The Quantum Distribution of Galton board with Interactions

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Abstract

The Galton board is often used in classrooms as a demonstration experiment for the probability distribution of independent events, and the falling particles exhibit a normal distribution in the lower slots. However, when many particles fall simultaneously, the influence between particles cannot be ignored, and the distribution function in demonstration experiments usually deviates from the Gaussian distribution function. In this paper, we use simulation experiments to study the impact of interactions on distribution laws under different flow rates. We use a simplified model to parameterize α the strength of the interaction, which describes the impact of particle numbers on the direction of particles. The simulation experiment found that the flow rate N_{sm} (number of particles falling simultaneously) and the interaction factor α jointly alter the distribution function. After introducing interactions, the Gaussian distribution, the broadened Gaussian distribution, as well as combinations of multiple Gaussian distributions, cannot fit the experimental data well. The Fermi-Dirac distribution unexpectedly conforms to the experiment, which reflects the "quantum" effect of this classic toy experiment. Particles appear as a whole in certain locations, and the probability is affected by interactions. Probability interpretation and wholeness are the essence of quantum physics. The exclusivity between particles leads to negative chemical potential in the distribution function, and the temperature parameter increases with interaction intensity α and flow rate N_{sm}. The relationships between parameters can be expressed as a conjectural formula within a large parameter range.

Full Text

Preamble

The Quantum Distribution of Galton Board with Interactions

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The Galton board is often used in classrooms as a demonstration experiment for the probability distribution of independent events, where falling particles show a normal distribution in the lower slots. However, when many particles fall simultaneously, the influence between particles cannot be ignored, and the distribution function in demonstration experiments usually deviates from the Gaussian distribution. In this paper, we use simulation experiments to study the impact of interactions on distribution laws under different flow rates. We employ a simplified model to parameterize α as the strength of the interaction, which describes how particle numbers affect the direction of particles. The simulation experiments found that the flow rate N_{sm} (number of particles falling simultaneously) and the interaction factor α jointly change the distribution function. After introducing interactions, neither the Gaussian distribution, broadened Gaussian distribution, nor combinations of multiple Gaussian distributions can fit the experimental data well. Unexpectedly, the Fermi-Dirac distribution conforms to the experiment, which reflects the "quantum" effect of this classic toy experiment.

Particles appear as a whole in certain locations, and their probability is affected by interactions. Probability interpretation and wholeness are the essence of quantum physics. The exclusivity between particles leads to negative chemical potential in the distribution function, and the temperature parameter increases with the interaction intensity α and flow rate N_{sm}. The relationships between parameters can be expressed as a conjecture formula within a large parameter range.

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Introduction

The Galton board [1] is commonly used as a toy experiment to demonstrate probability distributions of independent events. This simple model can also be used to study metal electron gas systems after considering dynamic details [2, 5, 8]. Beyond physics research, Galton boards are also applied in medical and traffic studies [9, 10]. We adopt the following simplified model to discuss the working principle of the Galton board. As shown in [Figure 2: see original paper], the Galton board device consists of evenly spaced vertical slots below and regularly arranged horizontal bars in the middle. The upper container holds a large number of particles. When the small outlet at the bottom of the container is opened, numerous particles flow downward, colliding with the horizontal bars at each layer and falling into the slots. The Galton board simulates the trajectory of particle motion, counting the number of particles falling into each slot and their proportion to the total number of particles.

With the development of numerical calculation software, the Galton board has been easily approached using numerical simulation experiments [4], which can even calculate the dynamic trajectory of falling particles [6, 7]. In our simulation experiments, particles move left or right by one grid according to a set probability for each layer they fall through, which is equivalent to one binomial selection distribution event. If particles fall individually or interactions between particles are ignored, the probabilities of left and right shifts are equal. In this case, after passing through m-layer horizontal bars, one particle falls into one of m slots and the position distribution satisfies the binomial distribution:

$$P_i = \frac{m!}{(i-m)!i!2^m}$$

where P_i is the probability of particles falling into the i-th slot among total m slots. When the number of layers/slots in the Galton board is large enough ($m \geq 9$), the distribution approximates a Gaussian distribution function:

$$f(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

Here, $x = (i-m)/2$ is the slot position where particles enter. The standard deviation $\sigma = 1.76$ when $m = 13$ and $\sigma = 1.45$ when $m = 9$, according to the theoretical formula:

$$\sigma = \sqrt{\frac{m-1}{2\pi^2 \ln m}}$$

We simulated this process using MATLAB and obtained results consistent with theory.

Introduction and Parameterization of Interparticle Squeezing Effects

The probability of particles moving left or right is influenced by the number of particles on both sides, with fewer particles and more vacancies corresponding to higher probability. To simplify the problem, we assume that the probability of selecting left or right movement is influenced by the relative difference in particle numbers on the two sides. If the number of particles in the same grid is N_M, with N_L on the left and N_R on the right, then the probability of left shifting will increase by:

$$\Delta p_L = \frac{\alpha}{2}\frac{N_R - N_L}{N_L + N_M + N_R}$$

where the factor α represents the strength of the interaction, determined by the ratio of space between horizontal bars to particle diameter. The total probability of left and right shifting is normalized as $p_L + p_R = 1$. Therefore, the probabilities are:

$$p_L = \frac{1}{2} + \frac{\alpha}{2}\frac{N_R - N_L}{N_L + N_M + N_R}$$

$$p_R = \frac{1}{2} - \frac{\alpha}{2}\frac{N_R - N_L}{N_L + N_M + N_R}$$

When α is zero, the distribution functions for $N_{sm} = 1$ and $N_{sm} = 10$ are the same Gaussian function. We simulated the process of N_{sm} particles falling simultaneously using MATLAB, as shown in [Figure 3: see original paper]. If α is zero, meaning the particles are tiny and interactions between particles are negligible, then regardless of how many particles fall simultaneously, the probabilities are $p_L = p_R = 1/2$ and the distribution function is the same as that of free particles passing through the Galton board. The simulation experiment results verify this conclusion, as shown in [Figure 4: see original paper], where for simultaneous falling particle numbers $N_{sm} = 1$ and $N_{sm} = 5$, the distribution functions are identical Gaussian functions.

When both N_{sm} and α increase, the simulation distribution obviously deviates from the Gaussian function and cannot be fitted by a broadened Gaussian distribution, as shown in [Figure 5: see original paper]. Initially, we assumed the simulation result consisted of two extended Gaussian distributions with left/right shifts in the mean value. We used two symmetric Gaussian functions with center value shift to fit the simulation results:

$$f(x) = \exp\left(-\frac{(x+\mu)^2}{2\sigma^2}\right) + \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Two shifted Gaussian functions provide a much better fit to the simulation results when N_{sm} is large and α is not close to 1, as shown in [Figure 6: see original paper]. However, two-Gaussian fitting is not effective when N_{sm} is small and $\alpha > 0.7$. Meanwhile, the best-fitting parameters and σ exhibit complex functional curves with different N_{sm} and α values, as shown in [Figure 7: see original paper]. The and σ values of the Gaussian function increase with N_{sm} and α parameters, indicating that the number of particles falling simultaneously and the interaction factor cause particles to disperse on both sides due to collision repulsion during the falling process.

In summary, the extended Gaussian distribution cannot adequately describe the experimental results.

Simulation Models Considering Exclusivity

Considering the exclusivity caused by interactions and the wholeness of particles, we attempt to fit the experiment using the Fermi-Dirac distribution function:

$$f(x) \propto \frac{1}{1 + b\exp\left(\frac{E-E_F}{1+b}\right) + 1}$$

where $E = \frac{x^2}{2\sigma^2}$. The temperature parameter $1 + b$ and Fermi energy (chemical potential) E_F in this function are obtained through experimental fitting. The exclusivity between particles leads to negative chemical potential when N_{sm} is larger, and the temperature parameter increases with the interaction intensity α and flow rate N_{sm}. Fermi-Dirac distribution functions can fit experiments well within the full parameter range for $N_{sm} = 2 \to \infty$ and $\alpha = 0 \to 1$. The best-fitting Fermi-Dirac distribution curves compared with experiments are shown in [Figure 9: see original paper] and [FIGURE:??].

Although the Fermi-Dirac distribution function can describe experiments well, the fitting parameters, especially Fermi energy E_F, exhibit significant fluctuations, as shown in [Figure 10: see original paper]. The reason for these fluctuations is that most particles are distributed within the 3σ range (about 7 slots), which leads to insufficient available data for fitting. To find the relationship between the Fermi-Dirac distribution parameters b, $-E_F$ and experimental parameters α and N_{sm}, more slot numbers m are needed. When the slot number $m = 25$, the fluctuation of parameter b is significantly reduced. The parameter b exhibits a simple linear dependence on the interaction intensity α when flow rate $N_{sm} > 10$.

The “temperature” parameter b can be simply expressed as:

$$b = \kappa_b \alpha e^{-\eta_b \alpha}$$

Here, the slope κ_b slowly increases with the flow rate N_{sm} according to the conjecture formula:

$$\kappa_b = 1.99(1 - \gamma^2); \quad \gamma = \frac{m-1}{N_{sm} + m - 2}$$

The curves of are simple linear functions when parameter $N_{sm} > 10$, as shown in [Figure 12: see original paper]. When $N_{sm} \leq 10$, can be simply expressed as:

$$\xi = \kappa_\xi \alpha e^{\eta_\xi \alpha}$$

The exponential factor η_b rapidly decreases to zero with increasing N_{sm}:

$$\eta_b = 120\beta^4; \quad \beta = \frac{N_{sm} - 1}{N_{sm} + m - 2}$$

When N_{sm} is large, Fermi energy E_F is not sensitive to the quality of fitting. Otherwise, the fitted Fermi energy E_F is not accurate and has large fluctuations, as shown in [Figure 11: see original paper]. We need to find a new parameter that is sensitive to fitting. We find that the ratio of the FD function's second derivative to the function value at the origin is a good parameter. The new sensitive parameter (the ratio minus 1) is defined as:

$$\xi = b + (1 + b)e^{-E_F/(1+b)}$$

The curves of are simple linear functions when parameter $N_{sm} > 10$, as shown in [Figure 12: see original paper]. When $N_{sm} \leq 10$, can be simply expressed as:

$$\xi = \kappa_\xi \alpha e^{\eta_\xi \alpha}$$

When the flow rate N_{sm} is large, these curves are simple linear functions ($\eta_\xi = 0$) of α and the slope $\kappa_\xi \simeq 2.5$. Therefore, parameters b and can be determined using the above simple linear functions of α for $N_{sm} > 10$. The values of E_F can be obtained using the conjectured b and as:

$$E_F = (1 + b) \ln \left(\frac{\xi - b}{1 + b} \right)$$

The above simple linear relationship conjecture formulas can fit the experiment well when the flow rate $N_{sm} > 10$, as shown in [Figure 13: see original paper].

There are still some fluctuations in the curve when $N_{sm} < 10$, which come from uncertain fitting of E_F. To determine the slope κ_ξ and exponential factor η_ξ when N_{sm} is small, we need to eliminate these fluctuations as much as possible. Therefore, we fixed the b parameters using the above conjecture formula, fitted the experimental data again to obtain smooth curves of E_F and .

As shown in [Figure 14: see original paper], the curves can be simply expressed as $\xi = \kappa_\xi \alpha e^{\eta_\xi \alpha}$. The slope κ_ξ increases with the flow rate N_{sm} from 2 to 10 according to:

$$\kappa_\xi = (2.0 + 5.66\beta e^{0.12\beta})\kappa_b; \quad \beta = \frac{N_{sm} - 1}{N_{sm} + m - 2}$$

The exponential factor η_ξ is small and changes according to the following polynomial of $1/N_{sm}$:

$$\eta_\xi = 7.6\beta^2 - 14.5\beta^3$$

In [Figure 15: see original paper], it can be seen that the above conjecture formula for small N_{sm} fits the experimental data well.

Now we can use a similar conjecture formula to express the distribution for $m = 17$. The parameter b is also expressed as:

$$b = \kappa_b \alpha e^{-\eta_b \alpha}; \quad \eta_b = 120\beta^4; \quad \beta = \frac{N_{sm} - 1}{N_{sm} + m - 2}$$

When $N_{sm} > m/2$, the parameter is still linear with α and the slope κ_ξ is a constant 2.6, which is slightly fine-tuned. We fixed the b parameters using the above conjecture formula, fitted the experimental data for $m = 17$ again to obtain smooth curves of E_F and .

When $N_{sm} < m/2$, κ_b becomes a tiny value and the curvature of must be considered, as shown in [Figure 17: see original paper]. can still be expressed as $\xi = \kappa_\xi \alpha e^{\eta_\xi \alpha}$. Here, κ_ξ becomes:

$$\kappa_\xi = (2.17 + 2.36\beta e^{1.53\beta})\kappa_b; \quad \beta = \frac{N_{sm} - 1}{N_{sm} + m - 2}$$

The exponential factor η_ξ changes according to the following polynomial of $1/N_{sm}$:

$$\eta_\xi = 12\beta^2 - 22\beta^3$$

In [Figure 18: see original paper], it can be seen that the above conjecture formulas for small N_{sm} fit the experimental data well for $m = 17$.

When $N_{sm} > m/2$, the parameter b is still linear with α and the slope κ_b slowly increases with the flow rate N_{sm} according to:

$$\kappa_b = 1.68(1 - \gamma^2); \quad \gamma = \frac{m - 1}{N_{sm} + m - 2}$$

Only one coefficient changed from 1.99 for $m = 25$ to 1.68 for $m = 17$.

The Fermi-Dirac distribution degenerates into a Gaussian distribution function when $-E_F/(1 + b) > 2.5$. [Figure 19: see original paper] demonstrates the dependence of $-E_F/(1+b)$ on experimental parameters α and N_{sm} for $m = 17$. The Fermi-Dirac distribution degenerates into a Gaussian distribution function when α is small. The upper limit on α for degradation increases with N_{sm}. When N_{sm} reaches 1000, the distribution degenerates when $\alpha < 0.3$.

Increasing the number of slots m leads to growth of $-E_F/(1 + b)$, making degradation more likely to occur. As seen in [Figure 20: see original paper], the distribution even degenerates when $\alpha < 0.5$ if m increases to 25. This phenomenon indicates that m represents the number of particle states, and when the number of states is large, the states form an almost continuous spectrum. The quantum distribution degenerates to the classical limit, and the Fermi-Dirac distribution becomes the classical Boltzmann distribution, which is equivalent to the Gaussian distribution. To verify the classical limit of large m, we directly extended m to 101. The conjecture formula remains valid, with only the κ_b coefficient adjusted to:

$$\kappa_b = 2.4(1-\gamma^2); \quad \gamma = \frac{m-1}{N_{sm}+m-2}$$

while $m = 101$. Meanwhile, when $N_{sm} > 20$, the parameter is still linear with α and the slope κ_ξ is a constant 2.5, which is the same as for $m = 25$. The large values of $-E_F/(1+b)$ in [Figure 21: see original paper] indicate that classical correspondence does occur in the case of a continuous spectrum. The distribution function returns to a broadened Gaussian distribution, and the broadening ratio of σ^2 has a simple linear relationship with the interaction factor α.

Real Galton Board Experiment

Thus far, all studies have been based on simulation experiment data. To verify whether the above analysis results match real Galton board experiments, a real instrument with $m = 17$ was used to obtain actual data. It is difficult to measure the interaction intensity of particles in a real Galton board, and the flow rate of particles falling simultaneously is also difficult to control accurately. Therefore, the values of α and N_{sm} can only be determined based on experimental data.

The real Galton board is much more complex than the ideal model. Particles falling alone do not follow a binomial distribution because the step size of particle movement is not fixed at \$±\$0.5. The Gaussian distribution is still regarded as the probability distribution function when particles fall alone, but the standard deviation $\sigma \neq 2.02$ and needs to be measured. We first use small particle flow to obtain the standard deviation. The true standard deviation is 2.3, measured from the particle distribution data in [Figure 22: see original paper].

Real experimental devices can only control the flow rate within a rough range, so N_{sm} is not fixed. The particle interaction factor α also cannot be determined as a constant independent of N_{sm}, unlike in the ideal model. Therefore, we can only roughly determine the interaction factor α corresponding to the flow rate N_{sm} within a certain range from the experimental results. The basic standard deviation measured for this experimental device is $\sigma = 2.3$, which corresponds to the value $m = 22$ in the ideal model. According to the conclusion of the simulation experiment, the parameter b is linear with α when $N_{sm} > m/2$, and the slope κ_b slowly increases with the flow rate N_{sm} according to the conjecture formula for $m = 22$:

$$\kappa_b = 1.9(1-\gamma^2); \quad \gamma = \frac{m-1}{N_{sm}+m-2}$$

Based on the real experimental results in [Figure 23: see original paper], the flow rate N_{sm} is approximately 20-50. According to the conjecture formula from the simulation experiment, the equivalent interaction factor of this device is α 0.6−0.7 when $N_{sm} \simeq 20-50$.

Attraction and Bose-Einstein Distribution

The interesting results of simulation experiments of Galton boards with exclusivity between particles can be easily extended to other hypothetical models. It is possible to imagine situations where particles have attractive interactions. We can conveniently set α to negative values to parameterize the toy model. Like the exclusive Galton board, when attraction is present, the probability distribution deviates from the Gaussian distribution, as seen in Figure 24: see original paper. It is reasonable to assume that the particle distribution law would follow Bose-Einstein distribution functions. From Figure 24: see original paper, it can be seen that the Bose-Einstein distribution function fits the experiment much better.

We continue to use $1+b$ as the temperature parameter in the distribution functions, and as the chemical potential. Attraction causes b to be negative, with values close to -1 representing the appearance of Bose-Einstein condensation:

$$f(x) \propto \frac{1}{1+b\exp\left(\frac{E-\mu}{1+b}\right)-1}; \quad E = \frac{x^2}{2\sigma^2}$$

The distribution width $1+b$ is very narrow when attraction $-\alpha$ is strong and particle number N_{sm} is large, meaning almost all particles condense into the same state. The distribution square-width $1+b$ (also the temperature parameter) decreases with $-\alpha$ rapidly and linearly until $1+b$ reaches near zero. The parameter chemical potential $-\mu$ over "temperature" ($kT = 1+b$) represents the difference between quantum distribution and classical distribution functions. From Figure 26: see original paper, it can be seen that Bose-Einstein distributions cannot degenerate into Gaussian distributions when $-\alpha$ is in the middle range.

Discussion and Conclusions

Through simulation and real experiments, we found that the distribution of particles in Galton boards cannot be simply represented by a Gaussian distribution or even a binomial distribution. When the number of slots m is not too large, the particle's position states are discrete. And when the flow rate N_{sm} is not very large, particles appear as a whole at certain positions. The wholeness and exclusivity of particles result in the distribution law following the Fermi-Dirac function, reflecting quantum effects. When the flow rate N_{sm} is maximum, there are enough particles that can be distributed to different positions according to the probability distribution, which is equivalent to particles being divisible, therefore returning to the classical distribution. This simple model can simulate the diffusion process of exclusive fermions, such as electron gas in metals.

This simple toy model can also simulate Bose-Einstein condensation, just by adjusting the parameter α to negative values for attractions. In the model, the number of layers/slots m represents the number of particle states or the evolution time of diffusion, and the number of particles falling simultaneously N_{sm} represents the total particle number in the system. As m and n increase, the particle distribution exhibits a simple statistical pattern. At the same time, when the interaction strength is not too large, the distribution parameters exhibit a linear dependence on the strength. In the extreme case of large numbers, the distribution function and its parameters can be obtained by solving the functional extremum to obtain the distribution with maximum entropy [11].

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Note: Figure translations are in progress. See original paper for figures.