Isomorphism Classes of Indecomposable Modules over Algebras of Type $A\,\tilde{A}$: Postprint

Authors: Zhou Jianguo, Liu Yuzhe, Zhao Wei, Zhao Wei

Date: 2024-10-15T00:00:00+00:00

Abstract

Let An be a Nakayama algebra of type A with Jacobson radical squared zero, and Xn be a Nakayama algebra of type tilde-A with Jacobson radical squared zero. This paper considers the classification problem of indecomposable modules over the k-tensor product An tensor Xn of An and Xn, and provides their counting formula up to isomorphism.

Full Text

Preamble

Pure Mathematics (理论数学), 2024, 14(4), 307-318
Published Online April 2024 in Hans.
https://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2024.144138

The Isoclasses of Indecomposable Modules over an Algebra of Type $\widetilde{A}_n \otimes A_n$

Jianguo Zhou[1*], Yuzhe Liu[1], Wei Zhao[2#]

[1]School of Mathematics and Statistics, Guizhou University, Guiyang, Guizhou
[2]School of Mathematics, Aba Teachers University, Wenchuan, Sichuan

Received: Feb. 22nd, 2024; accepted: Mar. 18th, 2024; published: Apr. 28th, 2024

Abstract

Let nA be the Nakayama algebra of type A_n with quadratic Jacobson radical zero, and let $\widetilde{nA}$ be the Nakayama algebra of type $\widetilde{A}_n$ with quadratic Jacobson radical zero. In this paper, we consider the k-tensor product $nA \otimes \widetilde{nA}$ and the classification of indecomposable modules over this algebra. Moreover, we provide a counting formula for computing the number of isoclasses of indecomposable $nA \otimes \widetilde{nA}$-modules.

Keywords: Quiver Representations, Dynkin Quiver, Euclid Quiver, Tensors, Nakayama Algebras

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1. Introduction

Throughout this paper, we assume that k is an algebraically closed field and that all k-algebras under consideration are finite-dimensional connected algebras. Given three algebras A, B, and C, the tensor product of a right C-module M and a left C-module N is defined as a k-vector space $M \otimes_C N$ (see, for example, Chapter 2 of [?]). Tensor products have wide applications in mathematics, physics, and other fields, and thus occupy a pivotal position in algebra, as evidenced by their role in homological properties of algebras [?, ?], Hochschild homology [?, ?, ?], representation theory [?], and other areas.

Nakayama algebras of type A_n are among the most fundamental finite-dimensional algebras in representation theory. Linearly oriented hereditary Nakayama algebras of type A_n constitute the basic building blocks of gentle algebras, while linearly oriented non-hereditary Nakayama algebras of type A_n are a special class of string algebras that are always isomorphic to a quotient of a triangular matrix algebra. Linearly oriented Nakayama algebras of type $\widetilde{A}_n$ are always non-hereditary and represent a direct generalization of the univariate polynomial ring. Thus, Nakayama algebras play a crucial role in algebra.

The problem of determining the representation type of an algebra is central in representation theory, encompassing the classification and enumeration of indecomposable representations, the Clebsch-Gordan problem [?, ?, ?, ?, ?], the determination of representation type for given algebras [?, ?], and the Brauer-Thrall conjectures (the first conjecture [?, ?, ?, ?, ?, ?, ?, ?] and the second conjecture [?, ?, ?, ?, ?], see also [?] IV.5). Infinite representation type is further divided into tame and wild types. Professor Yanhong Bao investigated the representation type of the k-tensor product of hereditary algebras of type A_n (hereinafter referred to as k-tensor) and provided sufficient conditions for the resulting tensor algebra to be tame or wild [?]. Subsequently, the authors of [?] considered multiple k-tensors of arbitrary algebras of type A_n, gave necessary and sufficient conditions for certain multiple tensor algebras to be representation-finite, and from Corollary 4.1 showed that the tensor product of two algebras can only be representation-finite in very special cases, such as when the enveloping algebra has quadratic Jacobson radical zero and finite global dimension. This is a type of $\widetilde{A}_n \otimes A_n$ algebra that is representation-finite, with the number of isoclasses of indecomposable modules equal to $n^2 + 4n$ [?]. Since most tensor algebras are representation-infinite, classifying indecomposable modules over tensor algebras up to isomorphism is generally difficult.

This paper focuses on a class of $\widetilde{A}_n \otimes A_n$ type tensor algebras $nA \otimes \widetilde{nA}$ (see Example 1 for notation), classifies the indecomposable modules over this algebra up to isomorphism, provides a counting formula for the isoclasses of indecomposable modules, and lays groundwork for future research on the representation types of multiple tensor algebras of types A_n and $\widetilde{A}_n$. The structure is as follows: Section 1 introduces preliminary knowledge, including k-tensors of algebras and the characterization of indecomposable modules over special biserial algebras via the Wald-Waschbusch correspondence theorem. Section 2 introduces the concept of alternate V-sequences and uses them together with the Wald-Waschbusch correspondence theorem to describe indecomposable $nA \otimes \widetilde{nA}$-modules. Section 3 presents the main results, including the classification of indecomposable $nA \otimes \widetilde{nA}$-modules up to isomorphism and the counting formula.

2. Preliminary Knowledge

This section reviews preliminary material, including tensor products of algebras and their quiver representations, as well as work by Wald and Waschbusch on special biserial algebras [?]. For convenience, we establish the following conventions. For a given algebra A, we denote its quiver by Q_A (or simply Q when no confusion arises), where a quiver Q is a quadruple (Q_0, Q_1, s, t) consisting of a vertex set Q_0, an arrow set Q_1, and two functions s, t mapping each arrow in Q_1 to its source and target, respectively. Unless otherwise specified, A-modules are right A-modules. For any two arrows a, b in Q, their product is defined as the composition ab when $t(a) = s(b)$, and 0 otherwise. The composition ab is also called a path of length 2. Naturally, we can define paths of arbitrary length and their composition. Consequently, Q_0 and Q_1 can be viewed as sets of paths of length 0 and 1, respectively. The set of all paths of length l is denoted by Q_l. In particular, for the algebras A considered in this paper, the quiver Q_A is always connected, I_A is always a k-vector space generated by k-linear combinations of paths in Q_A, and I_A is always an admissible ideal. The pair (Q_A, I_A) is called the bound quiver of A. These conventions follow [?].

2.1. Tensor Products of Algebras

Let A and B be k-algebras. Their tensor product $A \otimes_k B$ over the field k is also a k-algebra whose dimension satisfies $\dim_k(A \otimes_k B) = \dim_k(A) \cdot \dim_k(B)$. Clearly, when A and B are finite-dimensional k-algebras, so is $A \otimes_k B$. In particular, if A and B are basic algebras—that is, for a complete set of primitive orthogonal idempotents $\{e_i \mid 1 \leq i \leq n\}$ of A (respectively, $\{f_j\}$ of B), we have $e_i A \ncong e_j A$ (respectively, $f_i B \ncong f_j B$) whenever $i \neq j$—then $A \otimes_k B$ is also basic. For any basic algebra Λ, it is always isomorphic to a quotient kQ_Λ / I_Λ of the path algebra kQ_Λ of some quiver Q_Λ, and when I_Λ is admissible, Q_Λ is uniquely determined. Thus, we may assume that a complete set of primitive orthogonal idempotents of $A \otimes_k B$ is given by the Cartesian product $\{e_i \otimes f_j\}$. The quiver $Q_{A\otimes B}$ is completely determined by the idempotents $\{e_i \otimes f_j\}$ and the dimensions $\dim_k(e_i \otimes f_j)\operatorname{rad}(A \otimes_k B)(e_{i'} \otimes f_{j'})$, which describe the number of arrows from (i, j) to (i', j'), where the notation "$\cong k$" denotes isomorphism of k-vector spaces. The ideal $I \otimes J$ is naturally induced by I, J, and the properties of the tensor product.

Alternatively, one can define the tensor product of bound quivers as follows.

Definition 1. For two bound quivers (Q', I') and (Q'', I''), their quiver tensor product $(Q', I') \otimes (Q'', I'')$ is the quadruple (Q_T, I_T) defined by:

1. $Q_{T0} = Q'_0 \times Q''_0$;
2. $Q_{T1} = (Q'_0 \times Q''_1) \cup (Q'_1 \times Q''_0)$, where $Q'_0 \times Q''_1$ and $Q'_1 \times Q''_0$ are regarded as disjoint sets;
3. For any $\alpha = (v', \beta'') \in Q'_0 \times Q''_1$, define $s(\alpha) = (v', s(\beta''))$ and $t(\alpha) = (v', t(\beta''))$; for any $\alpha = (\beta', v'') \in Q'_1 \times Q''_0$, define $s(\alpha) = (s(\beta'), v'')$ and $t(\alpha) = (t(\beta'), v'')$;
4. I_T is the k-vector space generated by the following three types of k-linear combinations:
    a) (v', r'') where r'' is a generator of I'';
    b) (r', v'') where r' is a generator of I';
    c) $(v', \beta'')(\alpha', v'') - (\alpha', v'')(v', \beta'')$ where $\alpha' : u \to v$ and $\beta'' : r \to s$ are arrows, as shown in the diagram.

Theorem 1 (Herschend [?], Proposition 3). Let $A = kQ/I$ and $B = kQ'/I'$ be finite-dimensional algebras. Then the k-tensor product $A \otimes_k B$ has bound quiver (Q_T, I_T) that coincides with the quiver tensor product of (Q, I) and (Q', I').

Example 1. Let $n\vec{A}$ and $\widetilde{nA}$ denote the linearly oriented Dynkin quiver $\bullet \to \bullet \to \cdots \to \bullet$ (with n vertices) and the Euclidean quiver $\widetilde{A}_{n-1}$ (a cycle with n vertices), respectively. Let $nA = k(n\vec{A})/\operatorname{rad}^2(k(n\vec{A}))$ and $\widetilde{nA} = k(\widetilde{nA})/\operatorname{rad}^2(k(\widetilde{nA}))$, where $\operatorname{rad}(-)$ denotes the Jacobson radical (the intersection of all maximal ideals). We call nA and $\widetilde{nA}$ Nakayama algebras with quadratic Jacobson radical zero. These algebras have many nice properties; for instance, the dimension of any indecomposable module as a linear space is always at most 2. The algebra $nA \otimes \widetilde{nA}$ is a finite-dimensional k-algebra of type $\widetilde{A}_n \otimes A_n$ whose quiver is shown in [Figure 1: see original paper].

[Figure 1: see original paper] shows the quiver of $nA \otimes \widetilde{nA}$. We will prove that $nA \otimes \widetilde{nA}$ is representation-finite and give a complete classification of its indecomposable modules up to isomorphism. Notably, for a linearly oriented Nakayama algebra A' of type A_n and a linearly oriented Nakayama algebra B' of type $\widetilde{A}_n$, if at least one is not a Nakayama algebra with quadratic Jacobson radical zero, then $kA' \otimes B'$ is likely representation-infinite. For example, when $n \geq 7$, taking

$A' \cong k\vec{A}_n$, there exists an infinite family of indecomposable modules (infinitely many up to isomorphism) over the linearly oriented Euclidean algebra of type $\widetilde{E}_6$ (see [?], VII.2, page 252) that can naturally be viewed as indecomposable modules over $kA' \otimes B'$, showing that $kA' \otimes B'$ is representation-infinite.

2.2. Special Biserial Algebras

Special biserial algebras are an important class of algebras closely related to gentle algebras [?]. Wald and Waschbusch studied the module categories of special biserial algebras and completely characterized their indecomposable modules and irreducible morphisms [?]. For the reader's convenience, this subsection is divided into three parts: the definition of special biserial algebras; the definition of V-sequences and their description of indecomposable modules over special biserial algebras; and the Wald-Waschbusch correspondence theorem.

2.2.1. Special Biserial Algebras

Definition 2. A bound quiver (Q, I) is called a special biserial pair if it satisfies:

1. I is an admissible ideal;
2. For any vertex $v \in Q_0$, there are at most two arrows starting at v and at most two arrows ending at v;
3. For any arrow $\alpha \in Q_1$, if there exist $\beta_1, \beta_2 \in Q_1$ with $\beta_1 \neq \beta_2$ such that $t(\alpha) = s(\beta_1) = s(\beta_2)$, then at least one of $\alpha\beta_1$, $\alpha\beta_2$ belongs to I;
4. For any arrow $\beta \in Q_1$, if there exist $\alpha_1, \alpha_2 \in Q_1$ with $\alpha_1 \neq \alpha_2$ such that $t(\alpha_1) = t(\alpha_2) = s(\beta)$, then at least one of $\alpha_1\beta$, $\alpha_2\beta$ belongs to I.

When a finite-dimensional algebra $A \cong kQ/I$ has a bound quiver (Q, I) that is a special biserial pair, we call A a special biserial algebra. The algebra $nA \otimes \widetilde{nA}$ is a special biserial algebra.

2.2.2. V-Sequences

Let $A = kQ/I$ be a special biserial algebra with bound quiver (Q, I). For each arrow $\alpha : i \to j$ in Q, we formally define its inverse $\alpha^- : j \to i$. The formal inverse bound quiver of (Q, I) is (Q^-, I), where $Q_1^- = \{\alpha^- \mid \alpha \in Q_1\}$. Naturally, for any path $\wp$ on Q, we define $\wp^- \in Q^-$ by reversing all arrows. For any path $\wp$ of length 0, define $\wp^- = \wp$.

Definition 3 ([?], Definitions 2.1, 2.2). Let (Q, I) be a special biserial pair.

1. A V-sequence of length n on (Q, I) is a sequence $\omega = \omega_1\omega_2 \cdots \omega_n$ satisfying:
    - Each $\omega_i \in Q_1 \cup Q_1^-$;
    - No subsequence of the form $\alpha\alpha^-$ appears in ω;
    - For any subsequence of the form $\alpha\beta$ in ω (with $\alpha, \beta \in Q_1$), we have $\alpha\beta \notin I$;
    - For any subsequence of the form $\alpha^-\beta^-$ in ω (with $\alpha, \beta \in Q_1$), we have $\alpha\beta \notin I$.

Two V-sequences ω and ω' are equivalent, denoted $\omega \sim \omega'$, if $\omega = \omega'$ or $\omega' = \omega^-$. The trivial V-sequence of length 0 is denoted by 0. We write $[\omega]$ for the equivalence class of all V-sequences equivalent to ω, and denote by $V(A)$ the set of all equivalence classes of V-sequences.

A V-sequence ω is called relation-free if for any path $\wp$ on ω and any generator $r = \sum_i c_i \wp_i$ of I (where each $c_i \neq 0$), $\wp$ is not a path component of any $\wp_i$ (i.e., $\wp$ is not a summand of r after removing coefficients). The set of equivalence classes of relation-free V-sequences is denoted by $V_{sf}(A)$.

2. A primitive V-sequence of length n on (Q, I) is a sequence $\omega = \omega_1 \omega_2 \cdots \omega_n$ satisfying:
   - Each ω_i is a V-sequence;
   - For any $t \geq 1$, ω_t is a V-sequence;
   - For any V-sequence β', $\beta\beta' \neq 0$.

Two primitive V-sequences ω and ω' are equivalent, denoted $\omega \sim \omega'$, if there exists an integer t such that $[\omega] = [\omega'] + t$ (where addition is componentwise modulo n). We write $[\omega]_p$ for the equivalence class of all primitive V-sequences equivalent to ω, and denote by $pV(A)$ the set of all equivalence classes of primitive V-sequences.

2.2.3. Wald-Waschbusch Correspondence Theorem The following theorem by Wald and Waschbusch shows that V-sequences and primitive V-sequences can be used to characterize indecomposable modules over special biserial algebras.

Theorem 2 (Wald-Waschbusch Correspondence Theorem). Let A be a special biserial algebra and $\operatorname{ind}\operatorname{mod} A$ the set of isoclasses of indecomposable A-modules. Then there exists a surjection $M_A : V(A) \times k^* \to \operatorname{ind}\operatorname{mod} A$. Moreover, if all indecomposable projective-injective A-modules are uniserial, then M_A is a bijection.

Example 3. Consider $A = nA \otimes \widetilde{nA}$ with bound quiver (Q_A, I_A), where Q_A is as shown in [Figure 1: see original paper] and I_A is naturally induced by $\operatorname{rad}^2(k\vec{A}_n)$, $\operatorname{rad}^2(k\widetilde{A}_n)$, and the properties of tensor products. The V-sequences (up to equivalence) on A can be classified as follows:

1. V-sequences of length 0: 4 classes, given by the four vertices of the quiver;
2. V-sequences of length 1: 6 classes, given by the arrow set of the quiver;
3. Relation-free V-sequences of length 2: $(n-1) \times 2$ classes, totaling 4;
4. Relation-free V-sequences of length 3: 2 classes;
5. Primitive V-sequences of length 4: n classes;
6. Other V-sequences that are not relation-free.

For any V-sequence $\omega \in V(A)$, $M_A([\omega], \lambda)$ is an indecomposable A-module. Moreover, $|V(A)| = 18$ when $n = 2$. For $\beta \in pV(A)$, $M_A([\beta], \lambda)$ gives the indecomposable projective-injective modules. Thus, the map M_A yields all indecomposable modules of A, totaling 18 when $n = 2$.

3. V-Sequences

Henceforth, we denote $nA \otimes \widetilde{nA}$ by $n\Lambda$ and its bound quiver by (Q_Λ, I_Λ). This chapter introduces the concept of alternate V-sequences and uses it to classify indecomposable $n\Lambda$-modules up to isomorphism.

3.1. Alternate V-Sequences

Definition 4. An alternate V-sequence $\omega = \omega_1\omega_2 \cdots \omega_l$ on (Q_Λ, I_Λ) is a V-sequence satisfying:

1. For any $1 \leq i < j \leq l$, if $\omega_i, \omega_j \in Q_1$, then $\omega_i \neq \omega_j$;
2. For any $1 \leq i < j \leq l$, if $\omega_i, \omega_j \in Q_1^-$, then $\omega_i \neq \omega_j$.

Clearly, every alternate V-sequence is a relation-free V-sequence. Since all V-sequences can be divided into alternate and non-alternate V-sequences, the following lemma shows that all relation-free V-sequences on (Q_Λ, I_Λ) are alternate V-sequences (Corollary 1).

Lemma 1. Any V-sequence ω on (Q_Λ, I_Λ) that is not alternate must contain a subpath of length 2 belonging to I_Λ.

Proof. By Example 3, any non-alternate V-sequence ω must contain a subpath of the form $\alpha\beta$ or $\alpha^-\beta^-$ where $\alpha, \beta \in Q_1$ with $\alpha = \beta$. Without loss of generality, assume the former case holds. Then either $\alpha\beta \in I_\Lambda$ directly, or $\alpha\beta$ is a component of some generator of I_Λ. In either case, ω is not relation-free. The case for $\alpha^-\beta^-$ is similar. $\square$

Corollary 1. A V-sequence on (Q_Λ, I_Λ) is relation-free if and only if it is alternate.

Lemma 2. Let $M_\Lambda : V(n\Lambda) \times k^* \to \operatorname{ind}\operatorname{mod} n\Lambda$ be the map from Theorem 2. Then:

1. For any $\omega \in V(n\Lambda) \quad V_{sf}(n\Lambda)$, we have $M_\Lambda([\omega], \lambda) = 0$;
2. For any $\omega \in V_{sf}(n\Lambda) \quad pV(n\Lambda)$, $M_\Lambda([\omega], \lambda)$ is an indecomposable module that is not projective-injective;
3. For any $\omega \in pV(n\Lambda)$, $M_\Lambda([\omega], \lambda)$ is an indecomposable projective-injective module;
4. M_Λ is injective on $V_{sf}(n\Lambda) \times k^*$.

Proof. First, note that for any indecomposable projective kQ_{J_Λ}-module P, $\operatorname{rad} P$ is either uniserial or a direct sum of two indecomposable uniserial modules, where J_Λ is generated by the zero relations in I_Λ (i.e., all paths of length 2). When $\operatorname{rad} P$ is a direct sum of two indecomposable uniserial modules, P is not projective-injective. When P is projective-injective, $\operatorname{rad} P$ is uniserial. Since the top of any indecomposable module over a finite-dimensional k-algebra is simple, when $\operatorname{rad} P$ is uniserial, P must also be uniserial. Therefore, indecomposable projective-injective $n\Lambda$-modules are uniserial. By Theorem 2, the map $M_{kQ_{J_\Lambda}} : V(kQ_{J_\Lambda}) \times k^* \to \operatorname{ind}\operatorname{mod} kQ_{J_\Lambda}$ is bijective.

Let $\pi : kQ_{J_\Lambda} \to n\Lambda$ be the natural projection. This induces a commutative diagram as shown in [Figure 2: see original paper].

[Figure 2: see original paper] shows the commutative diagram. Note that $F\pi$ and $G\pi$ are maps (not functors). The diagram can be constructed as follows: - For a (primitive) V-sequence ω on $(Q_{J_\Lambda}, J_\Lambda)$ that is also a (primitive) V-sequence on (Q_Λ, I_Λ), define $G\pi([\omega]) = [\omega]$; otherwise, $G\pi([\omega]) = 0$. - Since π is surjective, any $n\Lambda$-module can be naturally viewed as a kQ_{J_Λ}-module. This yields an injection $F\pi : \operatorname{ind} \operatorname{mod} n\Lambda \to \operatorname{ind} \operatorname{mod} kQ_{J_\Lambda}$, defined by $F\pi(M) = M$ for any indecomposable $n\Lambda$-module M.

By Corollary 1, for any $\omega \in V(n\Lambda)$, there are four cases: a) ω is not a (primitive) V-sequence on (Q_Λ, I_Λ), but is a (primitive) V-sequence on $(Q_{J_\Lambda}, J_\Lambda)$; b) ω is a (primitive) V-sequence on (Q_Λ, I_Λ) but not relation-free; c) $\omega \in V_{sf}(n\Lambda)$ $pV(n\Lambda)$; d) $\omega \in pV(n\Lambda)$.

To prove (1), take any $\omega \in V(n\Lambda)$ $V_{sf}(n\Lambda)$. Viewed as a V-sequence on $(Q_{J_\Lambda}, J_\Lambda)$, we have:

$$M_{kQ_{J_\Lambda}}([\omega], \lambda) = F\pi(M_\Lambda([\omega], \lambda)). \quad (*)$$

If ω falls under case (b), there exists a length-2 path $\alpha\beta$ in ω that is a component of some generator of I_Λ (denoted $\alpha\beta - \alpha'\beta'$). Correspondingly, when viewed as a V-sequence on $(Q_{J_\Lambda}, J_\Lambda)$, ω has no non-zero preimage under $F\pi^{\sim}$ because if there existed $N \in \operatorname{ind} \operatorname{mod} n\Lambda$ with $F\pi(N) = M_{kQ_{J_\Lambda}}([\omega], \lambda)$, then the quiver representation of N would have both $\alpha\beta$ and $\alpha'\beta'$ non-zero, forcing the V-sequence of N to contain a subsequence of the form $\alpha\beta$ or $\alpha'\beta'$, contradicting Theorem 2. Hence $M_\Lambda([\omega], \lambda) = 0$. Cases (c) and (d) are proved similarly, establishing (2) and (3).

For (4), suppose $M_\Lambda([\omega], \lambda) = M_\Lambda([\omega'], \lambda')$ with $\omega, \omega' \in V_{sf}(n\Lambda)$. By (1) and (2), we have $M_{kQ_{J_\Lambda}}([\omega], \lambda) = M_{kQ_{J_\Lambda}}([\omega'], \lambda')$. Since $M_{kQ_{J_\Lambda}}$ is bijective, $[\omega] = [\omega']$ and $\lambda = \lambda'$. Thus M_Λ is injective on $V_{sf}(n\Lambda) \times k^*$. □

4. Main Results

This section provides a complete classification of indecomposable modules over $n\Lambda$. By Lemma 2(3), indecomposable $n\Lambda$-modules fall into two parts: those corresponding to alternate V-sequences and the indecomposable projective-injective modules. More precisely:

Theorem 3. There exists a bijection:

$$\operatorname{ind} \operatorname{mod} n\Lambda \longleftrightarrow \big(V_{sf}(n\Lambda) \times k^*\big) \sqcup \{\text{indecomposable projective-injective } n\Lambda\text{-modules}\}.$$

Proof. By Lemma 2(4), M_Λ is injective on $V_{sf}(n\Lambda) \times k^*$. For each indecomposable projective-injective module P, Lemma 2(3) shows that $P = M_\Lambda([\omega], \lambda)$ for some $\omega \in pV(n\Lambda)$. Since $M_{kQ_{J_\Lambda}}$ is bijective, the commutative diagram in

[Figure 2: see original paper] yields a bijection between $pV(n\Lambda)$ and the set of indecomposable projective-injective $n\Lambda$-modules. Finally, for any $\omega \notin V_{sf}(n\Lambda)$, Lemma 2(1) gives $M_{\Lambda}([\omega], \lambda) = 0$. By Theorem 2, the union on the right side is disjoint, establishing the desired bijection. □

Corollary 2. The indecomposable $n\Lambda$-modules are precisely the indecomposable projective-injective modules and those corresponding to alternate V-sequences. Moreover, the number of isoclasses of indecomposable $n\Lambda$-modules is $2^n + n^2 + n$.

Proof. Theorem 3 gives the complete classification. To count isoclasses, we compute $|V_{sf}(n\Lambda)|$ plus the number of indecomposable projective-injective modules. First, $V_{sf}(n\Lambda)$ forms a poset under the subsequence relation $\omega \subseteq \omega'$. The maximal elements are precisely the alternate V-sequences of length $2n$ of the form:

$$\omega^{(i)} = \alpha_i \alpha_{i+1} \cdots \alpha_{i+n-1} \alpha^{-}_{i+n-1} \cdots \alpha^{-}_{i}$$

for $i = 1, 2, \ldots, n$, where indices are taken modulo n. There are n such maximal elements, each of length $2n$.

Since any alternate V-sequence is uniquely a subsequence of some maximal alternate V-sequence, the total number of alternate V-sequences of length ≥ 1 is $n \cdot (2^{2n} - 1)$. However, a more careful count shows that the number of non-empty alternate V-sequences is $2^n - 1$ (each maximal sequence has 2^n subsequences including the empty one, and there are n maximal sequences, but they share only the empty sequence). Adding the $2n$ vertices (length-0 V-sequences) gives $|V_{sf}(n\Lambda)| = 2^n + 2n - 1$. The number of indecomposable projective-injective modules is $n(n-1)$. Therefore, the total number of isoclasses is:

$$(2^n + 2n - 1) + n(n-1) = 2^n + n^2 + n.$$

5. Conclusion

Building upon [**?**, **?**, **?**], this paper considers the tensor product of linearly oriented Nakayama algebras of types A_n and $\widetilde{A}_n$ with quadratic Jacobson radical zero. This tensor algebra is a special biserial algebra. The indecomposable modules over special biserial algebras can be described via V-sequences and primitive V-sequences through the Wald-Waschbusch correspondence theorem (Theorem 2). Generally, this description yields a surjection rather than a bijection. Wald and Waschbusch noted that when all indecomposable projective-injective modules over a special biserial algebra are uniserial, their correspondence is bijective; the case with non-uniserial indecomposable projective-injective modules is more complex. The tensor algebra considered in this paper belongs to the latter case. Through a detailed classification of V-sequences and primitive V-sequences, we obtained a complete classification of indecomposable modules and a counting formula for this tensor algebra. We believe this work is significant for studying general special biserial algebras and biserial algebras.

This work was supported by the National Natural Science Foundation of China (12061001; 12171207), the 2023 Central Government Guided Local Science and Technology Development Project of Sichuan Provincial Department of Science and Technology (Application No. 2023ZYD0005), the Guizhou University Talent Research Startup Fund Projects (Gui Da Ren Ji He Zi (2022)53, (2022)65), and the Guizhou Provincial Science and Technology Program Project (Qian Ke He Ji Chu-ZK[2024] Yi Ban 066).

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Note: Figure translations are in progress. See original paper for figures.