Postprint: Co-orbital Motion at Arbitrary Inclination

Authors: Lu Rui, Hanlun Lei, Zhou Liyong

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Abstract

Co-orbital bodies share the same semi-major axis as the perturbing body and are in a 1:1 mean motion resonance. The Trojan bodies of several planets in the Solar System are co-orbital objects residing in tadpole orbits, among which the orbital dynamics and origin of some high-inclination Trojans remain incompletely understood. By employing a newly developed expansion of the disturbing function applicable to 1:1 mean motion resonance, we investigate co-orbital motion in three-dimensional space, computing the resonance center and resonance width of co-orbital orbits for various initial orbital elements, and analyzing the relationship between orbit type and initial orbital elements. Furthermore, by comparing and validating the results obtained from the analytical method against those from numerical methods, we obtain a global picture of co-orbital motion across a broad parameter space of initial orbital elements.

Full Text

Preamble

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On the Co-orbital Motion of Any Inclination

LU Rui[1], LEI Han-lun[1,2], ZHOU Li-yong[1,2]†

(1 School of Astronomy and Space Science, Nanjing University, Nanjing 210023)

(2 Key Laboratory of Modern Astronomy and Astrophysics, Ministry of Education, Nanjing 210023)

Abstract

Co-orbital motion occurs when a celestial body shares the same semi-major axis as a perturbing body, placing them in a 1:1 mean motion resonance. Trojan asteroids of several planets in the Solar System represent co-orbital objects in tadpole orbits, yet the orbital dynamics and origin of some high-inclination Trojans remain incompletely understood. Using a newly developed perturbation function expansion applicable to 1:1 mean motion resonance, we investigate co-orbital motion in three-dimensional space, calculating resonance centers and widths for various initial orbital elements and analyzing the relationship between orbital types and initial conditions. Results from our analytical approach are compared with and validated against numerical methods, yielding a comprehensive picture of co-orbital motion across a broad parameter space of initial orbital elements.

Key words celestial mechanics: circular restricted three-body problem, methods: analytical and numerical, co-orbit

1 Introduction

The circular restricted three-body problem possesses five equilibrium points, all located in the orbital plane (xy-plane) of the two primary bodies m and m'. Two of these, L_4 and L_5, form equilateral triangle configurations with the primaries and are termed triangular equilibrium points. When the mass parameter $\mu = m'/(m + m')$ falls below the Routh critical value (~ 0.03852), motion near L_4 and L_5 is linearly stable, and their nonlinear stability (except at the two unstable mass parameter values $\mu = 0.02429$ and 0.01351) was established following the development of the KAM theorem.

In 1906, astronomers observed the first small body moving near Jupiter's L_4 point, providing observational confirmation of this theoretical framework. Such objects moving near the Sun-planet triangular points later became known as Trojan asteroids (hereafter simply "Trojans"). With advancing observational capabilities, Trojans have been discovered for multiple planets: approximately 10,000 Jupiter Trojans, 32 Neptune Trojans, 9 Mars Trojans, 1 Uranus Trojan, and 2 Earth Trojans.

Bodies moving near equilibrium points share the semi-major axis of the secondary primary, hence their motion is termed "co-orbital." Common co-orbital orbit types include tadpole orbits around a triangular point (L_4 or L_5), horseshoe orbits encircling both triangular points and the collinear point L_3, and quasi-satellite orbits near the secondary primary, with tadpole orbits being the most common. These orbit types are illustrated in [Figure 1: see original paper] using a rotating coordinate system that matches the angular velocity of the primaries' circular motion. The examples in [Figure 1: see original paper] assume coplanar orbits with small initial eccentricities. For highly eccentric or highly inclined orbits, hybrid types combining these categories may also appear.

Originally, only objects in Sun-planet tadpole orbits were called Trojans, but since tadpole and horseshoe orbits can transform under perturbations, objects in horseshoe orbits are now often also referred to as Trojans. Trojans exhibit complex and intriguing dynamical characteristics, and their origins and fates may provide important clues to the early history of the Solar System.

Most studies of Trojans and co-orbital motion employ numerical simulations. Some works focus on individual Trojans, performing numerical simulations of observed orbits to analyze their orbital stability and infer their origins, as seen in detailed analyses of newly discovered Earth Trojans and Neptune Trojans. Other studies conduct global investigations of stability regions near planetary triangular points, most notably for Jupiter Trojans, Neptune Trojans, and Earth Trojans. With the discovery of numerous exoplanetary systems, some research has moved beyond specific Solar System planets to examine the long-term stability of triangular points for more general mass parameters, and even to study the dynamical characteristics of planets themselves in Trojan orbital configurations, including potentially habitable Earth-mass planets in such orbits and their observable signatures.

Beyond numerical methods, many studies employ simplified restricted three-body models through analytical or semi-analytical approaches to investigate Trojan dynamical evolution. In Sun-planet-Trojan systems, Trojans share the same mean motion frequency as the planet, resulting in a 1:1 mean motion resonance (MMR) that subjects them to strong planetary perturbations. This 1:1 MMR significantly influences Trojan dynamical evolution and long-term stability. Traditional expansions of the perturbation function contain Laplace coefficients that diverge when the two bodies are at equal distances from the central body, making them unsuitable for handling 1:1 MMR. Moreover, such expansions typically rely on small eccentricity e and inclination i, limiting their applicability to cases with large e and i.

For high-inclination problems, perturbation functions can be constructed for specific inclinations. For instance, for Centaurs and Trans-Neptunian Objects in polar orbits with $i \approx 90^\circ$, expansions can use e and $\cos i$ as small parameters. For retrograde motion with $i \approx 180^\circ$, e and $\cos(i/2)$ serve as expansion parameters. For arbitrary inclinations, a reference inclination i_r can be established such that $\sin(i - i_r)$ is small, enabling perturbation function expansion. Using this approach, Namouni et al. calculated resonance widths and centers for resonances such as 2:1, 3:1, and 1:2. However, their perturbation functions still employed Laplace coefficients and thus could not handle the 1:1 resonance.

To address co-orbital motion, alternative expansion methods avoiding Laplace coefficients have been developed, with Namouni, Namouni et al., and Nesvorný et al. using semi-analytical methods to numerically average the system Hamiltonian, investigating long-term evolution of co-orbital motion for large e and i. These studies typically focus on specific parameter ranges rather than providing systematic descriptions of co-orbital motion in continuous three-dimensional parameter space.

This paper utilizes a recently developed perturbation function expansion to treat the 1:1 MMR problem in the spatial circular restricted three-body model. This method avoids Laplace coefficients and can effectively describe co-orbital motion while being applicable to arbitrary inclinations, with only a moderate restriction on eccentricity ($e < 0.6627$ to ensure convergence). We analyze co-orbital motion within parameter ranges $e \in (0, 0.5)$ and $i \in (0^\circ, 90^\circ)$, focusing particularly on motion near triangular points. The paper structure is as follows: Section 2 briefly introduces the perturbation function expansion and numerical model, Section 3 uses both perturbation function expansion and numerical integration to calculate how resonance centers and tadpole orbit occurrence ranges vary with orbital elements e, i, and argument of periapsis ω, and Section 4 presents our conclusions.

2 Methods and Models

We discuss motion near triangular points within the circular restricted three-body problem framework, where two primary bodies (the Sun and Jupiter in this study) orbit each other circularly under mutual gravitation, while a massless third body (the Trojan) moves in their gravitational field. When treating the 1:1 MMR, we aim to express the perturbation function R affecting the small body as a function of $(a, e, i, \omega, \Omega, \sigma)$, where a is the semi-major axis, Ω is the longitude of ascending node, and σ is the 1:1 MMR resonant angle, typically defined as the difference in mean longitudes: $\sigma = \lambda - \lambda' = (\omega + \Omega + M) - (\omega' + \Omega' + M')$, with M being the mean anomaly and primed quantities referring to the planet's orbital elements.

In the restricted three-body model, the small body's motion around the first primary is perturbed by the second primary's gravity. Since gravity can be expressed as the gradient of a perturbing potential, we introduce the perturbation function:

$$R = Gm' \left(\frac{1}{\Delta} - \frac{r \cos\psi}{r'^2} \right)$$

where G is the gravitational constant, r and r' are distances from the first primary to the small body and second primary respectively, ψ is the angle between the small body and m' as seen from m, and Δ is the distance between the small body and m':

$$\Delta = \left(r^2 + r'^2 - 2rr' \cos\psi\right)^{1/2}$$

First, the perturbation function is expanded in powers of e (requiring $e < 0.6627$ for convergence). Then we define:

$$\Theta = \frac{2aa'}{(a + a')^2} = \frac{4\alpha}{(1 + \alpha)^2}$$

where a and a' are the semi-major axes of the small body and m', and $\alpha = a/a'$ is their ratio. Clearly, $\Theta \in [0,1]$, and when $a = a'$, $\Theta = 1/2$. Defining $\Theta_c = 2\alpha_0/(1+\alpha_0)^2$ (where α_0 is the initial semi-major axis ratio), $\delta = \Theta - \Theta_c$ becomes a small parameter for further expansion. The detailed expansion procedure is described in reference [27]; here we present only the perturbation function after averaging over the fast variable λ' (still denoted as R for simplicity):

$$R = \frac{Gm'}{a'} \sum_{n>0} \sum_{k>0} \sum_{q>0} \sum_{l>0} \sum_{m>0} \sum_{t>0} \sum_{t_1>0} \sum_{t_2>0} \kappa_0 \frac{(2k-1)!!}{(2k)!!} \binom{l}{t} \binom{2k-l}{t_1} \binom{l-t}{t_2} \times$$

$$(-1)^{n-m+k-q} 2^{q-l} (1-\Theta_c)^{1/2+k} \left[D^n f_q(\alpha)\right]_{\alpha_0} X^{2t-l-2t_1+2t_2}_{m,(l-2t_1-2t_2)}(e) \sin^{2l}\frac{i}{2} \tan^{2t}\frac{i}{2} \cos\left[(2t-l-2t_1+2t_2)\sigma + 2(l-t-\right.$$

$$-\frac{Gm'}{a'}\left(\frac{a}{a'}\right)\left[X^{-1}_{1,1}(e)\sin^2\frac{i}{2}\cos(\sigma-2\omega) - X^{1}_{1,1}(e)\cos^2\frac{i}{2}\cos\sigma\right]$$

where $n, k, q, l, m, t, t_1, t_2$ are summation indices, N and $k_{\max}$ are truncation orders for e and δ (taken as 4 and 30 respectively in this work), $X^{\zeta}_{\xi,\eta}(e)$ are Hansen coefficients, D is the derivative operator with subscript n indicating the nth derivative with respect to α, $f_q(\alpha) = (1+\alpha)^{2q+1}$, and κ_0 is given by:

$$\kappa_0 = \frac{(-1)^{n-m+k-q} 2^{q-l} (2k-1)!!}{(2k)!!} \binom{l}{t} \binom{2k-l}{t_1} \binom{l-t}{t_2} (1-\Theta_c)^{1/2+k}$$

The following set of canonical variables can be adopted to express the perturbation function:

$$P_1 = \Lambda - \Lambda', \quad Q_1 = \sigma$$

$$P_2 = \Lambda - \Lambda_p, \quad Q_2 = -\omega$$

$$P_3 = \Lambda_p, \quad Q_3 = \Omega$$

$$P_4 = \Lambda' + P_1, \quad Q_4 = \lambda'$$

where Λ' is the angular momentum conjugate to the planet's mean longitude λ', and $\Lambda_p = \Lambda(1-e^2)^{1/2}\cos i$ gives the coupling relation between a, e, and i during long-term evolution. The averaged Hamiltonian for the 1:1 MMR system (expressed in conventional orbital elements for convenience) is:

$$H^* = -\frac{\mu^2}{2\Lambda^2} - n'\Lambda' - R(a, e, i, \sigma, \omega)$$

In this averaged system, Q_3 and Q_4 are cyclic coordinates, making their conjugate momenta P_3 and P_4 integrals of motion. Since this study focuses on motion near triangular points on resonance timescales, and ω varies much more slowly than the resonant angle, ω can often be approximated as constant, reducing the Hamiltonian system to an integrable one-degree-of-freedom system.

In addition to the semi-analytical calculations using the perturbation function expansion and Hamiltonian formulation, we also perform direct numerical integration of the equations of motion. The Swifter numerical integration package is employed with its Helio algorithm to integrate test particle orbits. Based on extensive testing, we adopt an integration duration of 50 planetary orbital periods with a timestep of 1/20 of the planetary period, maintaining relative position and velocity errors below 10^{-12}. The central body is the Sun, the planet moves on a circular orbit with unit semi-major axis, and the total mass of Sun and planet is normalized to unity with the planetary mass set to $10^{-3}M_{\odot}$ (Jupiter-like).

3 Resonance Structure

In the planar circular restricted three-body model, the two triangular points L_4 and L_5 serve as the two resonance centers of the 1:1 MMR, with resonant angles $\sigma = 60^{\circ}$ and $\sigma = 300^{\circ}$ respectively. L_4 and L_5 are perfectly symmetric about $\sigma = 180^{\circ}$, so we discuss only motion near L_4, with L_5 following analogously.

3.1 Resonance Center

Equilibrium points correspond to minima of the perturbation function, and their positions can be derived from the expansion expression. In reality, when the small body's orbit is neither circular nor coplanar, the resonance center shifts away from 60°. Therefore, to discuss the 1:1 MMR near triangular points, we must first determine the precise resonance center location.

As noted, resonance centers occur at minima of the perturbation function. [Figure 2: see original paper] illustrates the calculation for initial conditions $e_0 = 0.1, i_0 = 30^{\circ}, \omega_0 = 60^{\circ}$. The solid line shows the perturbation function R (left axis) versus initial resonant angle σ_0, with minima marked by dashed lines corresponding to L_4 and L_5. The curve is symmetric about $\sigma = 180^{\circ}$, where a local maximum occurs at L_3. For given (e_0, i_0, ω_0), this method yields the perturbation function minima that define the resonance centers.

Alternatively, resonant bodies exhibit libration in both σ and a, with minimum libration amplitude at the resonance center. Numerical simulations can identify resonance centers by locating these amplitude minima. The scattered points in [Figure 2: see original paper] show the semi-major axis libration amplitude

(Δa) from numerical integration on a logarithmic scale, reaching minima near $\sigma_0 = 60^\circ$ and 300°, consistent with the analytical perturbation function minima.

Both methods reveal how the resonance center (identified by its resonant angle σ_c) varies with orbital elements e_0, i_0, and ω_0. [Figure 3: see original paper] summarizes several representative cases, comparing numerical and analytical results. Panel (a) shows σ_c versus inclination for fixed eccentricity. The resonance center first decreases then slightly increases, reaching an extremum near $i_0 = 70^\circ$–80° ($\sigma_c \sim 56^\circ$). For small $e_0 = 0.1$, numerical and analytical results agree excellently; for larger $e_0 = 0.3$, slight differences appear due to the averaging process in the analytical method and convergence limitations at higher eccentricities, though discrepancies remain acceptable.

Panel (b) displays σ_c versus eccentricity for $i_0 = 10^\circ$ and 30°. Here σ_c increases significantly with eccentricity, reaching $\sigma_c \sim 85^\circ$ for small inclinations. The agreement between methods mirrors panel (a): excellent for low i_0 and e_0, with minor differences at larger values. Panels (c)–(d) examine the argument of periapsis influence, showing that for $e_0 = 0.1, i_0 = 30^\circ$ and $e_0 = 0.3, i_0 = 10^\circ$, σ_c varies by only $\sim 1^\circ$ across ω_0, indicating minimal impact. Therefore, we fix $\omega_0 = 60^\circ$ hereafter without loss of generality. Namouni et al. derived analytical expressions for L_4 and L_5 positions using the Hill three-body problem, finding resonance center locations depend primarily on e and i, consistent with our results.

With ω fixed at 60°, we compute resonance centers across the (e_0, i_0) plane, presented in [Figure 4: see original paper]. Blank regions indicate where tadpole orbits vanish, while dashed lines mark the tadpole existence boundary from numerical methods. At low inclinations, increasing eccentricity drives the resonance center away from $\sigma_c = 60^\circ$ toward L_3. For fixed eccentricity, this deviation slightly decreases with higher inclination. From the perturbation function perspective, this shift minimizes R at the resonance center, while geometrically it prevents close approaches to the perturbing planet.

3.2 Resonance Width

For zero eccentricity and inclination, a body at the resonance center experiences force balance with fixed resonant angle and semi-major axis. When initial conditions deviate from equilibrium or have non-zero e and i, both σ and a oscillate with certain amplitudes. The resonance width is defined as the range of semi-major axis variation that maintains resonant motion, reflecting resonance strength. Resonant angle amplitude can also serve as a width indicator. This study defines resonance width as the boundary values of semi-major axis a during the body's motion.

3.2.1 Calculation Method With ω fixed, Hamiltonian (9) describes a one-degree-of-freedom system with conjugate variables P_1, Q_1 forming the phase space. Plotting Hamiltonian level curves in the (σ, a) plane reveals the global resonance geometry. [Figure 5: see original paper] shows such curves for $e_0 = 0.1, i_0 = 30^\circ, \omega_0 = 60^\circ$. Solid lines are level curves with background grayscale indicating $|H|$ magnitude (darker = larger). The dashed separatrix divides motion types: interior libration around resonance centers corresponds to tadpole orbits, exterior circulation represents non-resonant motion, and the narrow region between them with libration about $\sigma = 180^\circ$ and amplitude $< 360^\circ$ represents horseshoe orbits. The horseshoe region is small, indicating low probability and making it peripheral to this study.

The resonance center appears at the minimum $|H|$, with square markers indicating the $\sigma = 60^\circ$ center. The tadpole resonance width Δa is marked by dotted lines and calculated as:

$$\Delta a = a_{\text{sep}} - a_0 = \frac{[R(\sigma_u) - R(\sigma_s)]}{\partial H^*/\partial a}$$

where a_{sep} is the semi-major axis on the separatrix corresponding to the resonance center, satisfying $H^*(a_{\text{sep}}, \sigma_s) = H^*(a_0, \sigma_u)$. Here σ_s and σ_u represent stable and unstable equilibrium resonant angles, with $R(\sigma_s)$ and $R(\sigma_u)$ being the corresponding perturbation function values. The stable equilibrium is the resonance center $\sigma_s = \sigma_c$, while the unstable equilibrium is near L_3, corresponding to the perturbation function maximum near $\sigma_0 = 180^\circ$ in [Figure 2: see original paper]. Using equations (9) and (10) yields the analytical resonance width Δa. Numerical width can also be obtained by integrating test particle equations of motion and finding the maximum allowed semi-major axis deviation.

3.2.2 Influence of Orbital Inclination [Figure 6: see original paper] compares analytical and numerical resonance widths for $e_0 = 0.1, 0.3$, and 0.5, showing width versus inclination. The horizontal axis gives initial semi-major axis a_0, the vertical axis shows initial inclination i_0, with resonant angles set to the computed centers $\sigma_0 = \sigma_c$ and $\omega_0 = 60^\circ$. Analytical Δa values from (10) appear as red points at $1 \pm \Delta a$. Numerical results sample 50 points in $a_0 \in (0.9, 1.1)$ and 90 points in $i_0 \in (0^\circ, 90^\circ)$, with color indicating semi-major axis variation range (dark to light blue = small to large amplitude). Near resonance centers, amplitudes are small, increasing with distance until a jump occurs at the tadpole-horseshoe transition (yellow points from substituting horseshoe equilibrium into (10)). This jump location marks the maximum tadpole amplitude, whose distance from the center defines the resonance width.

The numerical boundary matches the analytical width excellently. For small e (0.1, 0.3), agreement is good, while for $e_0 = 0.5$, slight discrepancies arise due to poorer perturbation function convergence at high eccentricity. Notably, at $e_0 = 0.5$, the tadpole region shrinks dramatically, essentially disappearing for $i_0 > 30^\circ$. Overall, inclination has modest effect, with tadpole resonance width narrowing slightly as inclination increases.

3.2.3 Influence of Orbital Eccentricity Unlike inclination, initial eccentricity significantly affects 1:1 resonance stability and configuration, with larger eccentricity yielding smaller resonance width. [Figure 7: see original paper] summarizes width versus eccentricity for $i_0 = 0°, 30°$, and $60°$. Since resonant angle amplitude $\Delta\sigma$ shows clearer transitions between orbit types than semi-major axis amplitude, numerical results use $\Delta\sigma$ to distinguish configurations. Due to convergence limitations, analytical results are shown only to $e_0 = 0.5$, while numerical results extend to $e_0 = 0.99$.

As eccentricity increases, the tadpole region shrinks and vanishes at critical values. In the planar case ($i_0 = 0°$), tadpole orbits disappear near $e_0 = 0.88$, replaced by Lyapunov orbits around L_3. Horseshoe orbits appear in a narrow region just exterior to tadpole orbits for $e_0 < 0.10$. For higher inclinations ($i_0 = 30°, 60°$), tadpole orbits vanish at lower eccentricities ($e_0 = 0.49$ and 0.44 respectively). In these high-inclination cases, mixed tadpole-quasi-satellite orbits appear at high eccentricities (marked by +), while horseshoe orbits occur only in the narrow region exterior to low-eccentricity tadpole orbits. Above the dashed lines, small-amplitude ($\Delta\sigma$) motion at high eccentricity represents quasi-satellite orbits. At high inclinations where tadpole orbits exist only at low eccentricity, analytical and numerical widths remain consistent.

Given this agreement at low eccentricity, we use the analytical method to compute tadpole resonance widths across the (e_0, i_0) plane, summarized in [Figure 8: see original paper]. Color indicates Δa magnitude, with blank regions lacking tadpole orbits. The figure shows no tadpole orbits at high inclination or high eccentricity. In regions where tadpole orbits disappear ($e_0 = 0.5, i_0 > 30°$ in [Figure 6: see original paper]; $i_0 = 30°, e_0 > 0.49$ and $i_0 = 60°, e_0 > 0.44$ in [Figure 7: see original paper]), bodies remain in stable 1:1 MMR with regular amplitude distributions, but the orbit type becomes quasi-satellite. While focusing on tadpole orbits, we illustrate this transition in [Figure 9: see original paper] with Hamiltonian level curves for $e_0 = 0.50$ at $i_0 = 20°$ and $40°$.

These high-eccentricity level curves differ markedly from low-eccentricity cases (e.g., [Figure 5: see original paper] for $e_0 = 0.1$). At $e_0 = 0.50$, new fixed points near $\sigma = 0°$ dominate large regions, representing quasi-satellite orbit centers near the planet. For a given eccentricity, the phase space structure varies with inclination: at $i_0 = 20°$, the original L_4/L_5 tadpole centers remain visible, while at $i_0 = 40°$, the L_4 center disappears and orbits librate about the $\sigma \sim 0°$ center as quasi-satellites. Quasi-satellite resonance widths can be calculated analytically, with results marked by orange points in [Figure 6: see original paper] ($e_0 = 0.5$ panel). [Figure 9: see original paper] overlays actual trajectories (orange points) from integrated test particles onto the level curves, showing good agreement despite the truncated, averaged Hamiltonian. Notably, the left panel ($i_0 = 20°$) shows trajectories encircling both tadpole and quasi-satellite centers, corresponding to the "mixed" orbits marked by "+" in [Figure 7: see original paper].

3.3 Influence of Argument of Periapsis

[Figure 9: see original paper] reveals asymmetry between L_4 and L_5 regions, with L_4 tadpole orbits disappearing at smaller inclinations. This arises from our adopted asymmetric $\omega_0 = 60^\circ$ ($\omega_0 = 0^\circ$ would be symmetric). While [Figure 3: see original paper] shows minimal ω_0 dependence at small eccentricity, resonance structure does vary with ω_0 at large e. Though not studied in depth here, [Figure 10: see original paper] illustrates this effect with Hamiltonian phase diagrams for $e_0 = 0.1$ and 0.5 at $\omega_0 = 0^\circ, 60^\circ$, and 120°.

At $\omega_0 = 0^\circ$, L_4 and L_5 regions are symmetric regardless of eccentricity. For $e_0 = 0.1$ (top row), ω_0 variation produces negligible effect, with only minor differences between L_4 and L_5 regions. However, at $e_0 = 0.5$ (bottom row), $\omega_0 = 60^\circ$ eliminates the L_4 tadpole region (yielding quasi-satellite orbits), while $\omega_0 = 120^\circ$ restores L_4 tadpole orbits but eliminates those at L_5.

4 Conclusions

Co-orbital motion represents 1:1 mean motion resonance with a planet, influencing Trojan formation and stability, planetary orbital evolution, and interplanetary dust distribution. Co-orbital bodies experience strong planetary perturbations, and traditional perturbation expansions fail due to divergent Laplace coefficients at equal semi-major axes, particularly for high inclinations and eccentricities. Using a newly developed perturbation function expansion applicable to arbitrary inclinations and 1:1 resonance, combined with numerical integration, we have conducted a global study of co-orbital motion properties, providing complete coverage of resonance centers, widths, and orbit type variations for inclinations $(0^\circ, 90^\circ)$ and eccentricities $(0, 0.5)$.

We have determined how resonance center σ_c varies with orbital inclination and eccentricity, quantitatively mapping its position in the (e_0, i_0) plane. We have analyzed the replacement of tadpole orbits by quasi-satellite and mixed-type orbits at high inclination and eccentricity, explaining the mechanisms behind these transitions. We have quantified tadpole resonance width variations with inclination and eccentricity, delineating the parameter space where tadpole co-orbital motion occurs. We have also examined the influence of argument of periapsis ω, finding negligible effect at small eccentricity but significant impact at large eccentricity, where different ω values lead to different resonant configurations.

All conclusions are supported by both perturbation function analysis and direct numerical integration, with mutual validation confirming the applicability of this expansion method for co-orbital motion.

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Note: Figure translations are in progress. See original paper for figures.