

# A Method for Monotonic Interval Estimation and Non-Zero Real Root Solving of Series Functions Based on Lagrange Inversion

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## Abstract

This paper utilizes the Lagrange inversion series method to study the monotonic intervals in a neighborhood of  $x=0$  for series of the form  $y=a_1x+a_2x^2+a_3x^3+\dots+a_nx^n+\dots$ , and then presents a computational method for one non-zero real root for more general series equations.

## Full Text

### A Lagrange Inversion-Based Method for Estimating Monotone Intervals and Solving Non-zero Real Roots of Series Functions

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## Abstract

This paper employs the Lagrange inversion series method to investigate the monotone intervals in the neighborhood of series functions of the form  $y = a_{1x} + a_{2x}^2 + \dots + a_{nx}^n + \dots$ . Subsequently, for more general series equations, we present a computational method for determining one of their non-zero real roots.

**Keywords:** Lagrange inversion series; Real root

## 1. Monotone Intervals and Real Roots of Derivatives for Series Equations with Non-zero Leading Coefficient

We begin by considering the series:

$$y = a_{1x} + a_{2x}^2 + \dots + a_{nx}^n + \dots \quad (1)$$

For its convergence domain, the Lagrange inversion series formula is given by. Let us denote  $\psi$ , then the inversion series of equation (1) satisfies  $x = \dots$ . For notational simplicity, we express the coefficients (which are constants) as  $c_n$ . Therefore, the inversion series of equation (1) can be simplified as:

$$x = h(y) = c_{1y} + c_{2y}^2 + \dots + c_{ny}^n + \dots \quad (4)$$

Since our focus is on the monotone intervals and real root behavior of the inversion series, we assume that the convergence domain of equation (1) with respect to  $x$  contains the convergence domain of  $x$  determined by equation (4).

**Lemma 1:** If a series of the form (1) is analytic in a neighborhood of and possesses a Lagrange inversion series (4), then  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$  exists.

*Proof:* As the series is analytic in a neighborhood of and possesses a Lagrange inversion series, there exists a neighborhood [2] satisfying the required conditions. Because the series converges, when  $y$  is sufficiently close to 0, we have  $\lim_{n \rightarrow \infty} \frac{c_{n+1}y^{n+1}}{c_n^n y} = \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} y$ . Consequently,  $\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}$  must exist.

In a neighborhood of, with  $y$  fixed, consider the ratio of consecutive terms in the inversion series (4):  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}y^{n+1}}{c_n^n y} \right| = |y| \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$ . Three cases arise: 1)  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = 0$ ; 2)  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = \infty$ ; 3)  $0 < \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| < \infty$ . The second case leads to divergence for all  $y$  except  $y = 0$  and is therefore excluded. Combining the first and third cases yields the existence of the limit.

**Lemma 2:** If  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| > 0$  and  $|y| < \left| \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} \right|$ , then the ratio between consecutive terms of the series  $x = h(y)$  exceeds 1. Conversely, when  $|y| > \left| \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} \right|$ , we have  $|y| \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| > 1$ . Under these conditions,  $x$  and  $y$  have a one-to-one correspondence satisfying  $y \rightarrow 0$  as  $x \rightarrow 0$ .

Now consider the derivative of the series (1):

$$y' = a_1 + 2a_{2x} + \dots + na_{nx}^{n-1} + \dots$$

**Conclusion 1:** A non-zero real root of the derivative is given by  $\left( \left| \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} \right|, h \left( \left| \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} \right| \right) \right)$ .

*Proof:* By Lemma 2, we know that a real root of the derivative occurs at this point (assuming the derivative changes sign). Suppose there exists another real root satisfying  $|y| < \left| \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} \right|$ . Then we would have  $|y| = 2 \left| \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} \right| = 0$ , which leads to a contradiction as  $x = h(y)$  would be discontinuous in the convergence domain. Therefore, the identified root is indeed the smallest real root.

**Conclusion 2:** For the inversion series (4), when  $y = \psi(x) \in \left( - \left| \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} \right|, \left| \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} \right| \right)$ , the series converges and is monotone.

*Proof:* 1) First, we prove convergence. When  $y \in \left(-\left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|, \left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|\right)$ , we have  $0 < |y| \lim_{n \rightarrow \infty} \left|\frac{c_{n+1}}{c_n}\right| < 1$ . By the limit definition, there exists  $\epsilon > 0$  such that for sufficiently large  $n$ ,  $\left|\frac{c_{n+1}y^{n+1}}{c_n^n}\right| < 1 - \epsilon$ . Therefore, the series  $x = h(y)$  converges absolutely.

When  $y \in \left[0, \left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|\right)$ , the terms have the same sign and the ratio between consecutive terms satisfies  $0 < \left|\frac{c_{n+1}y^{n+1}}{c_n^n}\right| < 1 - \epsilon$ , ensuring convergence.

When  $y \in \left(-\left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|, 0\right]$ , the terms alternate in sign and form an alternating series with decreasing magnitude, which converges by the Leibniz criterion [4].

- 2) Next, we prove monotonicity. If  $x = h(y)$  were not monotone on this interval, there would exist  $y_1, y_2$  such that  $h(y_1) = h(y_2)$  with  $y_1 \neq y_2$ , implying  $\psi(x)$  has a discontinuity in its convergence domain, which contradicts the analytic properties.

**Computational Example:** Consider the quadratic equation  $x^2 + 2x - y = 0$ . Its Lagrange inversion series is:

$$x = y - y^2 + 2y^3 - 5y^4 + \dots + (-1)^{n-1} \frac{(2n-2)!}{(n-1)!n!} y^n + \dots$$

We compute  $\lim_{n \rightarrow \infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n \rightarrow \infty} \frac{(2n)!/n!(n+1)!}{(2n-2)!/(n-1)!n!} = 4$ . Therefore, the monotone interval containing the zero is  $y \in [0, 1/4)$ . The derivative of the series  $x^2 + 2x$  is  $2x + 2$ , and its non-zero real root is at  $x = -1$ , corresponding to  $y = -1$ .

## 2. Monotone Intervals and Real Roots of Derivatives for Series Equations with Zero Leading Coefficient

Now consider the case where the leading coefficient  $a_1 = 0$  in equation (1), giving a series of the form:

$$y = a_{2x}^2 + a_{3x}^3 + \dots + a_{nx}^n + \dots \quad (6)$$

For the convergence domain, we can transform this into two series:

$$x = \sqrt{y/a_2 + \dots} \quad \text{and} \quad x = -\sqrt{y/a_2 + \dots}$$

Applying the inversion formula (2) to both expressions yields:

$$x = h_1(y) = c_{1y}^{1/2} + c_{2y} + \dots \quad \text{and} \quad x = h_2(y) = -c_{1y}^{1/2} - c_{2y} - \dots$$

The coefficients  $c_n$  are determined by equation (3). From Conclusion 2, when  $y \in \left(-\left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|, \left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|\right)$ , both series converge. A monotone interval containing the zero is therefore  $\left(-\left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|, \left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|\right)$ .

For the derivative of series (6), the non-zero real root is given by  $\lim_{n \rightarrow \infty} \left|\frac{c_n}{c_{n+1}}\right|$ .

### 3. Monotone Intervals Near Zeros and Real Root Solving for Series Equations

Based on the preceding analysis, for any given series equation without constant term:

$$y = a_{1x} + a_{2x}^2 + \dots + a_{nx}^n + \dots \quad (13)$$

If it possesses a Lagrange inversion series of the form:

$$x = h(y) = c_{1y} + c_{2y}^2 + \dots + c_{ny}^n + \dots$$

Then under the condition  $c_1 \neq 0$ : 1) Its monotone interval is  $y \in \left(-\left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|, \left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|\right)$ . 2) Its non-zero real root can be determined from the inversion series of the derivative series.

If  $c_1 = 0$ : 1) Its monotone interval is  $y \in \left(-\left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|, \left|\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}\right|\right)$ . 2) Its non-zero real root can be determined from the inversion series of the transformed series.

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*Note: Figure translations are in progress. See original paper for figures.*

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