Studying the Equilibrium Points of the Modified Circular Restricted Three-body Problem: The Case of Sun–Haumea System (Postprint)

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Abstract

We intend to study a modified version of the planar Circular Restricted Three-Body Problem (CRTBP) by incorporating several perturbing parameters. We consider the bigger primary as an oblate spheroid and emitting radiation while the small primary has an elongated body. We also consider the perturbation from a disk-like structure encompassing this three-body system. First, we develop a mathematical model of this modified CRTBP. We have found there exist five equilibrium points in this modified CRTBP model, where three of them are collinear and the other two are non-collinear. Second, we apply our modified CRTBP model to the Sun–Haumea system by considering several values of each perturbing parameter. Through our numerical investigation, we have discovered that the incorporation of perturbing parameters has resulted in a shift in the equilibrium point positions of the Sun–Haumea system compared to their positions in the classical CRTBP. The stability of equilibrium points is investigated. We have shown that the collinear equilibrium points are unstable and the stability of non-collinear equilibrium points depends on the mass parameter of the system. Unlike the classical case, non-collinear equilibrium points have both a maximum and minimum limit of for achieving stability. We remark that the stability range of in non-collinear equilibrium points depends on the perturbing parameters. In the context of the Sun–Haumea system, we have found that the non-collinear equilibrium points are stable.

Full Text

Preamble

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Studying the Equilibrium Points of the Modified Circular Restricted Three-body Problem: The Case of Sun–Haumea System

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Abstract

We investigate a modified version of the planar Circular Restricted Three-Body Problem (CRTBP) by incorporating several perturbing parameters. We consider the larger primary as an oblate spheroid that emits radiation, while the smaller primary has an elongated body. We also include the perturbation from a disk-like structure encompassing this three-body system. First, we develop a mathematical model of this modified CRTBP.

We find that five equilibrium points exist in this modified CRTBP model, with three being collinear and the other two non-collinear. Second, we apply our modified CRTBP model to the Sun–Haumea system by considering various values for each perturbing parameter. Through numerical investigation, we discover that incorporating perturbing parameters shifts the equilibrium point positions of the Sun–Haumea system compared to their locations in the classical CRTBP. We investigate the stability of these equilibrium points and show that the collinear equilibrium points are unstable, while the stability of non-collinear equilibrium points depends on the mass parameter of the system. Unlike the classical case, the non-collinear equilibrium points exhibit both maximum and minimum limits for to achieve stability. We note that the stability range of for non-collinear equilibrium points depends on the perturbing parameters. In the context of the Sun–Haumea system, we find that the non-collinear equilibrium points are stable.

Key words: celestial mechanics – Kuiper Belt: general – planets and satellites: dynamical evolution and stability

1. Introduction

Celestial mechanics plays an important role in understanding the dynamics of solar system bodies (see, e.g., Murray & Dermott 1999; Souchay & Dvorak 2010; Lei 2021; Pan & Hou 2022). One fundamental problem in celestial mechanics is the Circular Restricted Three-Body Problem (CRTBP), which investigates the motion of an infinitesimal object under the gravitational influence of two primaries that orbit their common center of mass in circular paths.

The CRTBP has several applications, including deep space exploration and satellite navigation. The classical version assumes the primaries are point masses and considers only their mutual gravitational interaction. In the planar case, five equilibrium points exist: three collinear points (L_1, L_2, and L_3) and two noncollinear points (L_4 and L_5) (Murray & Dermott 1999). To create more realistic CRTBP models, researchers have modified the classical version by incorporating additional parameters.

Stellar objects, including the Sun, emit radiation that exerts pressure on objects in its path. Numerous studies have considered radiation pressure as an additional force in the restricted three-body problem (see, e.g., Haque & Ishwar 1995; Ishwar & Elipe 2001; Kushvah et al. 2007; Kushvah 2008a; Das et al. 2009; Yousuf & Kishor 2019; Patel et al. 2023). The first study on this topic was conducted by Radzievskii (1950). Chernikov (1970) extended this work by considering the relativistic Poynting–Robertson effect. Simmons et al. (1985) examined the radiation pressure force across all ranges of values. More recently, Idrisi (2017) and Idrisi & Ullah (2018) considered the effect of planetary albedo on CRTBP as a consequence of solar radiation pressure force.

Since stars and planets are not perfectly spherical, another important consideration in CRTBP is the oblateness of the primaries. Early studies on the impact of an oblate primary were published by Danby (1965) and Sharma & Subba Rao (1978, 1986). More recently, the effect of oblateness on CRTBP dynamics has been studied in detail by several authors (see, e.g., Markellos et al. 1996; Douskos & Markellos 2006; Safiya Beevi & Sharma 2012; Abouelmagd et al. 2013; Zotos 2015; Yousuf et al. 2022). Some authors have also considered the combined effects of oblateness and radiation force. For instance, Singh & Ishwar (1999) studied triangular equilibrium points when both primaries are oblate and emit radiation. This work was extended by Singh (2009) for the nonlinear stability of L_4. AbdulRaheem & Singh (2006) studied the linear stability of equilibrium points in this context. Other researchers, such as Nurul Huda et al. (2015), Dermawan et al. (2015), and Mia et al. (2023), considered the effects of oblateness and radiation force in the Elliptic Restricted Three-Body Problem.

Our solar system contains various types of celestial bodies, including elongated objects such as certain asteroids, comets, and dwarf planets. These bodies can be approximated as finite straight segments. Previous CRTBP studies have been enriched by assuming one or both primaries have elongated shapes. Initially, Riaguas et al. (1999, 2001) analyzed the dynamics of a two-body problem with one primary as a finite straight segment. These works were extended by Jain & Sinha (2014), Kaur et al. (2020), and Kumar et al. (2019) into the restricted three-body problem, assuming both or one primary has an elongated shape. More recently, Verma et al. (2023a) examined the perturbed restricted three-body problem where the smaller primary has an elongated shape and the larger primary is oblate and emits radiation. Verma et al. (2023b) considered the effects of a finite straight segment and oblateness to study the dynamics of the restricted 2 + 2 body problem.

Meanwhile, the effect of a disk-like structure as a perturbing force near a three-body system has been well studied by several authors (see, e.g., Jiang & Yeh 2004; Kushvah 2008b; Kushvah et al. 2012; Kishor & Kushvah 2013; Mahato et al. 2022a). Jiang & Yeh (2004) considered CRTBP by analyzing the influence of a disk-like structure near the three-body system. Yousuf & Kishor (2019) analyzed the combined effects of a disk-like structure, oblateness, and albedo on CRTBP. Mahato et al. (2022a) extended the classical CRTBP by considering both a disk-like structure and an elongated body. Mahato et al. (2022b) investigated the stability of equilibrium points within a perturbed restricted 2 + 2 body problem framework, taking into account the influence of a disk-like structure.

This study aims to obtain the collinear and non-collinear equilibrium points and investigate their stability within a modified CRTBP framework that incorporates radiation pressure, oblateness, a finite straight segment, and a disk-like structure. We extend the work of Yousuf & Kishor (2019) by assuming the small primary is a finite straight segment rather than an oblate body. This also extends Mahato et al. (2022a) since we consider the effects of oblateness and radiation from the larger primary.

We apply our modified CRTBP model to the Sun–Haumea system, assuming the Sun is the larger primary with an oblate shape that emits radiation, while Haumea is the smaller primary with an elongated body. We also consider the Kuiper Belt as a disk-like structure surrounding the Sun–Haumea system. Haumea was selected as our case study due to its unique characteristics, which have captured scientific attention since its discovery in 2003. Haumea's surface is predominantly covered by water ice (Barkume et al. 2006; Pinilla-Alonso et al. 2009; Noviello et al. 2022), and evidence suggests the presence of organic material on its surface (Lacerda et al. 2008; Gourgeot et al. 2016). Recently, Haumea was discovered to have a ring and two satellites named Namaka and Hi'iaka (Ortiz et al. 2017). Moreover, previous studies have proposed Haumea as a destination for space missions in the coming decades (see, e.g., Grundy et al. 2009; Sanchez et al. 2014).

Beyond the Sun–Haumea system, this modified CRTBP model can be applied to other cases. For instance, many exoplanetary systems have been discovered with dust particle disks or asteroid belts believed to be similar to the Kuiper Belt or main asteroid belt in our solar system (see, e.g., Greaves et al. 1998; Matrà et al. 2019). Previous studies have also explained the presence of extra-solar asteroids or dwarf planets near host stars (see, e.g., Jura 2003; Dufour et al. 2010). Furthermore, several space exploration missions have been devoted to studying small solar system bodies near the main belt or Kuiper Belt region. Since many solar system bodies have irregular shapes, it is reasonable to study the combined effects of perturbations from a disk, an elongated body, and an oblate radiating body on the motion of an infinitesimal mass in CRTBP.

The structure of this paper is as follows. In the next section, we present the mathematical formulation of the dynamical model. The positions and stability of equilibrium points are elucidated in Section 3. Section 4 describes the implementation of the dynamical model in the Sun–Haumea system. Finally, a conclusion is provided in Section 5. Here, MATLAB's Symbolic Toolbox is used to conduct certain algebraic calculations and find numerical solutions.

2. Mathematical Formulation of the Dynamical System

In this work, we consider a system where an infinitesimal mass moves under the influence of a larger primary with mass m_1 and a smaller primary with mass m_2. The primaries orbit their common center of mass in circular paths. We treat the larger primary as a radiation source with an oblate spheroid shape, while the smaller primary has an elongated shape. The unit of time is normalized such that the Gaussian gravitational constant equals one. The mass parameter is represented by $= m_2/(m_1 + m_2)$, where $m_1 = 1 -$ and $m_2 =$. For the restricted three-body problem, it is convenient to introduce a rotating coordinate system Oxy. The primaries are located on the x-axis with the distance between them chosen as the unit of length. The coordinates of the larger primary, smaller primary, and third body are (, 0), (− 1, 0), and (x, y), respectively.

The oblateness factor of the larger primary can be represented by $A = (AE^2 - AP^2)/5R^2$, where AE and AP represent the equatorial and polar radii, respectively, and R is the effective radius when assuming the primary to be spherical. Meanwhile, the radiation force F acts opposite to the gravitational force and diminishes with distance. The total force acting on the larger primary can be written as Fg − F = qFg, hence q = 1 − (F /Fg). Here q is called the mass reduction factor, where $0 < 1 - q \leq 1$.

The smaller primary is assumed to be a finite straight segment with length 2l. We also consider the effect of a disk-like structure surrounding the system. Following Miyamoto & Nagai (1975), the dimensionless planar potential of the disk-like structure is given by

$$\Phi_d = -\frac{M_b}{\sqrt{r^2 + T^2}},$$

where M_b is the total mass of the disk-like structure, $r^2 = x^2 + y^2$ is the radial distance of the infinitesimal mass, and T = a + b is the sum of flatness and core parameters.

Let the distances of the primaries to the center of mass be s_1 and s_2. Building on previous works such as Kushvah (2008b), Yousuf & Kishor (2019), and Mahato et al. (2022a), the motion of the primaries is given by

$$n^2 s_1 = \frac{GM_1}{R^2} + \frac{GM_2}{R^2} + \frac{M_b s_1}{(s_1^2 + T^2)^{3/2}},$$

$$n^2 s_2 = \frac{GM_1}{R^2} + \frac{GM_2}{R^2} + \frac{M_b s_2}{(s_2^2 + T^2)^{3/2}},$$

where R = s_1 + s_2 is the distance between primaries. Assuming R = 1, G = 1, and m_1 + m_2 = 1, the mean motion n of the system can be calculated by adding both equations in (1), approximating the expression $1/(1 - l^2)$ as $1 + l^2$ in series, and neglecting the term Al^2. Hence we have

$$n^2 = 1 + \frac{3}{2}A + \frac{M_b}{(1 + T^2)^{3/2}}.$$

The equations of motion of the third body in CRTBP are stated as follows:

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x},$$

$$\ddot{y} + 2n\dot{x} = \frac{\partial U}{\partial y},$$

where U is a pseudo-potential function:

$$U = \frac{n^2}{2}(x^2 + y^2) + \frac{q(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{q(1-\mu)A}{2r_1^3} + \frac{M_b}{\sqrt{r^2 + T^2}}.$$

Here

$$r_1 = \sqrt{(x-\mu)^2 + y^2}$$

is the distance of the third body to the small primary and

$$r_2 = \sqrt{(x - \mu + 1)^2 + y^2}$$

is the distance between the third body and the larger primary. It should be noted that this equation of motion differs from Yousuf & Kishor (2019) since we regard the small body as a finite straight segment.

3.1. Position of Equilibrium Points

The conditions for equilibrium points are $\ddot{x} = \ddot{y} = \dot{x} = \dot{y} = 0$. Hence we can deduce that $\Omega \ = \Omega \ = 0$, i.e.,

$$\frac{\partial U}{\partial x} = 0,$$

$$\frac{\partial U}{\partial y} = 0.$$

In the following, we solve Equations (5) and (6) to find the positions of equilibrium points.

The collinear points lie on the line connecting the primaries, thus we have y = 0. Equation (5) becomes

$$n^2 x - \frac{q(1-\mu)(x-\mu)}{|x-\mu|^3} - \frac{\mu(x-\mu+1)}{|x-\mu+1|^3} - \frac{3q(1-\mu)A(x-\mu)}{2|x-\mu|^5} - \frac{M_b x}{(x^2+T^2)^{3/2}} = 0.$$

To find the solution, we divide the region into three parts: ($-\infty$,   − 1 − l), (− 1 − l,), and (, ∞). Here L_1, L_2, and L_3 are the solutions located in ($-\infty$, − 1 − l), (− 1 − l,), and (, ∞), respectively. Hence we have:

$$x_{L1} < \mu - 1 - l,$$

$$\mu - 1 - l < x_{L2} < \mu,$$

$$x_{L3} > \mu.$$

These three equations have been solved numerically to find each collinear equilibrium point. Only real solutions are considered for the positions of equilibrium points.

Meanwhile, there are two non-collinear equilibrium points, L_4 and L_5. The additional condition for these points is y $\neq$ 0. Equations (5) and (6) can be rewritten as:

$$n^2 x - \frac{q(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x-\mu+1)}{r_2^3} - \frac{3q(1-\mu)A(x-\mu)}{2r_1^5} - \frac{M_b x}{(r^2+T^2)^{3/2}} = 0,$$

$$n^2 y - \frac{q(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} - \frac{3q(1-\mu)Ay}{2r_1^5} - \frac{M_b y}{(r^2+T^2)^{3/2}} = 0.$$

Neglecting higher-order terms of $_1$, $_2$, l^2, and A, we have:

$$n^2 - \frac{q(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3q(1-\mu)A}{2r_1^5} - \frac{M_b}{(r^2+T^2)^{3/2}} = 0,$$

where $r_1 = 1 + \epsilon_1$ and $r_2 = 1 + \epsilon_2$. The position of non-collinear equilibrium points (x_0, y_0) is given by:

$$x_0 = \frac{1}{2} - \mu + \frac{\epsilon_1 - \epsilon_2}{2},$$

$$y_0 = \pm\frac{\sqrt{3}}{2}\left(1 + \frac{\epsilon_1 + \epsilon_2}{2}\right).$$

Hence from Equation (10) we have:

$$\epsilon_1 + \epsilon_2 = \frac{3M_b}{2(1+T^2)^{5/2}} - \frac{3A}{2}.$$

Substituting Equation (11) into Equation (9) gives:

$$\frac{3}{2}(\epsilon_1 - \epsilon_2)(1-2\mu) = \frac{3M_b}{2(1+T^2)^{5/2}}\left(\frac{1}{2} - \mu\right) - \frac{3A}{2}\left(\frac{1}{2} - \mu\right) + \frac{15A}{4}.$$

In the classical case, the positions of these equilibrium points are located at $r_1 = 1$ and $r_2 = 1$. Since perturbations exist, we assume that r_1 and r_2 are perturbed by $_1$ and $_2$, respectively. Hence, in our case, we have (Mahato et al. 2022a):

$$r_1 = 1 + \epsilon_1,$$

$$r_2 = 1 + \epsilon_2.$$

The calculations of $_1$ and $_2$ are performed by substituting Equation (13) into Equations (12) and (11) and solving these equations. By approximating with series and neglecting higher-order terms, and putting the values of $_{1,2}$ into Equation (15), we get:

$$x_0 = \frac{1}{2} - \mu + \frac{5A}{4(1-2\mu)} - \frac{M_b(\frac{1}{2} - \mu)}{2(1+T^2)^{5/2}(1-2\mu)} + \frac{l^2}{2},$$

$$y_0 = \pm\frac{\sqrt{3}}{2}\left[1 + \frac{M_b}{2(1+T^2)^{5/2}} - \frac{A}{2}\right].$$

If the perturbation parameters are not considered, Equation (16) reduces to the classical version where $x_0 = \frac{1}{2} - \mu$ and $y_0 = \pm\frac{\sqrt{3}}{2}$.

3.2. Linear Stability

Let us assume a small displacement from an equilibrium point by defining:

$$x = x_0 + u, \quad y = y_0 + v,$$

where the subscript "0" corresponds to the equilibrium points. The equation of motion for this small displacement is expressed as:

$$\ddot{u} - 2n\dot{v} = \Omega^0_{xx} u + \Omega^0_{xy} v,$$

$$\ddot{v} + 2n\dot{u} = \Omega^0_{yx} u + \Omega^0_{yy} v,$$

where

$$\Omega_{xx} = \frac{\partial^2 \Omega}{\partial x^2}, \quad \Omega_{yy} = \frac{\partial^2 \Omega}{\partial y^2}, \quad \Omega_{xy} = \Omega_{yx} = \frac{\partial^2 \Omega}{\partial x \partial y}.$$

Here Ω_0 means the pseudo-potential evaluated at the equilibrium points, which is constant. Equation (18) has general solutions:

$$u(t) = \sum_{i=1}^{4} \alpha_i e^{\lambda_i t},$$

$$v(t) = \sum_{i=1}^{4} \beta_i e^{\lambda_i t},$$

where α and β are constants while λ are the roots of the characteristic equation. Substituting Equation (22) into Equation (18) produces:

$$\sum_{i=1}^{4} (\lambda_i^2 \alpha_i - 2n\lambda_i \beta_i - \Omega^0_{xx} \alpha_i - \Omega^0_{xy} \beta_i) e^{\lambda_i t} = 0,$$

$$\sum_{i=1}^{4} (\lambda_i^2 \beta_i + 2n\lambda_i \alpha_i - \Omega^0_{yx} \alpha_i - \Omega^0_{yy} \beta_i) e^{\lambda_i t} = 0.$$

The first term on the left-hand side must be a singular matrix. Hence the determinant of this matrix must be zero:

$$\begin{vmatrix} \lambda^2 - \Omega^0_{xx} & -2n\lambda - \Omega^0_{xy} \\ 2n\lambda - \Omega^0_{yx} & \lambda^2 - \Omega^0_{yy} \end{vmatrix} = 0,$$

which yields the characteristic equation:

$$\lambda^4 + b\lambda^2 + c = 0,$$

where

$$b = 4n^2 - \Omega^0_{xx} - \Omega^0_{yy},$$

$$c = \Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{xy})^2.$$

This is a quadratic equation in λ^2. The solution gives:

$$\lambda^2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

If all obtained λ are purely imaginary, the motion exhibits stable periodic behavior in the vicinity of the equilibrium points. However, if at least one λ has a real or complex form, the third body is unstable since u and v will increase exponentially with time. We can investigate the stability behavior by examining the signs of b and c. The system is stable if b > 0, $b^2 - 4c > 0$, and b > √|c|, since this case produces purely imaginary λ .

4. The Case of Sun–Haumea System

In this work, we model the Sun–Haumea system through the framework of the restricted three-body problem with the Sun as the larger primary and Haumea as the smaller primary. We also consider the Kuiper Belt in this system. We assume Haumea has a circular orbit and moves in the same plane as the Kuiper Belt. The mass of the smaller primary is a combination of Haumea's mass and the masses of its satellites Namaka and Hi'iaka. The Sun has a mass of approximately 1.989×10^{30} kg. Haumea has a length of 2300 km for its largest axis and a mass of 4×10^{21} kg (Ragozzine & Brown 2009). Meanwhile, Namaka and Hi'iaka have masses of 1.79×10^{18} kg and 17.9×10^{18} kg, respectively (Ortiz et al. 2017). Hence we have   = 2×10^{-9} and l = 3.5×10^{-7}.

Following Yousuf & Kishor (2019), we assume the Sun has A = 2.6×10^{-11}, while the Kuiper Belt has T = 0.11 and M_b = 3×10^{-7}. According to Sharma (1987), the photogravitational parameter q can be expressed in CGS units as q = 1 − (5.6×10^{-5}/a), where a and   are the radius and density of a moving body, respectively. Assuming a spacecraft has a = 700 cm and   = 0.05 g cm^{-3}, we obtain 1 − q = 1.6×10^{-6}.

We calculated the positions of the collinear equilibrium points in the Sun–Haumea system. By substituting the system parameters into Equation (8) and solving it numerically, we found L_1, L_2, and L_3. Table 1 shows the positions of the collinear equilibrium points. Here we vary each perturbation parameter to examine its impact on the equilibrium point positions. For L_1, the position moves closer to the primaries if A and $1 - q$ increase. Decreasing A and increasing $1 - q$ cause L_2 to move closer to the larger primary. The position of L_3 becomes nearer to the primaries if the larger primary emits stronger radiation pressure. According to Table 1, the positions of collinear equilibrium points also depend on the values of M_b and l. Increasing M_b and decreasing l make L_1's location nearer to the smaller primary. Increases in both M_b and l cause L_2's position to become closer to the larger primary. L_3 moves closer to the primaries as M_b increases.

The positions of non-collinear equilibrium points are calculated from Equation (16). Table 2 shows these positions for various parameter values. When no perturbing factors are present, the triangular points have the same coordinates as in the classical case. The inclusion of perturbation parameters shifts the locations of non-collinear equilibrium points. Increasing A moves these equilibrium points closer to the small primary. In contrast, reducing q or increasing M_b shifts the equilibrium positions toward the larger primary. The positions also move closer to the larger primary as l increases.

We now analyze the linear stability of each equilibrium point in the Sun–Haumea system. Collinear equilibrium points lie on the abscissa, so we have $\Omega\ ^0 = 0$. To study stability, we divide the abscissa into three regions: L_1 $(-\infty,\ \ -1-l)$, L_2 $(\ -1-l,\ )$, and L_3 $(\ ,\infty)$, and calculate the signs of b and $b^2 - 4c$ numerically for each region.

First, we estimate stability using the perturbation parameters of the Sun–Haumea system. As shown in Figure 1, both pure real and pure imaginary characteristic roots exist for   between 0 and 0.5. Hence, all collinear equilibrium points in the Sun–Haumea system are unstable. Furthermore, we performed calculations by varying the perturbation parameters. Table 3 displays these results. All regions have $b < 0$ and $b^2 - 4c > 0$, which means they produce two real pairs and two pure imaginary pairs. This shows that even when perturbation parameter values are changed, the collinear equilibrium points remain unstable.

Next, we investigate the stability of non-collinear equilibrium points in the Sun–Haumea system. We discuss only L_4 since the dynamics of L_5 are nearly identical. In the classical case, non-collinear equilibrium points are stable under the condition $27\ (1 - \ ) < 1$, which yields   < _c, where the critical mass _c = 0.038520896504551. This critical mass can be calculated by solving $b^2 - 4c = 0$. In this modified CRTBP, we numerically calculate the roots by solving Equation (24). When perturbing parameters are considered, the stability of non-collinear equilibrium points exhibits both a maximum limit (_c) and a minimum limit (_o) for the mass parameter, which differs from the classical case. For the Sun–Haumea system, we find _c = 0.0385208896007 and _o = 1.386×10^{-12}. Since the Sun–Haumea system has $= 2 \times 10^{-9}$, we conclude that it possesses stable non-collinear equilibrium points.

Figure 2 displays stability comparisons for several cases with different perturbing parameters in the Sun–Haumea system. The stability range depends on the parameters A, q, l, and M_b. The characteristic roots are purely imaginary if _o < < _c. The considered perturbation parameters alter the stability range in . Increasing A or decreasing q reduces the size of the stability region. The stability region shifts toward larger values when M_b and l increase.

5. Conclusion

We have investigated the dynamics of an infinitesimal mass under the gravitational influence of two primaries. Our study assumes the smaller primary is an elongated body, while the larger primary is oblate and emits radiation. Additionally, we have considered the presence of a disk surrounding the three-body system. We find five equilibrium points in this modified CRTBP: three collinear and two non-collinear. Our numerical exploration of the Sun–Haumea system reveals that including perturbing parameters causes a displacement in the positions of the Sun–Haumea system's equilibrium points relative to their positions in the classical CRTBP. We have shown that the collinear equilibrium points are unstable, while the non-collinear equilibrium points are stable. Moreover, we have ascertained that the collinear equilibrium points remain unstable across several possible ranges of perturbing parameters. In contrast, the non-collinear equilibrium points are conditionally stable with respect to . When accounting for perturbing parameters, we find both upper and lower limits of for achieving stability of non-collinear equilibrium points. The stability region for depends on the perturbing parameters.

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