Note on "Vacuum stability of a general scalar potential of a few fields"

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Abstract

The purpose of this letter is to point out that some conclusions in the paper (Eur. Phys. J. C { bf 76}, 324(2016)) are incomplete, and to give complete and improved conclusions. The analytic necessary and sufficient conditions are given for the boundedness-from-below conditions of general scalar potentials of two real scalar fields $ phi_1$ and $ phi_2$ and the Higgs bonson $ mathbf{H}$.

Full Text

Preamble

Note on "Vacuum stability of a general scalar potential of a few fields"

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Abstract

This note provides analytic necessary and sufficient conditions for the boundedness-from-below of general scalar potentials involving two real scalar fields $_1$ and $_2$ and the Higgs boson H.

Keywords: Boundedness-from-below; Positive definiteness; Scalar potentials; Analytical expression.

Introduction

Kannike [**?**] presented the boundedness-from-below conditions for general scalar potentials of two real scalar fields $_1$ and $_2$ and the Higgs boson H, given by:

$$V(\phi_1, \phi_2, |H|) = \lambda_H|H|^4+\lambda_{H20}|H|^2\phi_1^2+\lambda_{40}\phi_1^4+\lambda_{31}\phi_1^3\phi_2+\lambda_{H11}|H|^2\phi_1\phi_2+\lambda_{H02}|H|^2\phi_2^2+\lambda_{13}\phi_1\phi_2^3+\lambda_{04}\phi_2^4+\lambda_{22}\phi_1^2\phi$$

This is equivalent to establishing analytic necessary and sufficient conditions for $V(\phi_1, \phi_2, |H|) > 0$ for all ϕ_1, ϕ_2, H.

For two real scalar fields ϕ_1 and ϕ_2 and the Higgs boson H, a general scalar potential $V(\phi_1, \phi_2, |H|)$ can be rewritten as:

$$V(\phi_1, \phi_2, |H|) = \lambda_H |H|^4 + M^2(\phi_1, \phi_2)|H|^2 + V(\phi_1, \phi_2)$$

where

$$M^2(\phi_1, \phi_2) = \lambda_{H20}\phi_1^2 + \lambda_{H02}\phi_2^2 + \lambda_{H11}\phi_1\phi_2$$

and

$$V(\phi_1, \phi_2) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4$$

Applying the well-known positivity conditions for the quadratic polynomial $p(t) = at^2 + bt + c$ with $t = |H|^2 \geq 0$ (a classical result established centuries ago), we have $V(\phi_1, \phi_2, |H|) > 0$ for all ϕ_1, ϕ_2, H (with $a = \lambda_H > 0$) if and only if for all ϕ_1, ϕ_2:

$$c = V(\phi_1, \phi_2) > 0$$

and either

$$b = M^2(\phi_1, \phi_2) \geq 0$$

or

$$4ac - b^2 = 4\lambda_H V(\phi_1, \phi_2) - (M^2(\phi_1, \phi_2))^2 > 0$$

It is evident that $M^2(\phi_1, \phi_2)$ is a quadratic form in the two variables ϕ_1, ϕ_2. Consequently, the inequality $M^2(\phi_1, \phi_2) \geq 0$ is equivalent to the positive semi-definiteness of its coefficient matrix. By Sylvester's criterion, $M^2(\phi_1, \phi_2) \geq 0$ if and only if:

$$\lambda_{H20} \geq 0, \quad \lambda_{H02} \geq 0, \quad \lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 \geq 0$$

Thus, these conditions differ from Eqs.~(54) and (55) of Kannike [**?**].

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2 Boundedness-from-below Conditions

We now correct this mistake and present the analytic necessary and sufficient conditions for the boundedness from below of the scalar potential for two real scalar fields ϕ_1 and ϕ_2 and the Higgs doublet H.

From the conclusion in Eq.~(3), we first require the analytic necessary and sufficient conditions for $V(\phi_1, \phi_2) > 0$, where:

$$V(\phi_1, \phi_2) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4$$

It is clear that the discriminant $D \geq 0$ is a necessary condition for $V(\phi_1, \phi_2) > 0$. Such positivity conditions can be traced back to works by Rees [?], Lazard [?], Gadem and Li [?], Ku [?], and Jury and Mansour [?]. In 2005, Wang and Qi [?] improved upon these proofs and provided complete analytic necessary and sufficient conditions. For further applications of these results, see also Song and Qi [?]. Specifically, for all $(\phi_1, \phi_2) \neq (0, 0)$, the binary quartic homogeneous polynomial in Eq.~(5) satisfies $V(\phi_1, \phi_2) > 0$ if and only if one of the following holds:

$$\lambda_{40} > 0, \lambda_{04} > 0, D = 0, G = 0, R = 0 \text{ and } Q > 0;$$

$$D > 0 \text{ and either } Q \geq 0 \text{ or } R > 0;$$

where

$$I = \lambda_{40}\lambda_{04} - \lambda_{31}\lambda_{13} + \lambda_{40}\lambda_{22}\lambda_{04}$$

$$J = \lambda_{40}\lambda_{22}\lambda_{04} - \lambda_{40}\lambda_{13}^2 - \lambda_{31}^2\lambda_{04}$$

$$D = I^3 - 27J^2$$

$$R = \lambda_{40}\lambda_{04} - \lambda_{31}\lambda_{13}$$

$$Q = \lambda_{40}\lambda_{22}\lambda_{04} - \lambda_{31}^2\lambda_{04} - \lambda_{40}\lambda_{13}^2$$

Recently, Qi, Song, and Zhang [?] provided an alternative necessary and sufficient condition distinct from the above results in Eq.~(6).

Next, we present the revised version of conclusion Eq.~(68) from Kannike [?]. Let:

$$V'(\phi_1, \phi_2) = 4\lambda_H V(\phi_1, \phi_2) - (M^2(\phi_1, \phi_2))^2$$

We now establish the condition $V'(\phi_1, \phi_2) > 0$. Expanding $V'(\phi_1, \phi_2)$ yields:

$$V'(\phi_1, \phi_2) = (4\lambda_{40}\lambda_H - \lambda_{H20}^2)\phi_1^4 + (4\lambda_H\lambda_{31} - 2\lambda_{H20}\lambda_{H11})\phi_1^3\phi_2 + (4\lambda_H\lambda_{22} - 2\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2)\phi_1^2\phi_2^2 + (4\lambda_H\lambda_{13} - 2\lambda_{$$

with the primed coefficients defined as:

$$\lambda'_{40} = 4\lambda_{40}\lambda_H - \lambda_{H20}^2$$

$$\lambda'_{31} = 4\lambda_H\lambda_{31} - 2\lambda_{H20}\lambda_{H11}$$

$$\lambda'_{22} = 4\lambda_H\lambda_{22} - 2\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2$$

$$\lambda'_{13} = 4\lambda_H\lambda_{13} - 2\lambda_{H02}\lambda_{H11}$$

$$\lambda'_{04} = 4\lambda_{04}\lambda_H - \lambda_{H02}^2$$

In terms of these coefficients, we define the following quantities:

$$I' = \lambda'_{40}\lambda'_{04} - \lambda'_{31}\lambda'_{13} + \lambda'_{40}\lambda'_{22}\lambda'_{04}$$

$$J' = \lambda'_{40}\lambda'_{22}\lambda'_{04} - \lambda'_{40}\lambda'^2_{13} - \lambda'^2_{31}\lambda'_{04}$$

$$D' = {I'}^3 - 27{J'}^2$$
$$R' = \lambda'_{40}\lambda'_{04} - \lambda'_{31}\lambda'_{13}$$
$$Q' = \lambda'_{40}\lambda'_{22}\lambda'_{04} - \lambda'^2_{31}\lambda'_{04} - \lambda'_{40}\lambda'^2_{13}$$

Applying the conclusion from Eq.~(6), we have $V'(\phi_1, \phi_2) > 0$ for all $(\phi_1, \phi_2) \neq (0, 0)$ if and only if one of the following holds:

$$\lambda'_{40} > 0, \lambda'_{04} > 0, D' = 0, G' = 0, R' = 0 \text{ and } Q' > 0;$$

$$D' > 0 \text{ and either } Q' \geq 0 \text{ or } R' > 0.$$

Combining Eq.~(3) with Eqs.~(6), (4), and (7), we establish the analytic necessary and sufficient condition for the boundedness from below of the scalar potential for two real scalar fields ϕ_1 and ϕ_2 and the Higgs doublet H. Specifically, $V(\phi_1, \phi_2, |H|) > 0$ for all ϕ_1, ϕ_2, H with $(\phi_1, \phi_2, H) \neq (0, 0, 0)$ if and only if:

$$\lambda_H > 0, \quad \lambda_{40} > 0, \quad \lambda_{04} > 0$$

and **either**

(i) $D = 0$, $G = 0$, $R = 0$ and $Q > 0$; **or** $D > 0$ and either $Q \geq 0$ or $R > 0$;

and **either**

(ii) $\lambda_{H20} \geq 0$, $\lambda_{H02} \geq 0$, and $4\lambda_{H20}\lambda_{H02} - \lambda^2_{H11} \geq 0$;

(iii) or $4\lambda_{40}\lambda_H - \lambda^2_{H20} > 0$, $4\lambda_{04}\lambda_H - \lambda^2_{H02} > 0$ and either $D' = 0$, $G' = 0$, $R' = 0$ and $Q' > 0$; or $D' > 0$ and $Q' \geq 0$, or $R' > 0$.

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