemma 3.5. If $d_{2r+2} \leq r-2$ and there exists an integer i, $1 \leq i \leq r-3$, such that $d_i \leq 2r-i-1$, then

$$\sigma(\pi) \leq (i-1)(n-1)+(2r+1-i+1)(2r-i-1)+(r-2)(n+1-2r-2) = i^2+i(n-4r-2)-(n-1)+(2r-1)(2r+2)+(r-$$

Since $n \geq 5r+19$, it is easy to see that $i^2+i(n-4r-2)$, considered as a function of i, attains its maximum value when $i = r-3$. Therefore,

$$\sigma(\pi) \leq (r-3)^2+(n-4r-2)(r-3)-(n-1)+(2r-1)(2r+2)+(r-2)(n-2r-1) = (r-1)(2n-r)-3(n-r)-n+5r+16$$

which is a contradiction.

Thus,

$$\sigma(K_{r+1} - Z, n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

for $n \geq 5r + 19$.

Acknowledgment

The authors thank the referees for many helpful comments.

References

[1] B. Bollobás, *Extremal Graph Theory*, Academic Press, London, 1978.
[2] P. Erdős, On sequences of integers no one of which divides the product of two others and some related problems, *Izv. Naustno-Issl. Mat. i Meh. Tomsk* 2 (1938), 74-82.
[3] P. Erdős and T. Gallai, Graphs with given degrees of vertices, *Math. Lapok* 11 (1960), 264-274.
[4] P. Erdős, M.S. Jacobson and J. Lehel, Graphs realizing the same degree sequences and their respective clique numbers, in *Graph Theory, Combinatorics and Application*, Vol. 1 (Y. Alavi et al., eds.), John Wiley and Sons, Inc., New York, 1991, 439-449.
[5] R.J. Gould, M.S. Jacobson and J. Lehel, Potentially G-graphic degree sequences, in *Combinatorics, Graph Theory and Algorithms*, Vol. 2 (Y. Alavi et al., eds.), New Issues Press, Kalamazoo, MI, 1999, 451-460.
[6] D.J. Kleitman and D.L. Wang, Algorithm for constructing graphs and digraphs with given valences and factors, *Discrete Math.* 6 (1973), 79-88.
[7] Chunhui Lai, A note on potentially $K_4 - e$ graphical sequences, *Australasian J. of Combinatorics* 24 (2001), 123-127.
[8] Chunhui Lai, An extremal problem on potentially $K_{p,1,1}$-graphic sequences, *Discrete Mathematics and Theoretical Computer Science* 7 (2005), 75-80.
[9] Chunhui Lai, Potentially $K_{p,1,1,\ldots,1}$-graphic degree sequences, *J. Zhangzhou Teachers College* 17(4) (2004), 11-13.
[10] Chunhui Lai, An extremal problem on potentially $K_m - C_4$-graphic sequences, *Journal of Combinatorial Mathematics and Combinatorial Computing* 61 (2007), 59-63.
[11] Chunhui Lai, An extremal problem on potentially $K_m - P_k$-graphic sequences, accepted by *International Journal of Pure and Applied Mathematics*.
[12] Chunhui Lai and Lili Hu, An extremal problem on potentially $K_{r+1} - H$-graphic sequences, accepted by *Ars Combinatoria*.
[13] Chunhui Lai and Yuzhen Sun, An extremal problem on potentially $K_{r+1} - (kP_2 \cup tK_2)$-graphic sequences, *International Journal of Applied Mathematics & Statistics* 14 (2009), 30-36.
[14] Jiong-Sheng Li and Zi-Xia Song, An extremal problem on the potentially P_k-graphic sequences, *Discrete Math.* 212 (2000), 223-231.
[15] Jiong-Sheng Li and Zi-Xia Song, The smallest degree sum that yields potentially P_k-graphical sequences, *J. Graph Theory* 29 (1998), 63-72.
[16] Jiong-sheng Li and Zi-Xia Song, On the potentially P_k-graphic sequences, *Discrete Math.* 195 (1999), 255-262.
[17] Jiong-sheng Li, Zi-Xia Song and Rong Luo, The Erdős-Jacobson-Lehel conjecture on potentially P_k-graphic sequence is true, *Science in China (Series A)*

41(5) (1998), 510-520.
[18] Rong Luo, On potentially C_k-graphic sequences, *Ars Combinatoria* 64 (2002), 301-318.
[19] P. Turán, On an extremal problem in graph theory, *Mat. Fiz. Lapok* 48 (1941), 436-452.
[20] Jianhua Yin and Jiongsheng Li, Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size, *Discrete Math.* 301 (2005), 218-227.
[21] Jianhua Yin, Jiongsheng Li and Rui Mao, An extremal problem on the potentially $K_{r+1} - e$-graphic sequences, *Ars Combinatoria* 74 (2005), 151-159.
[22] Mengxiao Yin, The smallest degree sum that yields potentially $K_{r+1} - K_3$-graphic sequences, *Acta Math. Appl. Sin. Engl. Ser.* 22 (2006), no. 3, 451-456.

Note: Figure translations are in progress. See original paper for figures.