A Multigrid Method for Semilinear Elliptic Problems Based on Symmetric Interior Penalty Discontinuous Galerkin Method

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Abstract

This paper introduces a new kind of multigrid approach for semilinear elliptic problems, which is based on the symmetric interior penalty discontinuous Galerkin (SIPDG) method. We first give an optimal error estimate of the SIPDG method for the problem. Then, we design a type of multigrid method, which is called the multilevel correction method, and derive a-priori error estimates. The primary idea of this method is to take the solution of the semilinear problem and utilize it to establish a sequence of solutions for associated linear boundary value problem on discontinuous finite element spaces and a newly defined low dimensional augmented subspace. Lastly, numerical experiments are offered to confirm the suggested method's precision and effectiveness.

Full Text

Preamble

A Multigrid Method for Semilinear Elliptic Problems Based on the Symmetric Interior Penalty Discontinuous Galerkin Method

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Abstract

This paper introduces a novel multigrid approach for semilinear elliptic problems based on the symmetric interior penalty discontinuous Galerkin (SIPDG) method. We first establish an optimal error estimate for the SIPDG method applied to the problem. Then, we design a multilevel correction method and derive a priori error estimates. The core idea of this method is to transform the solution of the semilinear problem into a sequence of solutions for associated linear boundary value problems on discontinuous finite element spaces and a newly defined low-dimensional augmented subspace. Finally, numerical experiments are presented to confirm the accuracy and effectiveness of the proposed method.

Mathematics Subject Classification 2020: 65N30, 65N55, 65B99
Keywords: Augmented subspace, multilevel correction method, discontinuous Galerkin method, semilinear elliptic problem, a priori error estimate

1. Introduction

It is well known that a significant portion of problems in scientific research and engineering practice are nonlinear. In particular, as contemporary science and technology advance, the scope of nonlinear problems continues to expand, creating an urgent need for efficient numerical solution techniques. Multigrid methods have reached a highly advanced stage of development for linear boundary value problems, yielding nearly comprehensive theoretical frameworks and solvers [8, 9, 12, 18, 25, 27]. Conversely, multigrid methods for nonlinear problems remain relatively underdeveloped and require further attention.

The traditional approach to applying multigrid methods involves first linearizing the nonlinear problem through some nonlinear iteration scheme, then using multigrid techniques to solve the resulting linearized equations. This is commonly referred to as an inner iteration (multigrid iteration) plus an outer iteration (nonlinear iteration). Although multigrid methods are highly efficient for the inner iteration, the total computational work is determined by the number of outer iteration steps. Consequently, when dealing with problems exhibiting strong nonlinearity that require numerous outer iteration steps, the overall computational cost becomes substantial despite the efficiency of the inner multigrid iteration. This limitation means that conventional multigrid algorithms cannot eliminate the dependence on outer iterations or make the entire computational process independent of nonlinear iterations.

The two-grid method, initially presented and analyzed in [26], represents one of the efficient techniques for handling nonlinear problems. Its fundamental idea is to use a coarse-grid space to obtain a rough approximation for solving the nonlinear problem, then utilize this coarse-grid solution as a starting point for the fine-grid space. Optimal error estimates can be achieved when the coarse and fine mesh sizes maintain a suitable proportional relationship. Later, numerous variations of the two-grid method were proposed and developed [31, 32].

In 2015, Lin and Xie [13] took the eigenvalue problem as an example and proposed a multigrid method called the multilevel correction method, based on a newly introduced augmented subspace. Subsequently, Xu et al. extended this method to nonlinear boundary value problems [14, 20, 21, 22, 23, 24]. The multilevel correction method constructs an augmented subspace in each correction step using a low-dimensional finite element space defined on the coarse grid. This augmented subspace iteration requires only the final finite element space on the finest mesh and the low-dimensional finite element space on the coarse mesh, transforming the solution of the nonlinear problem on the finest mesh into solutions of linear boundary value problems on the finest mesh and a nonlinear problem on the low-dimensional augmented subspace. Based on this augmented subspace iteration, the multilevel correction method provides an approach for constructing multigrid methods for nonlinear problems. However, the aforementioned references primarily investigate conforming finite element methods.

The discontinuous Galerkin (DG) method is an efficient numerical technique for solving partial differential equations, offering many attractive features including convenience for parallel computation, inherent flexibility for complex geometries through unstructured grids, and local mass conservation at the element level. First introduced in [3, 10], DG methods have been subsequently developed effectively for various problem types [4, 15, 19, 28].

In this paper, we take the semilinear elliptic problem as an example and present a multilevel correction method based on the DG method. We first provide an optimal error estimate for the DG method applied to the problem. Then, we design the multilevel correction method based on the augmented subspace iteration and derive a priori error estimates. Unlike existing multilevel correction literature based on conforming finite element methods, where coarse and fine spaces must adopt conforming finite element spaces with the same degree piecewise polynomial, our coarse and fine spaces are respectively selected as conforming and discontinuous finite element spaces. Moreover, the coarse and fine spaces can use piecewise polynomials of different degrees (see Theorem 3.1, Remarks 3.1 and 3.2 for details). Considering computational resource economy and efficiency, we choose the coarse space as the linear conforming finite element space. To the best of our knowledge, this is the first paper to investigate the multilevel correction method from the perspective of nonstandard finite element methods.

This paper is organized as follows. Section 2 introduces the SIPDG method for semilinear elliptic problems. Section 3 presents the augmented subspace iteration and its multilevel correction method, along with a priori error estimates. Finally, Section 4 provides numerical experiments to verify the accuracy and effectiveness of our proposed approach.

2. Discontinuous Galerkin Discretization

In this paper, we consider the following semilinear elliptic problem on a convex polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\partial\Omega$:

$$-\Delta u + f(x,u) = g, \quad \text{in } \Omega, \tag{2.1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{2.2}$$

where $f(x,u)$ is a nonlinear function with respect to u.

We now define several function spaces that will be used throughout this paper. We employ the Sobolev space $H^l(\Omega)$ with $l \geq 0$ (cf. [1]), with corresponding inner product and norm denoted by $(\cdot,\cdot)_l$ and $\|\cdot\|_l$, respectively. When $l = 0$, the space of square-integrable functions $L^2(\Omega)$ coincides with $H^0(\Omega)$. In this case, the inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The space $H^1_0(\Omega)$ is the subspace of $H^1(\Omega)$ containing functions with vanishing traces on $\partial\Omega$. Furthermore, the symbol C, with or without subscripts, denotes a generic positive constant that may vary depending on its context.

Assume that the domain Ω is partitioned by a shape-regular triangulation $\mathcal{T}_h$, i.e., $\mathcal{T}_h = \{K\}$, where h_K denotes the diameter of element K and $h = \max h_K$. Additionally, we use Γ to represent the set of all edges in $\mathcal{T}_h$, and e to denote edges in Γ. For the polygonal or polyhedral domain Ω, we have $\Gamma = \Gamma_I \cup \partial\Omega$, where Γ_I represents the set of all interior edges.

Next, we define the broken Sobolev space with $s \geq 1$:

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^s(K), \forall K \in \mathcal{T}_h\},$$

and the corresponding discontinuous finite element space:

$$V_h = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_r(K), \forall K \in \mathcal{T}_h\},$$

where $\mathcal{P}_r(K)$ is the set of polynomials of degree at most $r \geq 1$ defined on K.

We now define the average and jump operators required for the DG method. Assume $v \in H^1(\mathcal{T}_h)$ and that K_i and K_j $(i > j)$ are two neighboring elements of $\mathcal{T}_h$ sharing an interior edge $e = \partial K_i \cap \partial K_j \subset \Gamma_I$. The direction of the outward unit normal vector n_e is assumed to be from K_i to K_j. The average and jump of v on e are given by:

$$[v] = v|_{K_i} - v|_{K_j}, \quad \{v\} = \frac{1}{2}(v|_{K_i} + v|_{K_j}).$$

If $e \subset \partial K_i \cap \partial\Omega$ is a boundary edge, then:

$$[v] = v|_e, \quad \{v\} = v|_e.$$

The generic DG variational formulation of (2.1)-(2.2) is now presented as: Find $u \in H^2(\mathcal{T}_h)$ such that

$$a(u,v) + (f(x,u),v) = (g,v), \quad \forall v \in H^2(\mathcal{T}_h). \tag{2.3}$$

where

$$a(u,v) = \sum_{K\in\mathcal{T}h} \int_K \nabla u\cdot\nabla v\, dK - \sum_{e\in\Gamma} \int_e \{\nabla u\}\cdot n_e [v]\, ds - \sum_{e\in\Gamma} \int_e \{\nabla v\}\cdot n_e [u]\, ds + \sigma \sum_{e\in\Gamma} \int_e [u][v]\, ds,$$

$$(f(x,u),v) = \sum_{K\in\mathcal{T}h} \int_K f(x,u)v\, dK.$$

Here, σ is a sufficiently large positive constant.

Lemma 2.1. [29] If $u \in H_0^1(\Omega) \cap H^2(\Omega)$ is the solution to (2.1)-(2.2), then u is the solution to (2.3) and vice versa.

To ensure the existence and uniqueness of the solution to problems (2.1)-(2.2), we make the following assumption about the nonlinear term $f(x,\cdot)$.

Assumption A. The nonlinear function $f(x,\cdot)$ satisfies convexity and Lipschitz continuity conditions as follows:

$$\begin{cases} (f(x,w) - f(x,v), w - v) \geq 0, & \forall w, v \in H_0^1(\Omega) \cap H^2(\mathcal{T}_h), \\ (f(x,w) - f(x,v), \phi) \leq C_f \|w - v\| \|\phi\|, & \forall w, v, \phi \in H_0^1(\Omega) \cap H^2(\mathcal{T}_h), \end{cases} \tag{2.4}$$

where the norm $\|\cdot\|$ is defined for $v \in H_0^1(\Omega) \cap H^2(\mathcal{T}_h)$ as:

$$\|v\|^2 = \sum_{K\in\mathcal{T}h} \|\nabla v\|_{0,K}^2 + \sum_{e\in\Gamma} \int_e [v]^2\, ds.$$

The SIPDG scheme for (2.3) is then given as: Find $\bar{u}_h \in V_h$ such that

$$a(\bar{u}_h, v_h) + (f(x,\bar{u}_h), v_h) = (g, v_h), \quad \forall v_h \in V_h. \tag{2.5}$$

We present the following useful lemmas from [16] and [3].

Lemma 2.2. [16] Let e represent an edge or face of element $K \in \mathcal{T}_h$. Then there exists a constant C independent of h such that:

$$\|v\|_{0,e} \leq C h_K^{-1/2} \|v\|_{0,K}, \quad \forall v \in \mathcal{P}_1(K), \tag{2.6}$$

$$\|\nabla v \cdot n_e\|_{0,e} \leq C h_K^{-1/2} \|\nabla v\|_{0,K}, \quad \forall v \in \mathcal{P}_1(K), \tag{2.7}$$

$$\|v\|_{0,e} \leq C h_K^{-1/2} (\|v\|_{0,K} + h_K \|\nabla v\|_{0,K}), \quad \forall v \in H^1(K), \tag{2.8}$$

$$\|\nabla v \cdot n_e\|_{0,e} \leq C h_K^{-1/2} (\|\nabla v\|_{0,K} + h_K \|\nabla^2 v\|_{0,K}), \quad \forall v \in H^2(K). \tag{2.9}$$

Lemma 2.3. [3] There exists a constant $C_p > 0$ such that:

$$\|v\|^2 \leq C_p \|v\|^2, \quad \forall v \in H^1(\mathcal{T}_h).$$

The following lemma establishes the boundedness and coercivity of $a(\cdot, \cdot)$.

Lemma 2.4. [2] If σ is sufficiently large, there exist positive constants C_1 and C_2 independent of h such that:

$$a(v, v) \geq C_1 \|v\|^2, \quad \forall v \in V_h,$$

$$a(w, v) \leq C_2 \|w\| \|v\|, \quad \forall w, v \in H^2(\mathcal{T}_h).$$

We now introduce an interpolation operator $I_h u \in V_h$, available in [17], which satisfies the following estimates:

$$\|u - I_h u\| + h\|u - I_h u\| \leq C h^{r+1} \|u\|_{r+1}, \quad \forall u \in H^{r+1}(\mathcal{T}_h).$$

Furthermore, we define the elliptic projection $P_h w \in V_h$ that satisfies:

$$a(P_h w, v_h) = a(w, v_h), \quad \forall v_h \in V_h. \tag{2.10}$$

We present the following lemma regarding the approximation properties of the elliptic projection, derived from [30].

Lemma 2.5. For any $u \in H^{r+1}(\Omega)$ $(r \geq 1)$, the following error approximations hold:

$$\|u - P_h u\| \leq (1 + C) \inf_{v_h \in V_h} \|u - v_h\| \leq C_{p1} (1 + C) h^r \|u\|_{r+1},$$

$$\|u - P_h u\| \leq C_{p2} C_2 h \|u - P_h u\| \leq C_{p1} C_{p2} C_2 (1 + C) h^{r+1} \|u\|_{r+1}.$$

Proof. Due to the coercivity and continuity of $a(\cdot, \cdot)$, we obtain for any $v_h \in V_h$:

$$C_1\|v_h-P_hu\|^2 \leq a(v_h-P_hu, v_h-P_hu) = a(v_h-u, v_h-P_hu)+a(u-P_hu, v_h-P_hu) = a(v_h-u, v_h-P_hu) \leq C_2\|v_h-$$

Consequently:

$$\|v_h - P_hu\| \leq C\|v_h - u\|.$$

By the triangle inequality:

$$\|u - P_hu\| \leq \|u - v_h\| + \|v_h - P_hu\| \leq (1 + C)\|u - v_h\|.$$

Since $v_h \in V_h$ is arbitrary:

$$\|u - P_hu\| \leq (1 + C) \inf_{v_h\in V_h} \|u - v_h\| \leq (1 + C)\|u - I_hu\| \leq (1 + C)C_{p1}h^r\|u\|_{r+1}.$$

For the L^2-norm estimate, we consider the following dual problem: Find $\psi \in H^2(\Omega)$ such that:

$$a(v, \psi) = (v, u - P_hu), \quad \forall v \in H^2(\Omega).$$

Assume the solution of the dual problem depends continuously on $u-P_hu$, i.e., $\|\psi\|_{H^2(\Omega)} \leq C\|u - P_hu\|$. Then:

$$\|u-P_hu\|^2 = a(u-P_hu, \psi) = a(u-P_hu, \psi-I_h\psi) \leq C_2\|u-P_hu\|\|\psi-I_h\psi\| \leq C_2Ch\|\psi\|_2\|u-P_hu\| \leq C_2C_{p2}h\|u-P_h$$

Thus:

$$\|u - P_hu\| \leq C_{p2}C_2h\|u - P_hu\| \leq C_{p1}C_{p2}C_2(1 + C)h^{r+1}\|u\|_{r+1}.$$

Lemma 2.6. Under Assumption A, let u and $\bar{u}_h$ be the solutions of (2.3) and (2.5), respectively. Then:

$$\|u - \bar{u}_h\| \leq C_{p2}C_2(1 + C_pC_f)h\|u - \bar{u}_h\|,$$

$$\|u-\bar{u}_h\| \leq (1+C_fC_{p2}C_2h)(1+C) \inf_{v_h\in V_h} \|u-v_h\| \leq (1+C_fC_{p2}C_2h)C_{p1}(1+C)h^r\|u\|_{r+1}.$$

Proof. The uniqueness of solutions to problems (2.3) and (2.5) follows from [11] and [29]. From (2.3) and (2.5), we obtain the orthogonality property:

$$a(u-\bar{u}_h, v_h) + (f(x,u) - f(x,\bar{u}_h), v_h) = 0. \tag{2.11}$$

Using (2.4), (2.10), and (2.11), we derive:

$$a(P_hu-\bar{u}_h, P_hu-\bar{u}_h) \leq a(P_hu-\bar{u}_h, P_hu-\bar{u}_h)+(f(x,P_hu)-f(x,\bar{u}_h), P_hu-\bar{u}_h) = a(P_hu-u, P_hu-\bar{u}_h)+(f(x,P_hu \tag{2.12}$$

Applying Lemma 2.5 yields:

$$\|P_hu-\bar{u}_h\| \leq C_f\|u-P_hu\| \leq C_fC_{p2}C_2h\|u-P_hu\|. \tag{2.13}$$

Combining (2.13) with the triangle inequality gives:

$$\|u-\bar{u}_h\| \leq \|u-P_hu\|+\|P_hu-\bar{u}_h\| \leq (1+C_fC_{p2}C_2h)\|u-P_hu\| \leq (1+C_fC_{p2}C_2h)(1+C)\inf_{v_h\in V_h}\|u-v_h\| \leq (1+C_fC_{p2}$$

Using the triangle inequality and (2.13), we obtain:

$$\|u-\bar{u}_h\| \leq \|u-P_hu\|+\|P_hu-\bar{u}_h\| \leq \|u-P_hu\|+C_p\|P_hu-\bar{u}_h\| \leq (C_{p2}C_2h+C_pC_fC_{p2}C_2h)\|u-P_hu\| \leq C_{p2}C_2(1+C_p$$

This completes the proof.

3. A Multigrid Method for Semilinear Elliptic Problems

This section begins by presenting the augmented subspace iteration algorithm for the semilinear elliptic problem and deriving the a priori algebraic error estimate. This algorithm primarily transforms the solution of the semilinear problem into a sequence of solutions for associated linear boundary value problems on discontinuous finite element spaces and the semilinear problem on a newly defined low-dimensional augmented subspace. We then describe the details of the multilevel correction method and examine the a priori error based on the augmented subspace iteration and its algebraic error estimate.

To implement this multigrid approach, we define the following multigrid sequence. First, construct a coarse triangulation mesh $\mathcal{T}_H$ with mesh size H, on which the corresponding finite element space V_H must be created. Note that the coarse space V_H remains unchanged throughout the entire process. Then, generate the following sequence of triangulations $\mathcal{T}_{h_k}$ for the domain $\Omega \subset \mathbb{R}^d$.

Assume we have $\mathcal{T}_{h_1}$, which can be obtained from $\mathcal{T}_H$ through regular refinements or may be identical to $\mathcal{T}_H$, and let $\mathcal{T}_{h_k}$ be obtained from $\mathcal{T}_{h_{k-1}}$ via one step of regular refinement such that:

$$h_k = \frac{h_{k-1}}{\beta}, \tag{3.1}$$

where the refinement index is represented by the positive constant $\beta > 1$, which typically equals 2.

3.1. Augmented Subspace Iteration

To provide a clearer description of the algorithm, we design the coarse space V_H as the linear conforming finite element space:

$$V_H = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega, v|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_H\},$$

and V_{h_k} as the discontinuous finite element space:

$$V_{h_k} = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_r(K), \forall K \in \mathcal{T}_{h_k}\}.$$

Suppose we have an approximate solution $u^{(l)} \in V_{h_k}$ (where l denotes the iteration index). This approximate solution can be taken as the one previously obtained from mesh $\mathcal{T}_{h_{k-1}}$. The augmented subspace iteration algorithm, designated as Algorithm 1, aims to improve the accuracy of the given approximation $u^{(l)}$.

Algorithm 1: Augmented Subspace Iteration

1. **Solve a linear boundary value problem:** Seek $\tilde{u}^{(l+1)} \in V_{h_k}$ such that

$$a(\tilde{u}^{(l+1)}, v_{h_k}) = -(f(x, u^{(l)}), v_{h_k}) + (g, v_{h_k}), \quad \forall v_{h_k} \in V_{h_k}. \tag{3.2}$$

2. **Define the augmented subspace** $V_{H,h_k} = V_H + \text{span}\{\tilde{u}^{(l+1)}\}$ and solve the following semilinear elliptic equation: Seek $u^{(l+1)} \in V_{H,h_k}$ such that

$$a(u^{(l+1)}, v_{H,h_k}) + (f(x, u^{(l+1)}), v_{H,h_k}) = (g, v_{H,h_k}), \quad \forall v_{H,h_k} \in V_{H,h_k}. \tag{3.3}$$

To simplify notation, we condense these two steps and define $u^{(l+1)} = \text{SemilinearMG}(V_H, u^{(l)}, V_{h_k})$.

Theorem 3.1. Assume that $u^{(l)}$ satisfies the L^2-norm error estimate:

$$\|\bar{u}_{h_k} - u^{(l)}\| \leq C_{p2}C_2(1 + C_pC_f)H\|\bar{u}_{h_k} - u^{(l)}\|. \tag{3.4}$$

Then the solution $u^{(l+1)}$ obtained by executing Algorithm 1 satisfies the following convergence estimates:

$$\|\bar{u}_{h_k} - u^{(l+1)}\| \leq \gamma \|\bar{u}_{h_k} - u^{(l)}\|, \tag{3.5}$$

$$\|\bar{u}_{h_k} - u^{(l+1)}\| \leq (C_{p2}C_2 + C_pC_fC_{p2}C_2)H\|\bar{u}_{h_k} - u^{(l+1)}\|, \tag{3.6}$$

where $\gamma = (1 + C_fC_{p2}C_2H)(1 + C_2)C_fC_{p2}C_2(1 + C_pC_f)H$.

Proof. From (2.5) and (3.2), we obtain the following estimate:

$$a(\bar{u}_{h_k}-\tilde{u}^{(l+1)}, v_{h_k}) = (f(x, u^{(l)})-f(x, \bar{u}_{h_k}), v_{h_k}) \leq C_f\|u^{(l)}-\bar{u}_{h_k}\|\|v_{h_k}\| \leq C_fC_{p2}C_2(1+C_pC_f)H\|\bar{u}_{h_k}-u^{(l)}\|\|v_{h_k}\|. \tag{3.7}$$

Using Lemma 2.4 with $v_{h_k} = \bar{u}_{h_k} - \tilde{u}^{(l+1)}$ in (3.7) yields:

$$\|\bar{u}_{h_k} - \tilde{u}^{(l+1)}\| \leq C_fC_{p2}C_2(1 + C_pC_f)H\|\bar{u}_{h_k} - u^{(l)}\|. \tag{3.8}$$

Since V_H is the linear conforming finite element space, we have $V_H \subset V_{h_k}$, and combined with $\tilde{u}^{(l+1)} \in V_{h_k}$, we obtain $V_{H,h_k} \subset V_{h_k}$. This implies that the semilinear elliptic problem (3.3) defined in V_{H,h_k} is a finite-dimensional approximation of the semilinear elliptic problem (2.5) defined in V_{h_k}.

Define $P_{H,h_k} : V_{h_k} \to V_{H,h_k}$ as the projection operator satisfying:

$$a(P_{H,h_k}w_{h_k}, v_{H,h_k}) = a(w_{h_k}, v_{H,h_k}), \quad \forall w_{h_k} \in V_{h_k}, v_{H,h_k} \in V_{H,h_k}. \tag{3.9}$$

It is clear that:

$$\|\bar{u}_{h_k} - P_{H,h_k}\bar{u}_{h_k}\| \leq (1 + C) \inf_{v_{H,h_k} \in V_{H,h_k}} \|\bar{u}_{h_k} - v_{H,h_k}\| \leq (1 + C)\|\bar{u}_{h_k} - \tilde{u}^{(l+1)}\|,$$

$$\|\bar{u}_{h_k} - P_{H,h_k}\bar{u}_{h_k}\| \leq C_{p2}C_2H\|\bar{u}_{h_k} - P_{H,h_k}\bar{u}_{h_k}\|.$$

Let $w_h = P_{H,h_k}\bar{u}_{h_k} - u^{(l+1)}$. From (2.5) and (3.3), we have:

$$a(\bar{u}_{h_k} - u^{(l+1)}, v_{H,h_k}) + (f(x, \bar{u}_{h_k}) - f(x, u^{(l+1)}), v_{H,h_k}) = 0. \tag{3.10}$$

Using (2.4), (3.10), and (3.9), we obtain:

$$a(P_{H,h_k}\bar{u}_{h_k}-u^{(l+1)},w_h)+(f(x,P_{H,h_k}\bar{u}_{h_k})-f(x,u^{(l+1)}),w_h) \leq a(P_{H,h_k}\bar{u}_{h_k}-u^{(l+1)},w_h)+(f(x,P_{H,h_k}\bar{u}_{h_k})-f(x,\bar{u} \tag{3.11}$$

Therefore:

$$\|P_{H,h_k}\bar{u}_{h_k}-u^{(l+1)}\| \leq C_f\|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\| \leq C_fC_{p2}C_2H\|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|. \tag{3.12}$$

Combining (3.12) with the triangle inequality yields:

$$\|\bar{u}_{h_k}-u^{(l+1)}\| \leq \|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|+\|P_{H,h_k}\bar{u}_{h_k}-u^{(l+1)}\| \leq (1+C_fC_{p2}C_2H)\|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\| \leq (1+C_fC_{p2}C_2H)(1+ \tag{3.13}$$

Let $\gamma = (1+C_2)(1+C_fC_{p2}C_2H)C_fC_{p2}C_2(1+C_pC_f)H$. We obtain the desired result (3.5) from (3.8) and (3.13).

Next, using the triangle inequality and (3.12), we have:

$$\|\bar{u}_{h_k}-u^{(l+1)}\| \leq \|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|+\|P_{H,h_k}\bar{u}_{h_k}-u^{(l+1)}\| \leq \|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|+C_p\|P_{H,h_k}\bar{u}_{h_k}-u^{(l+1)}\| \leq (C_{p2}C_2+C_pC$$

This establishes the desired result (3.6). The proof is complete.

Remark 3.1. From the proof of Theorem 3.1, we observe that because V_H is chosen as the conforming finite element space, we have $V_H \subset V_{h_k}$. Combined with $\tilde{u}^{(l+1)} \in V_{h_k}$, this yields $V_{H,h_k} \subset V_{h_k}$. This shows that the semilinear elliptic problem (3.3) formulated in V_{H,h_k} is a finite-dimensional approximation to the semilinear elliptic problem (2.5) in V_{h_k}. Therefore, we can estimate the algebraic error between $\bar{u}_{h_k}$ and $u^{(l+1)}$. However, if V_H were selected as the discontinuous finite element space, then $V_H \subset H^2(\mathcal{T}_h)$. Together with $\tilde{u}^{(l+1)} \in H^2(\mathcal{T}_h)$, we would have $V_{H,h_k} \subset H^2(\mathcal{T}_h)$, which means that (3.3) would be a finite-dimensional approximation to (2.3). In that case, we could only estimate the error between u and $u^{(l+1)}$.

Remark 3.2. From (3.13), we can see that the degree of the piecewise polynomial in V_H does not affect the convergence order of the error, because we utilize the orthogonality of the projection operator (3.9) to transform the error from $\|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|$ to $\|\bar{u}_{h_k}-\tilde{u}^{(l+1)}\|$. Therefore, from the perspective of computational resource economy and efficiency, we select V_H as the linear finite element space.

3.2. Multilevel Correction Method

In this subsection, we construct a multigrid method for the semilinear elliptic problem based on the augmented subspace iteration algorithm defined in Algorithm 1. In the multilevel correction method, we first solve the semilinear elliptic problem in the initial space V_{h_1}. We then interpolate this solution into the finer space V_{h_2} and use it as the initial value for the augmented subspace iteration. Next, we perform Algorithm 1 in V_{h_2} until the approximate solution reaches the prescribed accuracy. We subsequently interpolate the solution from V_{h_2} into V_{h_3} and conduct the augmented subspace iteration as before. This process is repeated until we obtain the approximate solution in the final space V_{h_n}. The details of the multilevel correction method are presented in Algorithm 2.

Algorithm 2: Multilevel Correction Method

1. **Solve in the initial space V_{h_1}:** Seek $u_{h_1} \in V_{h_1}$ such that

$$a(u_{h_1}, v_{h_1}) + (f(x, u_{h_1}), v_{h_1}) = (g, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}.$$

2. **For $k = 2, \dots, n$, perform the augmented subspace iteration from Algorithm 1:**
   - Set $u^{(0)} = u_{h_{k-1}}$.
   - For $l = 0, \dots, p{-}1$, perform the iterations: $u^{(l+1)} = \text{SemilinearMG}(V_H, u^{(l)}, V_{h_k})$.
   - Define $u_{h_k} = u^{(p)}$.

Finally, we provide the approximate solution u_{h_n} in the finest space V_{h_n}.

Theorem 3.2. Let $\bar{u}_{h_n} \in V_{h_n}$ be the solution of (2.5), and let u_{h_n} be obtained from Algorithm 2. Then:

$$\|\bar{u}_{h_n} - u_{h_n}\| \le \frac{2\gamma^p\beta^r}{1-\gamma^p\beta^r} h_n^r \|u\|_{r+1}, \tag{3.14}$$

$$\|\bar{u}_{h_n} - u_{h_n}\| \le C_{p2}C_2(1 + C_pC_f)H\|\bar{u}_{h_n} - u_{h_n}\|, \tag{3.15}$$

where γ and β are defined in Theorem 3.1 and (3.1), respectively, and H is sufficiently small such that $\gamma^p\beta^r < 1$.

Proof. Solving the semilinear elliptic problem in the initial space V_{h_1}, which is the first stage of Algorithm 2, yields $u_{h_1} = \bar{u}_{h_1}$, indicating that u_{h_1} satisfies condition (3.4). Therefore, the requirement of Theorem 3.1 is satisfied when we use u_{h_1} as the initial value for Algorithm 1 in V_{h_2}. Consequently, we have the approximation:

$$\|\bar{u}_{h_2} - u_{h_2}\| = \|\bar{u}_{h_2} - u^{(p)}\| \le \gamma^p\|\bar{u}_{h_2} - u^{(0)}\| = \gamma^p\|\bar{u}_{h_2} - u_{h_1}\| = \gamma^p(\|\bar{u}_{h_2} - \bar{u}_{h_1}\| + \|\bar{u}_{h_1} - u_{h_1}\|). \tag{3.16}$$

In space V_{h_k}, we have the following approximation for u_{h_k}, analogous to (3.16):

$$\|\bar{u}_{h_k}-u_{h_k}\| = \|\bar{u}_{h_k}-u^{(l+1)}\| \leq \gamma\|\bar{u}_{h_k}-u^{(l)}\| \leq \gamma^p\|\bar{u}_{h_k}-u^{(0)}\| = \gamma^p\|\bar{u}_{h_k}-u_{h_{k-1}}\| \leq \gamma^p(\|\bar{u}_{h_k}-\bar{u}_{h_{k-1}}\|+\|\bar{u}_{h_{k-1}}-u_{h_{k-1}}\|) \tag{3.17}$$

Repeating (3.17), we obtain:

$$\|\bar{u}_{h_n}-u_{h_n}\| \leq \gamma^p\|\bar{u}_{h_n}-\bar{u}_{h_{n-1}}\|+\gamma^{2p}(\|\bar{u}_{h_{n-1}}-\bar{u}_{h_{n-2}}\|+\|\bar{u}_{h_{n-2}}-u_{h_{n-2}}\|) \leq \sum_{k=1}^{n-1}\gamma^{kp}\|\bar{u}_{h_{n-k+1}}-\bar{u}_{h_{n-k}}\| \leq \sum_{k=1}^{n-1}\gamma^{kp}(\|\bar{u}_{h_{n-k+}}$$

which yields the desired result (3.14). Similar to the proof of Theorem 3.1, we can establish:

$$\|\bar{u}_{h_n} - u_{h_n}\| \leq C_{p2}C_2(1 + C_pC_f)H\|\bar{u}_{h_n} - u_{h_n}\|,$$

which completes the proof.

Corollary 3.3. Using Algorithm 2, the approximate solution u_{h_n} satisfies the following error estimates:

$$\|u - u_{h_n}\| \leq Ch_n^r\|u\|_{r+1},$$

$$\|u - u_{h_n}\| \leq C(h_n^{r+1} + Hh_n^r)\|u\|_{r+1}.$$

Proof. This follows directly from Lemma 2.6 and Theorem 3.1.

4. Numerical Experiments

This section presents numerical examples to validate the theoretical analysis and demonstrate the efficiency of Algorithm 2. In practical computations, we note that the mesh size of the coarse conforming finite element space V_H is independent of the mesh size of the fine discontinuous finite element space. Furthermore, the computations for this work were performed on high-performance computers at the State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences. Each compute node features two 18-core 2.3 GHz Intel Xeon Gold 6140 processors and 192 GB of RAM. The linear systems arising from the coefficient matrix in Algorithm 2 were solved using the PETSc package [5, 6, 7]. Both computational complexity and convergence tests utilized 36 processors.

In our examples, we consider the following semilinear elliptic problem:

$$-\Delta u + \kappa u^3 = g, \quad \text{in } \Omega, \tag{4.1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{4.2}$$

where $\Omega = (0,1)^2$ and the parameter κ is chosen as 1 and 10 to test the problem under different nonlinear strengths. The right-hand side term g is selected according to the analytical solution $u = \sin(\pi x)\sin(\pi y)$.

We employ a uniform triangular mesh as the coarse mesh $\mathcal{T}_H$ with mesh size $H = 2/4$, which is identical to the initial mesh $\mathcal{T}_{h_1}$. The mesh $\mathcal{T}_{h_k}$ is then obtained from $\mathcal{T}_{h_{k-1}}$ $(k = 2, \dots, n)$ via one regular refinement step. For the discontinuous finite element space, we separately choose $r = 1$ and $r = 2$ in the following examples. The nonlinear equations from Algorithm 2 are solved using the fixed-point iteration method, which is terminated when the difference between successive approximations falls below the preset tolerance of 10^{-10}. To better illustrate the advantages of the proposed method, we compare Algorithm 2 with a direct method that uses the SIPDG method together with fixed-point iteration to solve problems (4.1)-(4.2) only on the final mesh without coarsening. We denote solutions with and without the "dir" superscript to represent the direct method and Algorithm 2, respectively.

4.1. The Case $r = 1$

In this case, the finite-dimensional spaces in Algorithm 2 are taken as:

$$V_H = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega, v|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_H\},$$

$$V_{h_k} = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_{h_k}\}, \quad k = 1, \dots, n.$$

Figures 1 and 2 depict the errors and computational time for both the direct method and Algorithm 2 with penalty parameter $\sigma = 5 \times 10^3$ and $\kappa = 1, 10$, respectively. These figures demonstrate that Algorithm 2 achieves optimal convergence rates and asymptotically linear computational complexity. Additionally, the results indicate that Algorithm 2 is accurate, as its approximate solution is nearly as precise as that obtained by the direct method. Meanwhile, Algorithm 2 proves more efficient than the direct method, as evidenced by its significant advantage in computational time.

[Figure 1: see original paper]

Figure 1: Errors and CPU time (in seconds) of Algorithm 2 for $r = 1$ and $\kappa = 1$.

[Figure 2: see original paper]

Figure 2: Errors and CPU time (in seconds) of Algorithm 2 for $r = 1$ and $\kappa = 10$.

4.2. The Case $r = 2$

In this subsection, we choose the finite-dimensional spaces as:

$$V_H = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega, v|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_H\},$$

$$V_{h_k} = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_2(K), \forall K \in \mathcal{T}_{h_k}\}, \quad k = 1, \dots, n.$$

Figures 3 and 4 show the errors and CPU time for both the direct method and Algorithm 2 with penalty parameter $\sigma = 1 \times 10^4$ and $\kappa = 1, 10$, respectively. These figures demonstrate that Algorithm 2 achieves optimal convergence orders while maintaining asymptotically linear computational complexity. Furthermore, Figures 3 and 4 reveal that Algorithm 2 is more efficient than the direct method without sacrificing accuracy. The results also confirm that selecting the linear conforming finite element space V_H provides an economical and effective approach for Algorithm 2, verifying Remark 3.2.

[Figure 3: see original paper]

Figure 3: Errors and CPU time (in seconds) of Algorithm 2 for $r = 2$ and $\kappa = 1$.

[Figure 4: see original paper]

Figure 4: Errors and CPU time (in seconds) of Algorithm 2 for $r = 2$ and $\kappa = 10$.

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