Zero Tensor Factors of k-Algebras

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Abstract

For a k-algebra A over a field k, this paper considers two problems: when the tensor product M N over algebra A is zero, under what conditions can we deduce that M = 0 or N = 0; and when the element m n in the tensor product M N over the algebra is zero, under what conditions can we deduce that m=0 or n = 0.

Full Text

Preamble

The Zero Tensor-Divisors of Algebras

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Abstract

Let A be a k-algebra defined over a field k. This paper addresses two questions: When does the tensor product $M \otimes_A N = 0$ imply $M = 0$ or $N = 0$ for right A-module M and left A-module N? We introduce the concepts of strong and weak zero-tensor-divisors and answer these questions using quiver methods. Specifically, for a connected algebra A whose quiver contains at most one loop, we obtain two main results: (1) A has strong zero-tensor-divisors if and only if its quiver has at least 2 vertices, which is also equivalent to its quiver not being a single loop; (2) If A has no weak zero-tensor-divisors, then A must be infinite-dimensional.

Keywords: quiver representations; tensors; zero-tensor-divisors; dimensions

Introduction

Tensor products are fundamental concepts in multilinear algebra, originally referring to tensor products of rings, with roots tracing back to Frobenius' s work on group representations. Tensors, particularly those of algebras and modules, occupy an extremely important position in mathematics, applied mathematics, and even physics, and thus have consistently attracted widespread attention. For a ring R, the tensor product $M \otimes_R N$ of a right R-module M and left R-module N is essentially a quotient group of an abelian group. Specifically, when M and N also possess left R-module and right R-module structures respectively (where R is also an R-module or right R-module, or when R is a finite-dimensional algebra), then $M \otimes_R N$ naturally has a left R-right R-bimodule structure and can thus be viewed as a left R-module. In this case, $R \otimes_R R$ is called the tensor ring of R. Systematic study of modules over tensor rings constitutes an important area in ring and algebra theory, including quiver representations of tensor algebras [**?**], the Clebsch-Gordan problem [**?**, **?**, **?**, **?**], representation type problems [**?**, **?**], and homological/Hochschild homological properties of algebras [**?**, **?**, **?**, **?**, **?**].

It is well known that when R is a commutative ring, left/right R-modules can naturally be viewed as left R-right R-bimodules (often simply called R-modules). In this case, the tensor product of R-modules is commutative up to isomorphism, i.e., $M \otimes_R N \cong N \otimes_R M$, with the isomorphism given naturally by $m \otimes n \mapsto n \otimes m$. On the other hand, viewing R as an R-module over itself, the tensor multiplication in $R \otimes_R R$ is given by multiplication in R. From this perspective, tensor multiplication is a generalized form of multiplication, and when R is a ring without zero divisors, $m \otimes n = 0$ if and only if $m = 0$ or $n = 0$. However, for a general tensor product $M \otimes_R N$ over a ring R, $m \otimes n = 0$ does not necessarily imply $m = 0$ or $n = 0$ (see Example 1(1) in this paper).

Thus, we can naturally pose the following questions:

Problem 1. Under what conditions on a ring R does $M \otimes_R N = 0$ imply $M = 0$ or $N = 0$ for any right R-module M and left R-module N?

Problem 2. Under what conditions on a ring R does $m \otimes n = 0$ imply $m = 0$ or $n = 0$ for any right R-module M, left R-module N, and any elements $m \in M$, $n \in N$?

This paper answers these questions in the case where R is a k-algebra. For convenience of exposition, we define: A **strong zero-tensor-divisor** of a ring R is a non-zero left R-module M (respectively, right R-module N) for which there exists a non-zero right R-module N (respectively, left R-module M) such that $M \otimes_R N = 0$. A **weak zero-tensor-divisor** of a ring R is a non-zero element $m \in M$ in a finitely generated right R-module M (respectively, non-zero element $n \in N$ in a finitely generated left R-module N) for which there exists a non-zero element $n \in N$ (respectively, $m \in M$) such that $m \otimes n = 0$ (see Definition 1 and Definition 2).

Furthermore, from this point forward we adopt the following conventions: $A = kQ/I$ is a basic algebra over field k (i.e., for a complete set of primitive orthogonal idempotents $\{e_i\}_{i=1}^n$ of A, we have $e_{iAe}i \cong k$ for all i), where Q is a finite connected quiver with Q_0 mapping vertices to starting and ending points of arrows, and I is an admissible ideal. All A-modules considered are finitely generated, and for an arrow α in Q, with $s(\alpha)$ and $t(\alpha)$ denoting its source and target respectively, their composition is denoted as $\alpha\beta$ when $t(\alpha) = s(\beta)$.

The main results of this paper are as follows:

Main Theorem. Let A be a k-algebra. 1. If A is finite-dimensional, then the following statements are equivalent: (a) A has strong zero-tensor-divisors; (b) A is non-simple, its quiver is connected and contains at most one loop; (c) The quiver of A either contains a loop as a proper subquiver, or contains no loops. 2. If A has no weak zero-tensor-divisors, then A is infinite-dimensional.

Part (1) of the Main Theorem answers Problem 2 in the case where A is a non-simple finite-dimensional algebra whose quiver is connected and contains at most one loop. Part (2) shows that any finite-dimensional algebra A always has weak zero-tensor-divisors, which means we have completely negated Problem 1 in the finite-dimensional case. It should be noted that non-simple algebras without weak zero-tensor-divisors do exist (see Example 5).

The structure of this paper is as follows: Section 1 introduces the concept of zero-tensor-divisors. Section 2 examines the existence of strong/weak zero-tensor-divisors on finite-dimensional k-algebras. Section 3 presents our main results.

1.1 Module Tensors

Given a ring R, a right R-module M, and a left R-module N, the R-tensor product $M \otimes_R N$ is an abelian group equipped with an R-bilinear map $T : M \times N \to M \otimes_R N$ such that for any given abelian group G and any R-bilinear map $f : M \times N \to G$, there exists a unique R-module homomorphism $\tilde{f} : M \otimes_R N \to G$ with $f = \tilde{f} \circ T$. Since tensor operations satisfy bilinearity, any element in $M \otimes_R N$ (without loss of generality, assume $M = \langle m_i \mid i \in I\rangle$ and $N = \langle n_j \mid j \in J\rangle$) can always be expanded as:

$$m \otimes n = \left(\sum_{i\in I} m_i r_i\right) \otimes \left(\sum_{j\in J} s_j n_j\right) = \sum_{i\in I, j\in J} m_i \otimes (r_i s_j) n_j = \sum_{i\in I, j\in J} m_i r_i \otimes s_j n_j.$$

Elements in $M \otimes_R N$ are always of the form $\sum_{i=1}^t m_i \otimes n_i$ where $m_i \in M$ and $n_i \in N$.

Example 1. (1) Consider the tensor product $M_{12} \otimes_R N_{21}$ where R is the algebra of 2×2 upper triangular matrices over a field k. There are three indecomposable right R-modules and three indecomposable left R-modules. The elements of M_{12} and N_{21} are cosets of the form $\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$ and $\begin{pmatrix} u & 0 \\ v & 0 \end{pmatrix}$ respectively. Thus

$m \otimes n = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \otimes \begin{pmatrix} u & 0 \\ v & 0 \end{pmatrix}$. Note that as a set, the zero vector in $M_{12} \otimes_R N_{21}$ (as a k-vector space) is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, the above expression equals zero. However, in M_{12} and N_{21}, neither m nor n is a zero element.

(2) By the arbitrariness of M_{12} and N_{21} in (1), we know that $M_{12} \otimes_R N_{21} = 0$ does not imply $M_{12} = 0$ or $N_{21} = 0$.

1.2 Zero-Tensor-Divisors

This section introduces the concept of zero-tensor-divisors.

Definition 1. Let R be a ring. If there exist finitely generated right R-module M and left R-module N such that $M \otimes_R N = 0$ but $M \neq 0$ and $N \neq 0$, then M and N are called **strong zero-tensor-divisors** of R. If R has no strong zero-tensor-divisors (i.e., for any finitely generated right R-module M and left R-module N, $M \otimes_R N = 0$ implies $M = 0$ or $N = 0$), then R is called a **ring without strong zero-tensor-divisors**; otherwise, R is called a **ring with strong zero-tensor-divisors**.

Remark. [**?**, Chapter 2, Proposition 2.45] provides a concrete construction of tensor products.

Example 2. (1) A field is a ring without strong zero-tensor-divisors. For any two finitely generated k-modules M and N over a field k, we have the k-linear isomorphism $M \otimes_k N \cong M \otimes_k N$.

(2) Similar to (1), one can show that any ring without zero divisors must be a ring without strong zero-tensor-divisors.

Definition 2. Let R be a ring. If there exist finitely generated right R-module M and left R-module N with non-zero elements $m \in M$ and $n \in N$ such that $m \otimes n = 0$, then m and n are called **weak zero-tensor-divisors** of R. If R has no weak zero-tensor-divisors (i.e., for any finitely generated right R-module M and left R-module N, if $m \otimes n = 0$ then $m = 0$ or $n = 0$), then R is called a **ring without weak zero-tensor-divisors**; otherwise, R is called a **ring with weak zero-tensor-divisors**.

Example 3. (1) Take R to be the upper triangular matrix algebra from Example 1(1). In Example 1(1), we constructed two non-zero elements in non-zero right R-module M_{12} and left R-module N_{21} whose tensor product is zero, showing that R is a ring with weak zero-tensor-divisors. The arbitrariness of construction in Example 1(2) further shows that R is also a ring with strong zero-tensor-divisors.

(2) A ring without weak zero-tensor-divisors must be a ring without zero divisors. Indeed, suppose a ring R without weak zero-tensor-divisors has zero divisors $r_1, r_2 \neq 0$ with $r_1 r_2 = 0$. When R is viewed as a right R-module and left R-module over itself, we have $r_1 \otimes r_2 \in R \otimes_R R$. However,

the isomorphism $R \otimes_R R \cong R$ given by $r \otimes r' \mapsto rr'$ yields $r_1 \otimes r_2 \mapsto r_1 r_2 = 0$. This constructs non-zero right R-module $r_1 R$ and non-zero left R-module Rr_2 with a weak zero-tensor-divisor, contradicting the assumption.

Proposition 1. A ring with strong zero-tensor-divisors must be a ring with weak zero-tensor-divisors.

Proof. Let R be a ring with strong zero-tensor-divisors. Then there exist finitely generated right R-module M and left R-module N with $M \neq 0$, $N \neq 0$ but $M \otimes_R N = 0$. Since $M \neq 0$ and $N \neq 0$, there exist non-zero elements $m \in M$ and $n \in N$. Then $m \otimes n \in M \otimes_R N = 0$, so $m \otimes n = 0$. This shows R is a ring with weak zero-tensor-divisors.

Remark. In fact, by Example 4 and Example 2(2), a ring without weak zero-tensor-divisors must be a ring without strong zero-tensor-divisors. This is the contrapositive of Proposition 1.

Proposition 2. Let k be a field and $R = k[x]$ the polynomial ring in one variable over k. 1. R is a ring with weak zero-tensor-divisors. 2. Furthermore, if k is algebraically closed, then R is a ring without strong zero-tensor-divisors.

Proof. In this proof, let k be a field and $R = k[x]$. Any finitely generated R-module M is a finite-dimensional k-vector space that can be represented as a pair (V, φ) via quiver methods, where V is a k-vector space, $\varphi \in \mathrm{End}_k(V)$, and the right R-action is given by $x \cdot v = \varphi(v)$.

(1) Consider the pair (V, φ) where φ is a nilpotent Jordan block $J_m(0)$ of size m with minimal polynomial x^m. Then for any $n < m$, we have $J_m(0)^n \neq 0$ but $J_m(0)^m = 0$. Thus there exists $0 \neq w \in V$ such that $J_m(0)^n(w) \neq 0$ but $J_m(0)^m(w) = 0$. Taking $0 \neq v \in V$, we have $x^n \cdot v \neq 0$ but $x^m \cdot (x^n \cdot v) = 0$. This constructs weak zero-tensor-divisors for R.

(2) When k is algebraically closed, any φ can be put into Jordan normal form, so finitely generated indecomposable R-modules correspond to quiver representations of the form $(V, J_m(\lambda))$. Take two non-zero representations $(V, J_m(\lambda))$ and $(W, J_n(\mu))$ giving indecomposable R-modules M and N. Let $\{e_1, \dots, e_m\}$ be the standard basis of V and $\{f_1, \dots, f_n\}$ the standard basis of W. Then there are right R-module isomorphisms $M \cong k[x]/((x-\lambda)^m)$ and left R-module isomorphisms $N \cong k[x]/((x-\mu)^n)$. For any non-zero finitely generated R-modules M and N, they have non-zero direct summands of the form $k[x]/((x-\lambda)^m)$ and $k[x]/((x-\mu)^n)$ (with R-action induced by Jordan blocks) satisfying $M \otimes_R N \neq 0$.

2 Zero-Tensor-Divisor Properties of Algebras

Let $A = kQ/I$ be a finite-dimensional basic k-algebra (Q a connected quiver, I an admissible ideal), with M and N finitely generated right and left A-modules respectively. By the unique decomposition theorem, we can write their complete direct sum decompositions as $M \cong \bigoplus_{i=1}^m M_i$ and $N \cong \bigoplus_{j=1}^n N_j$. Therefore, the study of $M \otimes_A N$ reduces to studying tensors of indecomposable right and left A-modules. Throughout this section, our discussion of tensors assumes we are working with indecomposable modules.

2.1 Strong Zero-Tensor-Divisors in Finite-Dimensional Algebras

We first present a lemma stating that under certain conditions, A-modules can naturally be viewed as A'-modules. For this, we introduce the concept of decoefficient components. Let $\sum_{i\in I} k_i\alpha_i$ be a generator of I, where I is an index set. A **decoefficient component** (or simply **sum component**) refers to the basis vectors corresponding to non-zero summands when this element is expressed as a vector in kQ.

Lemma 1. Let $A = kQ/I$ be a finite-dimensional algebra whose quiver Q contains a subquiver of type $\mathbb{A}_n$ such that for any generator $\sum_{i\in I} k_i\alpha_i$ of I (where I is an index set and $k_i \neq 0$ always holds), no subpath of Q is ever a sum component of I. Then there exists an embedding of categories $F : \text{mod-}A' \hookrightarrow \text{mod-}A$ that naturally views left/right A-modules as left/right A'-modules.

Proof. By assumption, for any generator of I, no subpath of Q appears as a sum component. Therefore, for any non-zero right A-module M, define $F(M)$ as the direct sum of M as a k-vector space (still denoted M) with a right A'-action naturally induced by the above condition, making $F(M)$ a right A'-module. For any $a \in A'$, the k-linear map $M \times A' \to M$ is zero. Otherwise, there would exist some index set I and an element $i \in I$ such that the action is non-zero, implying some subpath must be a sum component, contradicting the hypothesis. Thus the right A-action is simultaneously a right A'-action. This induces a functor $F : \text{mod-}A \to \text{mod-}A'$. The functor naturally embeds a full subcategory of mod-A into mod-A', yielding an embedding of k-categories. The left module case is dual.

Example 4. Let $A = kQ/I$ where Q is the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$ with $I = \langle \beta\alpha - \gamma\beta \rangle$. For any A-module M, let its quiver representation be (M_i, φ_α) where M_i are k-vector spaces. Then M can naturally be viewed as a right A'-module where $A' = kQ'/\langle \beta\alpha, \gamma\beta \rangle$ with $Q' = Q$. However, M cannot be viewed as a right A-module because the definition of I implies $\varphi_\gamma\varphi_\beta = \varphi_\beta\varphi_\alpha$, which need not hold for the induced vector space structure.

Lemma 2. Let $A = kQ/I$ be a finite-dimensional algebra whose quiver Q contains a subquiver of type $\mathbb{A}_2$. Then A is an algebra with strong zero-tensor-divisors.

Proof. We construct two non-zero indecomposable modules $M, N \in \text{indmod-}A$ with $M \otimes_A N = 0$. Let the $\mathbb{A}_2$ subquiver be $v \xrightarrow{\alpha} w$. Consider the right A-injective module $E(v)$ and left A-module $E(w)^\vee$. By Lemma 1' s embedding functor F, we have $F(E(v)) \otimes_A F(E(w)^\vee) = 0$, constructing right A-module M and left A-module N with $M \otimes_A N = 0$. The proof that $M, N \neq 0$ follows from the fact that left A-projective modules are determined by paths ending at vertices.

Thus, as k-vector spaces, using tensor properties we have isomorphisms showing $M \otimes_A N = 0$ while $M, N \neq 0$, proving A has strong zero-tensor-divisors.

Corollary 1. Let Q be a connected quiver containing at least two vertices. Then any finite-dimensional algebra $A = kQ/I$ is an algebra with strong zero-tensor-divisors.

Proof. Connectivity guarantees two vertices v, w connected by an arrow α. The subquiver $v \xrightarrow{\alpha} w$ is of type $\mathbb{A}_2$, so Lemma 2 applies.

Lemma 3. Let $A = kQ/I$ be a finite-dimensional algebra whose quiver Q contains a loop ω at vertex v, and suppose there exists a positive integer n such that $\omega^n \in I$ but $\omega^{n-1} \notin I$. Then there exists an embedding of categories $F : \text{mod-}A' \hookrightarrow \text{mod-}A$ where $A' = k\langle\omega\rangle/\langle\omega^n\rangle$, naturally viewing left/right A-modules as left/right A'-modules.

Proof. Any right A-module has a quiver representation (V, φ) where V is a k-vector space and $\varphi \in \text{End}_k(V)$ gives the right A-action via $x \cdot v = \varphi(v)$. The condition $\omega^n \in I$ ensures that $\varphi^n = 0$, making V a right A'-module. This induces a k-category embedding F that views each right A-module as a right A'-module.

Note the proof does not require discussing whether ω is a sum component of generators of I. If ω is a sum component, the construction in Lemma 3 still yields a right A'-module. If not, Lemma 3 holds for the same reasons as Lemma 1. In particular, when $A = k[x]/(x^2)$, the only indecomposable right modules are the simple module $S(1)$ and the projective-injective module $P(1) = A$. Viewed as right A'-modules, these correspond to right A'-simple and indecomposable modules, making Lemma 3 explicit.

Lemma 4. Let $A = kQ/I$ be a finite-dimensional algebra whose quiver Q is a single loop. Then A is an algebra without strong zero-tensor-divisors.

Proof. Let $F : \text{mod-}A' \hookrightarrow \text{mod-}A$ be the embedding functor from Lemma 3. Since $A' = k[x]/(x^n)$ is a commutative algebra, any left/right A-module is naturally a left/right A'-module. As F is an embedding, any non-zero finitely generated right A-module M and left A-module N satisfy $M \otimes_A N \cong F(M) \otimes_{A'} F(N) \neq 0$ by Proposition 2. Thus A has no strong zero-tensor-divisors.

Corollary 2. Let Q be a quiver (possibly disconnected) containing a loop. Then a finite-dimensional algebra $A = kQ/I$ is without strong zero-tensor-divisors if and only if Q itself is a single loop.

Proof. Lemma 4 gives the "if" direction, while Lemma 2 gives the "only if" direction.

Theorem 1. Let A be a non-simple finite-dimensional algebra whose quiver is connected and contains at most one loop. Then A is without strong zero-tensor-divisors if and only if $A \cong k[x]/I$ where I is an admissible ideal.

Proof. If A has no strong zero-tensor-divisors, Corollary 1 implies Q has only one vertex, say v, so Q must be a single loop. Thus $A \cong k[x]/I$. Conversely, if $A \cong k[x]/I$, then I being admissible implies Q is a loop, and Corollary 2 shows A has no strong zero-tensor-divisors.

2.2 Weak Zero-Tensor-Divisors in Finite-Dimensional Algebras

This section considers weak zero-tensor-divisors. In Section 2.1, we showed that for a connected finite-dimensional algebra A whose quiver has at most one loop, A has strong zero-tensor-divisors if and only if its quiver is not a loop (Theorem 1). By Proposition 1, we immediately obtain:

Corollary 3. If a connected finite-dimensional algebra A has a quiver that is not a loop, then A must have weak zero-tensor-divisors.

The following proposition addresses the case where A' s quiver is a loop, where $A \cong k[x]/I$ has a unique indecomposable projective left/right module. We use this to show A still has weak zero-tensor-divisors.

Proposition 3. Any finite-dimensional algebra $A = k[x]/I$ is an algebra with weak zero-tensor-divisors.

Proof. First, A has a unique indecomposable projective right/left A-module $P = A$. Note that A is a principal ideal domain, so there exists a decomposition of the form $A \cong k[x]/(f(x))$ where $f(x)$ is a polynomial of degree $t \geq 2$ over k. Clearly, in the algebra A, we have $(x-\lambda) \otimes (x-\lambda)^{t-1} = 0$ while $(x-\lambda) \neq 0$ and $(x-\lambda)^{t-1} \neq 0$. This constructs weak zero-tensor-divisors.

By Corollary 3 and Proposition 3, we obtain:

Theorem 2. Every finite-dimensional algebra is an algebra with weak zero-tensor-divisors.

3 Main Results

We now present the main conclusions of this paper.

Theorem 3. Let A be a k-algebra. 1. If A is finite-dimensional, then the following are equivalent: (a) A has strong zero-tensor-divisors; (b) A is non-simple, its quiver is connected and contains at most one loop; (c) The quiver of A either contains a loop as a proper subquiver, or contains no loops. 2. If A has no weak zero-tensor-divisors, then A is infinite-dimensional.

Proof. (1) (a) $\Rightarrow$ (c): By the hypothesis of at most one loop, this is clear. (c) $\Rightarrow$ (b): If the quiver contains no loop or contains a loop as a proper subquiver, then it is not a single loop, so A is not isomorphic to an algebra of the form $k[x]/I$. By Theorem 1, A has strong zero-tensor-divisors. (b) $\Rightarrow$ (a): If A has no strong zero-tensor-divisors, then by Corollary 1 its quiver has only one vertex, which must be a single loop, contradicting (b). (2) follows by taking the contrapositive of Theorem 2.

Note that non-simple algebras without weak zero-tensor-divisors exist, as shown in Example 5.

Example 5. Consider the polynomial ring $k[x_1, x_2, ...]$ in infinitely many variables. For any polynomials $f, g \in k[x_1, \dots, x_n]$, since k is a field, $k[x_1, \dots, x_n]$ is a domain, so $f \cdot g = 0$ implies $f = 0$ or $g = 0$. However, for general polynomial rings, $f \otimes g = 0$ does not necessarily imply $f = 0$ or $g = 0$.

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Note: Figure translations are in progress. See original paper for figures.