Proof of the Velocity and Acceleration Theorem for the Composite Motion of a Point Based on Rotation Matrices

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Abstract

Existing textbooks feature relatively complex proofs of the velocity and acceleration composition theorems for point motion, posing significant challenges to classroom instruction and student learning. This paper proposes a novel proof methodology based on rotation transformation matrices, intended to facilitate students' mastery, comprehension, and application of the velocity and acceleration composition theorems. By employing both the composition of motion approach and the rotation transformation matrix approach, analytical expressions relating a point's absolute motion, transport motion, and relative motion are established. Taking the first time derivative of these analytical expressions yields the velocity composition theorem for planar motion, parallel translation, and fixed-axis rotation. Subsequently, taking the second time derivative of the analytical expressions yields the acceleration composition theorem for planar motion, parallel translation, and fixed-axis rotation. The proof process presented herein is characterized by concise and coherent mathematical reasoning, clear and distinct physical concepts, and is highly suitable for pedagogical purposes.

Full Text

Preamble

Proof of Theorems of Composition of Velocity and Acceleration of a Particle Based on Rotation Matrices

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Abstract

The proofs of the theorems of velocity and acceleration for synthetic motions of points in existing textbooks are complicated, which creates significant difficulties for classroom teaching and student learning. This paper proposes a new proof method based on rotation transformation matrices to help students grasp, understand, and apply the velocity and acceleration synthesis theorems more quickly. Using the method of composition of motion and rotational transformation matrices, this paper establishes analytical relationships for the absolute, transport, and relative motions of a particle. Taking the first-order time derivative of these analytical relationships yields the velocity composition theorem for points in plane motion, translation, and rotation about a fixed axis. Further, taking the second-order time derivative yields the acceleration composition theorem for these same motion types. The proof process in this paper is mathematically simple and fluent, with clear and concise physical concepts, making it highly suitable for teaching.

Keywords: Rotational transformation matrix method, translation, rotation about a fixed axis, plane motion, theorem of composition of the velocity and acceleration of a particle

The theorem of composition of motion for points constitutes the primary content of kinematics in theoretical mechanics [1]. This theorem provides an effective and convenient means for analyzing complex composite motions and represents an important method for analyzing and solving problems related to the velocity and acceleration of moving objects. The composition of motion for a point involves six elements: (1) selecting a point M on the object as the moving point, (2) selecting a coordinate system fixed to the moving object as the moving frame, (3) selecting a coordinate system fixed to the ground as the fixed frame, (4) the motion of the moving point relative to the fixed frame is the absolute motion, (5) the motion of the moving point relative to the moving frame is the relative motion, and (6) the motion of the moving frame relative to the fixed frame is the transport motion. References [2-4] derived the velocity and acceleration composition theorems for points based on vector methods and Poisson relations.

The composition of motion for points has long been a challenging topic and point of debate in theoretical mechanics instruction, attracting considerable attention from scholars. Reference [5] compared and discussed the characteristics and applications of generalized coordinate methods and transport point methods. Reference [6] provided a simple and clear proof of the velocity and acceleration composition theorems by starting from the relationship between the absolute position vector of a moving point and that of an arbitrary point on the moving frame. Reference [7] employed arc coordinates to intuitively describe relative motion and derived the velocity and acceleration composition theorems. Reference [8] proposed an analytical method for representing transport velocity and acceleration, deriving the composition theorems without introducing the concept of relative derivatives. Reference [9] reviewed the history of Coriolis force and concluded that Coriolis inertial force preceded the acceleration composition theorem. Reference [10] introduced the concept of coupling displacement to improve the geometric derivation of the velocity composition theorem for composite motion commonly used in current textbooks. Reference [11] used the transport point tracking method to derive the velocity and acceleration composition theorems, offering clear concepts and physical interpretation that facilitate understanding. Reference [12] identified errors in the derivation of the velocity composition theorem in Voronkov's theoretical mechanics textbook and provided a mathematically correct derivation based on clarified concepts. Reference [13] simply and clearly derived the velocity and acceleration composition theorems for arbitrary points on a rigid body using matrix methods. Reference [14] proposed a novel method based on complex vectors and polar coordinates to derive the acceleration composition theorem for points when the transport motion is plane motion. Reference [15] presented a set of fast solution formulas for composite motion based on matrix transformation methods. In summary, scholars have explored various approaches—including vector methods, transport point methods, geometric methods, tracking methods, and coupling displacement methods—to develop intuitive, accurate, and convenient proofs of the composition of motion theorems from diverse perspectives.

This paper employs the coordinate transformation matrix A as a mathematical tool to define the analytical vector coordinate transformation relationship between the moving and fixed frames in space. From this analytical expression, the velocity composition theorem is obtained through first-order differentiation of the vector relationship, and the acceleration composition theorem through second-order differentiation. This method fundamentally reveals the sources of complexity and difficulty in composite motion, offering a novel perspective for examining the intrinsic mechanism of point composition of motion.

1. Relationship Between Absolute, Transport, and Relative Vectors

As shown in Figure 1 [Figure 1: see original paper], we establish a spatial coordinate system for the composition of motion of a point. Let M be the moving point, the moving coordinate system be , the fixed coordinate system be , the absolute motion be the motion of the moving point relative to the fixed frame, the relative motion be the motion of the moving point relative to the moving frame, and the transport motion be the motion of the moving frame relative to the fixed frame with rotation angle . The relationship between the absolute and relative motions of the moving point is given by:

The absolute position vector represents the displacement vector of point M relative to the fixed frame. The first time derivative of the absolute position vector represents the absolute velocity of the moving point, and the second time derivative represents the absolute acceleration.

The transport position vector represents the displacement vector of the moving frame's origin. The first time derivative of the transport position vector is the first transport velocity when the transport motion is translation, and the second time derivative is the first transport acceleration when the transport motion is translation.

The relative position vector's first time derivative is the first relative velocity, and its second time derivative is the first relative acceleration.

The rotation angle of the moving frame relative to the fixed frame is the transport rotation angle. The first time derivative of is the angular velocity vector of the moving frame's rotation, and the second time derivative is the angular acceleration vector of the moving frame's rotation.

Let the rotation transformation matrix A be the transformation matrix for rotation about the axis. Taking the first derivative of the rotation matrix yields the tangential rotation matrix, and taking the second derivative yields the normal rotation transformation matrix.

This section, based on the rotation transformation matrix, provides analytical relationships for absolute displacement, transport displacement, rotation matrix, and relative displacement; defines absolute, relative, and transport velocities; obtains definitions for absolute, relative, and transport accelerations; and also defines rotation angle, angular velocity, and rotational acceleration.

2. Proof of the Velocity Composition Theorem

According to the definition of velocity, the absolute velocity vector of moving point M is:

Expanding the second term on the right side of equation (14) yields:

From equation (15), the second transport velocity for transport motion as rotation about a fixed axis is:

Therefore, the total transport velocity is:

Equation (17) shows that the transport acceleration comprises two parts: the first part, , originates from the translational motion of the moving frame's origin, while the second part, , originates from the rotational motion of the moving frame. The geometric relationship of transport velocity is shown in Figure 2 Figure 2: see original paper.

Comprehensive analysis yields the relative velocity:

Combining equations (14)-(18), we obtain the velocity composition theorem for points when the transport motion is plane motion:

where the absolute velocity magnitude is | | and direction is along ; the first transport velocity magnitude is | | and direction is along ; the second transport velocity magnitude is | | and direction forms an angle with the first relative velocity. The geometric relationship of the velocity composition theorem (19) for plane transport motion is shown in Figure 2(a).

We now discuss two special cases of the velocity composition theorem (19). When the transport motion is pure translation, the corresponding rotation angle is constant, making the second transport velocity equal to zero:

Consequently, the velocity composition theorem (19) reduces to:

Thus, equation (22) represents the velocity composition theorem for translational transport motion, with its geometric relationship shown in Figure 2(b).

When the transport motion is pure rotation about a fixed axis, the corresponding transport position vector is constant:

In this case, the first transport velocity is zero, while the second transport velocity remains non-zero due to non-zero angular velocity. This yields another degenerate case of the velocity composition theorem (19):

Therefore, equation (25) represents the velocity composition theorem for rotational transport motion about a fixed axis, with the geometric relationship illustrated in Figure 2(c). Comparing the derived velocity composition theorem (19) with the proof process in textbook [1] reveals that equation (11) in this paper clearly demonstrates the essential origin of transport velocity and does not require introducing the concept of relative derivatives. From a mathematical perspective, the derivation process appears remarkably smooth and natural.

3. Proof of the Acceleration Composition Theorem

According to the definition of acceleration, the absolute acceleration of moving point M is:

Expanding the second term on the right side of equation (26) yields:

where the acceleration obtained by differentiating the transport velocity with respect to time includes: , the tangential third transport acceleration due to rotational transport motion; , the normal second transport acceleration due to rotational transport motion; , the first Coriolis acceleration generated by transport motion; and , the second Coriolis acceleration generated by relative motion.

The acceleration obtained by differentiating the relative velocity with respect to time includes: , the relative acceleration.

Combining equations (26)-(28), we obtain the acceleration composition theorem for points when the transport motion is plane motion:

where the absolute acceleration magnitude is | | and direction is along ; the first transport acceleration magnitude is | | and direction is along ; the third transport acceleration magnitude is | | and direction is along ; the second transport acceleration magnitude is | | and direction is perpendicular to the plane determined by and . Figure 3(a) illustrates the geometric relationship of the acceleration composition theorem (30).

The total transport acceleration is:

Correspondingly, Figure 3(b) shows the geometric relationship of the transport acceleration formula (31).

The total Coriolis acceleration is:

Figure 3(c) shows the geometric relationship of the Coriolis acceleration formula (32).

We now discuss special cases of the acceleration composition theorem (30). The first case is translational transport motion. When the transport motion is translation, the corresponding rotation angle is constant, resulting in zero angular velocity and angular acceleration:

Substituting equation (34) into (28), the second and third transport accelerations vanish. Substituting equation (34) into (29), the second Coriolis acceleration also vanishes. Comprehensive analysis shows that the acceleration composition theorem for translational transport motion becomes:

In equation (31), the first transport acceleration arises from the translational motion of the moving frame relative to the fixed frame, while the relative acceleration results from the motion of the moving point relative to the moving frame; all other acceleration terms degenerate to zero. Although the acceleration theorem formula (30) matches the expression in textbook [1], the formulas in this paper more clearly reveal the origins of transport acceleration and are easier to understand. Figure 3(d) illustrates the geometric relationship of this acceleration composition theorem.

When the transport motion is pure rotation about a fixed axis, the corresponding transport position vector is constant:

In this case, the first transport acceleration is zero:

Substituting equation (37) into the acceleration composition theorem (30) yields the acceleration composition theorem for rotational transport motion:

It should be noted that in equation (34), the transport acceleration partially originates from transport motion, while the second Coriolis acceleration and third transport acceleration both arise from the fixed-axis rotation of the transport motion, and the first Coriolis acceleration originates from relative motion. Figure 3(e) shows the geometric relationship of the acceleration composition theorem for rotational transport motion. Comparing theorem (38) with textbook [1] demonstrates that the expressions in this paper clearly explain the origins of transport acceleration, relative acceleration, and Coriolis acceleration.

This paper presents a proof method for the velocity and acceleration composition theorems based on transformation matrix A, yielding the following main conclusions: (1) Unlike vector relationship proofs in textbooks, the introduction of rotation transformation matrix A in this paper clearly reveals the essential origins of transport velocity and relative velocity. Simultaneously, the rotation transformation matrix A also reveals the essential origins of transport acceleration, relative acceleration, and Coriolis acceleration. (2) In the acceleration proof process, there is no need to distinguish whether the transport motion is translation or fixed-axis rotation, demonstrating the unified nature of this method for handling acceleration problems. (3) The derivation method in this paper offers advantages including clear mathematical reasoning, intuitive physical meaning, and smooth logical flow, dispelling the clouds that have long obscured the teaching of point composition of motion in theoretical mechanics and clarifying the subtleties in proving velocity and acceleration composition theorems, which is beneficial for understanding, mastering, and applying kinematics.

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Note: Figure translations are in progress. See original paper for figures.