Chaotic Motion of Charged Test Particles in Magnetized Schwarzschild Black Holes (Postprint)

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Abstract

When a Schwarzschild black hole is surrounded by an asymptotically uniform external magnetic field, the Hamiltonian system describing the motion of charged particles in the vicinity of the Schwarzschild black hole becomes non-integrable. Such relativistic Hamiltonian systems do not admit a two-part separable form with explicit analytical solutions, which presents difficulties for the construction and application of explicit symplectic algorithms. A series of recent works has proposed decomposing relativistic Hamiltonian systems into separable forms consisting of more than two parts with explicit analytical solutions, successfully resolving many challenges in constructing explicit symplectic algorithms for relativistic spacetimes. Recent work has addressed two questions: how the number of explicitly integrable separable parts in a Hamiltonian system influences the accuracy of long-term numerical integration, and which explicit symplectic algorithm exhibits the optimal long-term numerical performance, demonstrating that Hamiltonians with the minimum number of integrable separable parts—namely, a three-part split solution form—when applied to an optimized 4th-order partitioned Runge-Kutta explicit symplectic algorithm, can achieve the highest precision. Accordingly, the aforementioned numerical integration method was selected and employed to investigate the orbital dynamics of charged particles moving near a magnetized Schwarzschild black hole using Poincaré sections, maximal Lyapunov exponents, and fast Lyapunov indicators. The results demonstrate that: for specific particle energies and angular momenta, weaker external magnetic fields are unlikely to generate chaotic orbits; larger positive magnetic field parameters readily induce orbital chaos, and as the magnetic field strength increases, the degree of chaos in the orbits intensifies; a moderate increase in particle energy can also enhance the degree of chaos, whereas both negative magnetic field parameters and increasing particle angular momentum suppress chaos.

Full Text

Preamble

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Chaotic Motion of Charged Test Particles in a Magnetized Schwarzschild Black Hole

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Abstract

When an asymptotically uniform external magnetic field exists around a Schwarzschild black hole, the Hamiltonian system describing charged particle motion becomes nonintegrable. Such relativistic Hamiltonian systems lack a two-part separable form with explicit analytical solutions, posing significant challenges for constructing and applying explicit symplectic algorithms. Recent work over the past year has proposed decomposing relativistic Hamiltonian systems into more than two separable parts with explicit analytical solutions, successfully resolving many difficulties in constructing explicit symplectic algorithms for relativistic spacetimes. Our latest work addresses two key questions: how the number of explicit integrable separations in a Hamiltonian system affects long-term numerical integration accuracy, and which explicit symplectic algorithm delivers optimal long-term performance. We demonstrate that the minimal separable decomposition—namely a three-part splitting—combined with an optimized fourth-order Partitioned-Runge-Kutta explicit symplectic algorithm (PRK64) achieves the highest accuracy. Employing this numerical integration method, we investigate the orbital dynamics of charged particles near a magnetized Schwarzschild black hole using Poincaré sections, maximum Lyapunov exponents, and fast Lyapunov indicators. The results show that for specific particle energies and angular momenta, weak external magnetic fields rarely produce chaotic orbits, while larger positive magnetic field parameters readily induce chaos that intensifies with increasing field strength. Moderately increasing particle energy also enhances chaos, whereas negative magnetic field parameters and larger angular momenta suppress chaotic behavior.

Keywords: celestial mechanics, black hole, magnetic field, chaos, integrators

1 Introduction

In recent years, the successful detection of gravitational waves [1–2] and the release of images of the supermassive black hole M87* [3–4] and Sagittarius A* (Sgr A*) [5–6] have confirmed Einstein's general relativity predictions about black holes. The physical properties of black holes and tests of gravitational theory have consequently become hot research topics.

The Schwarzschild black hole represents a solution to Einstein's field equations. With four independent constants of motion—particle energy, angular momentum, the four-velocity relation, and a Carter-like constant (also called azimuthal motion) [7]—the Schwarzschild system is integrable. Although analytical solutions theoretically exist, they cannot be expressed in elementary functions and are only representable through elliptic integrals. Most observable black holes possess plasma accretion disks that generate magnetic fields externally. These fields are generally weak and have negligible effect on black hole spacetime geometry, yet the Lorentz force acting on particles with large charge-to-mass ratios significantly influences particle dynamics [8–12]. When an external electromagnetic field permeates Schwarzschild spacetime, the Hamilton-Jacobi equation often becomes non-separable, meaning no Carter-like constant exists, rendering the corresponding Hamiltonian system nonintegrable. Under certain conditions, charged particle motion around magnetized black holes exhibits chaotic phenomena [13–20]. Common methods for detecting orbital chaos include Poincaré sections, maximum Lyapunov exponents [21], and fast Lyapunov indicators [22].

Numerical methods are the most widely used approach for studying chaotic systems. For long-term integration of Hamiltonian systems, symplectic integrators that preserve Hamiltonian phase flow are naturally preferred, as they prevent secular growth of energy errors. Symplectic algorithms are classified as explicit [23–24] or implicit [25], with explicit-implicit hybrid algorithms also frequently employed [26–31]. Explicit algorithms offer computational efficiency advantages over implicit methods.

Hamiltonians corresponding to relativistic spacetimes generally lack a two-part separable form with explicit analytical solutions, making explicit symplectic algorithm construction extremely difficult. Consequently, previous applications of symplectic algorithms to relativistic spacetime problems were limited to implicit [25] or explicit-implicit hybrid forms [26–31]. While explicit symplectic-like methods based on midpoint permutations in extended phase space [14, 32–37] can handle non-separable relativistic spacetime problems, Professor Wu Xin's team has recently published a series of works in the Astrophysical Journal [17, 20, 38–44] proposing decomposition of relativistic Hamiltonians or time-transformed Hamiltonians into more than two explicitly integrable parts, successfully solving many challenges in constructing explicit symplectic algorithms for relativistic spacetimes. This approach can also be applied to Yoshida high-order explicit symplectic algorithms [45–46].

Two important questions remain: whether the number of explicit integrable separations affects long-term numerical accuracy, and which explicit symplectic algorithm delivers optimal performance. Our recent Astrophysical Journal publication [44] answers these questions, demonstrating that the minimal separable decomposition—specifically a three-part splitting—combined with an optimized fourth-order Partitioned-Runge-Kutta explicit symplectic algorithm (PRK64) [47] achieves the best accuracy. Note that this fourth-order PRK method, compared to the same-order Yoshida algorithm, involves more additional time coefficients and combinations of sub-Hamiltonian solutions, making the optimized PRK algorithm computationally more expensive than the Yoshida algorithm. However, the additional time cost is not excessive, and the method offers smaller truncation errors, making it highly recommended for practical applications.

This work follows the approach of our recent study [44], employing the fourth-order optimized PRK explicit symplectic method with a three-part Hamiltonian decomposition to investigate the orbital dynamics of charged particles near a magnetized Schwarzschild black hole using Poincaré sections, maximum Lyapunov exponents, and fast Lyapunov indicators.

2 Physical Model

This section presents the dynamical model describing charged particle motion around a Schwarzschild black hole with an asymptotically uniform magnetic field, followed by the symplectic algorithm suitable for solving the Hamiltonian system of magnetized Schwarzschild spacetime.

2.1 Schwarzschild Spacetime

In spherical coordinates (t, r, θ, ϕ), where t represents coordinate time and r, θ, ϕ denote three-dimensional spatial coordinates analogous to x, y, z in Cartesian coordinates, the Hamiltonian function governing the motion of a test particle with charge q near a magnetized Schwarzschild black hole is:

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = -m^2,$$

where q is the particle charge, $\mathbf{p}$ is the generalized momentum determined by the standard Hamiltonian canonical equations $\dot{x}^\mu = \partial H/\partial p_\mu$ [48]. Here p_μ, p_ν represent covariant generalized momenta, A_μ, A_ν represent electromagnetic four-vector potentials, and indices μ, ν denote arbitrary coordinate symbols from (t, r, θ, ϕ). The non-zero contravariant components of the Schwarzschild black hole metric are:

$$g^{tt} = 1/g_{tt} = -\left(1 - \frac{2}{r}\right), \quad g^{rr} = 1/g_{rr} = \left(1 - \frac{2}{r}\right)^{-1},$$

$$g^{\theta\theta} = 1/g_{\theta\theta} = \frac{1}{r^2}, \quad g^{\phi\phi} = 1/g_{\phi\phi} = \frac{1}{r^2 \sin^2\theta}.$$

The four-velocity $\dot{x}^\mu$ is the derivative of coordinate x^μ with respect to proper time τ. Two constant covariant momentum components are:

$$p_t = g_{tt}\dot{t} = -\left(1 - \frac{2}{r}\right)\dot{t} = -E,$$

$$p_\phi = g_{\phi\phi}\dot{\phi} + qA_\phi = r^2 \sin^2\theta\dot{\phi} + \frac{qBr^2 \sin^2\theta}{2} = L.$$

Here p_t, p_ϕ represent generalized momenta in the t and ϕ directions, B denotes magnetic field strength, E is the constant particle energy, and L is the constant particle angular momentum. The only non-zero covariant component of the electromagnetic field potential in equation (4) adopts the form from [49]:

$$A_\phi = \frac{Br^2 \sin^2\theta}{2}.$$

Substituting equations (3)–(5) into (1), the magnetized Schwarzschild Hamiltonian can be simplified to a 2-degree-of-freedom, 4-dimensional phase space system:

$$H = \frac{1}{2}g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = -\frac{1}{2}m^2,$$

which becomes:

$$H = -\frac{1}{2}\left(1 - \frac{2}{r}\right)^{-1} E^2 + \frac{1}{2}\left(1 - \frac{2}{r}\right)p_r^2 + \frac{1}{2r^2}p_\theta^2 + \frac{1}{2r^2 \sin^2\theta}\left(L - \frac{qBr^2 \sin^2\theta}{2}\right)^2.$$

Here p_r, p_θ represent generalized momenta in the r and θ directions, and $\beta = qB$. Equation (6) is obtained after nondimensionalization, with the speed of light c and gravitational constant G set to unity ($c = G = 1$). The dimensionless scaling uses black hole mass M and particle mass m: $t \to tM$, $r \to rM$, $B \to B/M$, $E \to mE$, $p_r \to mp_r$, $L \to mML$, $p_\theta \to mMp_\theta$, $q \to mq$, $H \to m^{2H}$. The test particle mass is m, and the gravitational source mass becomes geometric units: $M = 1$.

In addition to the two constants E and L, the Hamiltonian function (6) itself is always a conserved quantity. For timelike spacetime, this conserved quantity is $H = -\frac{1}{2}$. When no asymptotically uniform magnetic field exists outside the Schwarzschild black hole, the fourth integral of motion in Hamiltonian (6) exists, making the system strictly integrable. However, when such a magnetic field is present, this fourth integral disappears, rendering the system nonintegrable and potentially chaotic.

2.2 Algorithm Implementation

Reference [39] decomposed the Hamiltonian function (6) into four explicitly integrable parts to construct an explicit symplectic algorithm. Our recent work [44] demonstrates that the Hamiltonian (6) can also be decomposed into three or five explicitly integrable parts to build explicit symplectic algorithms, with the three-part decomposition showing the highest accuracy. Following this approach, the three-part splitting method is:

$$H = H_1 + H_2 + H_3,$$

where each partial Hamiltonian is:

$$H_1 = -\frac{1}{2}\left(1 - \frac{2}{r}\right)^{-1} E^2,$$

$$H_2 = \frac{1}{2}\left(1 - \frac{2}{r}\right) p_r^2 + \frac{1}{2r^2} p_\theta^2,$$

$$H_3 = \frac{1}{2r^2 \sin^2\theta}\left(L - \frac{\beta r^2 \sin^2\theta}{2}\right)^2.$$

Clearly, the three partial Hamiltonians H_1, H_2, and H_3 are all integrable with analytical solutions that are explicit functions of time t.

The evolution operators for solving the three parts are defined as $\mathcal{H}^t_{H_1}$, $\mathcal{H}^t_{H_2}$, and $\mathcal{H}^t_{H_3}$. Let h be the integration step size. Two first-order symplectic operators for Hamiltonian H are:

$$S_1^H = \mathcal{H}^h_{H_1} \times \mathcal{H}^h_{H_2} \times \mathcal{H}^h_{H_3},$$

$$S_1^{H*} = \mathcal{H}^h_{H_3} \times \mathcal{H}^h_{H_2} \times \mathcal{H}^h_{H_1},$$

where S_1^{H*} is the adjoint symplectic operator of S_1^H. The product of these two first-order symplectic operators forms a symmetric composition, yielding a second-order explicit symplectic method:

$$S_2^H = S_1^H \times S_1^{H*}.$$

Three second-order symmetric products can construct a fourth-order Yoshida explicit symplectic integrator [45]:

$$S_4^H = S_2^H(c_{1h}) \times S_2^H(c_{2h}) \times S_2^H(c_{1h}),$$

where $c_1 = 1/(2 - 2^{1/3})$ and $c_2 = 1 - 2c_1$. Furthermore, we can establish the optimized fourth-order Partitioned-Runge-Kutta PRK64 explicit symplectic algorithm [44, 47]:

$$\mathrm{PRK64} = S_1^{H\alpha_{12}} \times S_1^{H*\alpha_{11}} \times S_1^{H\alpha_{10}} \times S_1^{H*\alpha_9} \times S_1^{H\alpha_8} \times S_1^{H*\alpha_7} \times S_1^{H\alpha_6} \times S_1^{H*\alpha_5} \times S_1^{H\alpha_4} \times S_1^{H*\alpha_3} \times S_1^{H\alpha_2} \times S_1^{H*\alpha_1}.$$

From step $(n-1)$ to step n, the specific discrete difference scheme is as follows:

For $\mathcal{H}_{H_3}^h$:

$$r_{H_3} = \left[(r_{n-1}^2 - 3hp_{r,n-1})^2 / r_{n-1}\right]^{1/3},$$

$$p_{r,H_3} = p_{r,n-1}\left[(r_{n-1}^2 - 3hp_{r,n-1})/r_{n-1}^2\right]^{1/3}.$$

For $\mathcal{H}_{H_2}^h$:

$$e_1 = \cos\theta_{n-1}/r_{H_3} + p_{\theta,n-1}\sin\theta_{n-1},$$

$$e_2 = \sin\theta_{n-1}/r_{H_3} - p_{\theta,n-1}\cos\theta_{n-1},$$

$$f_1 = \arctan 2(e_2, e_1),$$

$$f_2 = \tan(\theta_{n-1} - f_1),$$

$$\theta_{H_2} = f_1 + \arctan\left[(e_2 + e_{1f}2)hp_{\theta,n-1} + f_2\right],$$

$$p_{\theta,H_2} = p_{\theta,n-1}(e_1\sin\theta_{n-1} - e_2\cos\theta_{n-1}),$$

$$r_{H_2} = 1/(e_1\cos\theta_{n-1} + e_2\sin\theta_{n-1}).$$

For $\mathcal{H}_{H_1}^h$:

$$p_{r,H_1} = p_{r,H_2} + h\left\{\left[L - \beta(r_{H_2})^2\sin^2\theta_{H_2}/2\right]^2/[(r_{H_2})^3\sin^2\theta_{H_2}] - \beta\left[L - \beta(r_{H_2})^2\sin^2\theta_{H_2}/2\right]/r_{H_2} - E^2/(r_{H_2} -$$

$p_{\theta,H_1} = p_{\theta,H_2} + h \left[-(\cos\theta_{H_2}\{L - [\beta(r_{H_2})^2 \sin^2\theta_{H_2}/2]^2\})/[(r_{H_2})^3 \sin^2\theta_{H_2}] - \{\beta\cos\theta_{H_2}[L - \beta(r_{H_2})^2 \sin^2\theta_{H_2}/2]\}\right.$

The operator $S_1^H = \mathcal{H}_{H_3}^h \times \mathcal{H}_{H_2}^h \times \mathcal{H}_{H_1}^h$ yields the updates:

$p^*_{r,n-1} = p_{r,n-1} + h \left[(L - \beta r_{n-1}^2 \sin^2\theta_{n-1})^2/(r_{n-1}^3 \sin^2\theta_{n-1}) - \beta(L - \beta r_{n-1}^2 \sin^2\theta_{n-1}/2)/r_{n-1} - E^2/(r_{n-1}-2)^2 \right]$

$p^*_{\theta,n-1} = p_{\theta,n-1} + h \left[-\cos\theta_{n-1}(L - \beta r_{n-1}^2 \sin^2\theta_{n-1}/2)^2/(r_{n-1}^3 \sin^2\theta_{n-1}) - \beta\cos\theta_{n-1}(L - \beta r_{n-1}^2 \sin^2\theta_{n-1}/2)/\sin\right.$

For the adjoint operator S_1^{H*}, we compute:

$$e_1^* = \cos\theta_{n-1}/r_{n-1} + p^*_{r,n-1}\sin\theta_{n-1}/p^*_{\theta,n-1},$$

$$e_2^* = \sin\theta_{n-1}/r_{n-1} - p^*_{r,n-1}\cos\theta_{n-1}/p^*_{\theta,n-1},$$

$$f_1^* = \arctan 2(e_2^*, e_1^*),$$

$$f_2^* = \tan(\theta_{n-1} - f_1^*),$$

$$\theta^*_{H_2} = f_1^* + \arctan\left[(e_2^* + e_1^* f_2^*) h p^*_{\theta,n-1} + f_2^*\right],$$

$$p^*_{\theta,H_2} = p^*_{\theta,n-1}(e_1^*\sin\theta_{n-1} - e_2^*\cos\theta_{n-1}),$$

$$r^*_{H_2} = 1/(e_1^*\cos\theta_{n-1} + e_2^*\sin\theta_{n-1}).$$

Finally:

$$r_n = \left\{[(r^*_{H_2})^2 - 3hp^*_{r,H_2}]^2/r^*_{H_2}\right\}^{1/3},$$

$$p_{r,n} = p^*_{r,H_2}\left\{[(r^*_{H_2})^2 - 3hp^*_{r,H_2}]/(r^*_{H_2})^2\right\}^{1/3}.$$

The coefficients $\alpha_1, \alpha_2, \cdots, \alpha_{12}$ are listed in . An optimized algorithm means that free coefficients among a_i, b_i minimize the sum of squares of coefficients in fifth-order (optimized fourth-order explicit symplectic algorithm) truncation error terms. Compared to non-optimized algorithms, this reduces truncation errors. Equations (12)–(16) represent explicit symplectic algorithms for solving Hamiltonian function (6) or (8).

Table 1 Correlation coefficients of PRK64 integrators

Integrators	Coefficients
PRK64	$\alpha_1 = \alpha_{12} = 0.079203696431196$
	$\alpha_2 = \alpha_{11} = 0.130311410182166$
	$\alpha_3 = \alpha_{10} = 0.222861495867608$
	$\alpha_4 = \alpha_9 = -0.366713268047426$
	$\alpha_5 = \alpha_8 = 0.324648188689706$
	$\alpha_6 = \alpha_7 = 0.109688477876750$

3 Numerical Analysis

We first evaluate algorithm performance, then select the superior algorithm combined with chaos indicators to study charged particle orbital dynamics, particularly examining how three dynamical parameters affect orbital chaos.

3.1 Algorithm Comparison

Using a time step of $h = 1$ and parameters: magnetic parameter $\beta = 8.9 \times 10^{-4}$, particle energy $E = 0.995$, and angular momentum $L = 4.6$. Initial conditions are $p_r = 0$, $\theta = \pi/2$. Once the initial radius r is given, the initial p_θ value is taken as positive and determined by equation (7). [Figure 1: see original paper] shows Hamiltonian errors for two fourth-order algorithms. Figure 1: see original paper displays Hamiltonian errors $\Delta H = H + 1/2$ for orbits with initial radius $r = 11$ integrated by both fourth-order algorithms, where the red curve represents the Yoshida symplectic algorithm S4 and the green curve represents PRK64. Both curves remain stable without growing over time. The explicit symplectic algorithm S4 maintains Hamiltonian error at approximately 10^{-9} to 10^{-8}, while the optimized explicit symplectic algorithm PRK64 achieves 10^{-13} to 10^{-12}, four orders of magnitude more accurate than S4. Figure 1: see original paper shows error plots for initial radius $r = 88$. Both fourth-order algorithms exhibit very high precision. The higher accuracy for larger initial radii occurs because the average orbital period is shorter, though after integrating to $\tau = 10^4$ both curves shift upward due to round-off errors exceeding truncation errors. Despite this long-term growth trend, accuracy remains around 10^{-13}. Note that the optimized explicit symplectic algorithm requires more computational time. lists CPU times for both algorithms, showing the optimized PRK method takes slightly longer than S4, but overall computation times remain short, with maximum CPU time under 3 minutes. Therefore, considering both accuracy and efficiency, the PRK64 algorithm is recommended for practical calculations.

Table 2 CPU times for the two integrators in Fig. 1

Integrators	$r = 11$	$r = 88$
S4	1'22"	1'53"
PRK64	1'48"	2'24"

3.2 Orbital Dynamics

The different long-term Hamiltonian error behaviors for the two orbits in [Figure 1: see original paper] stem from their distinct dynamical properties. The orbit with initial radius $r = 11$ corresponds to regular (quasi-periodic) motion, while $r = 88$ corresponds to chaotic motion—exhibiting exponential sensitivity to small changes in initial conditions. The regular or chaotic nature of orbits can be roughly observed through three-dimensional spatial configurations and planar projections.

Selecting initial orbital radius $r = 25$, particle energy $E = 0.995$, and angular momentum $L = 4$, Figure 2: see original paper–(c) shows three-dimensional orbital configurations for three different magnetic parameters β, with red indicating projections on the equatorial plane $\theta = \pi/2$. In Figure 2: see original paper with $\beta = 0$, the test particle motion is confined to the equatorial plane in circular periodic orbits around the black hole. In Figure 2: see original paper with $\beta = 1.2 \times 10^{-4}$, the three-dimensional orbit is non-planar, with its projection forming intersecting loops. In Figure 2: see original paper with $\beta = 0.01$, the three-dimensional orbit is also non-planar, but its projection consists of interlaced loops that appear more irregular and disordered than in Figure 2: see original paper. Whether these orbits are chaotic or regular cannot be easily determined from spatial configurations alone and requires additional methods.

3.2.1 Chaos Indicators Several methods exist for determining whether particle orbits around Schwarzschild black holes are chaotic, including Poincaré sections, Lyapunov exponents, and fast Lyapunov indicators.

Poincaré sections are suitable for studying 2-degree-of-freedom, 4-dimensional conservative systems and can determine the motion state. Using $\theta = \pi/2$ as the surface, with $\theta > \pi/2$ above and $\theta < \pi/2$ below, points on the surface can be obtained through linear interpolation. When only a few points or closed curves appear on the section, the system undergoes quasi-periodic motion; when points are randomly distributed across a region, the motion is chaotic.

The Lyapunov exponent measures the average exponential separation ratio of two nearby orbits over time, reflecting chaos strength. The maximum Lyapunov exponent provides accurate chaos detection, with calculation methods including variational and two-particle approaches [21]. This work uses the variational method:

$$\lambda = \lim_{\tau \to \infty} \frac{1}{\tau} \ln \frac{|\xi(\tau)|}{|\xi(0)|},$$

where $\xi(\tau)$ and $\xi(0)$ represent tangent vectors at time τ and initial time, respectively. A λ value stabilizing at a positive value indicates bounded chaotic orbits, while λ tending to zero indicates regular orbits.

The fast Lyapunov indicator (FLI) reveals chaos more quickly than the Lyapunov exponent and is defined as [50]:

$$\mathrm{FLI} = \ln |\xi(\tau)|.$$

Rapid FLI growth indicates bounded chaotic orbits, while slow growth indicates regular orbits.

[Figure 3: see original paper] shows Poincaré sections on the equatorial plane $\theta = \pi/2$. Figure 3: see original paper displays the section for the orbit in Figure 2: see original paper, showing closed loops that indicate quasi-periodic motion. Figure 3: see original paper shows the section for Figure 2: see original paper, with randomly distributed points across a region, indicating chaotic motion. Clearly, Poincaré sections distinguish regular from chaotic orbits more clearly and accurately than three-dimensional spatial configurations. The magnetic field destroys the fourth integral of motion, making the Hamiltonian system nonintegrable and fundamentally causing chaotic particle motion.

3.2.2 Dynamics of Particle Motion [Figure 4: see original paper] examines orbital evolution under varying magnetic parameter β. Panels (a)–(c) show Poincaré sections for three orbits: Orbit 1 (red, $r = 40$), Orbit 2 (blue, $r = 65$), and Orbit 3 (black, $r = 88$), plus additional orbits (green, $r = 11$; purple, $r = 20$). With fixed particle energy $E = 0.995$ and angular momentum $L = 4.6$, magnetic parameters are $\beta = 1.2 \times 10^{-4}$, $\beta = 5.6 \times 10^{-4}$, and $\beta = 8.9 \times 10^{-4}$ in the three subplots.

In Figure 4: see original paper with $\beta = 1.2 \times 10^{-4}$, all orbits exhibit regular quasi-periodic motion, appearing as closed loops on the section. Figure 4: see original paper confirms this through maximum Lyapunov exponents that trend toward zero without stabilizing after integration time 10^7, indicating regularity. Figure 4: see original paper shows all FLI values remain below 3 without exponential growth, characteristic of regular orbits.

In Figure 4: see original paper with $\beta = 5.6 \times 10^{-4}$, Orbit 2 shows weak chaos while Orbits 1 and 3 remain regular. Figure 4: see original paper and (h) confirm Orbits 1 and 3 as regular, while Orbit 2's Lyapunov exponent stabilizes at a positive value and its FLI grows exponentially, confirming chaos.

In Figure 4: see original paper with $\beta = 8.9 \times 10^{-4}$, Orbits 1 and 3 are chaotic while Orbit 2 forms five regular islands. Figure 4: see original paper and (i) confirm this classification through Lyapunov exponents and FLI. Thus, increasing the positive magnetic parameter β (i.e., increasing β^2) transitions particle motion from regular to chaotic, with chaos strength increasing accordingly, because the Lorentz force from the magnetic field enhances gravitational effects.

[Figure 5: see original paper] examines negative magnetic parameter β values. Five orbits with initial radii $r = 11, 20, 40, 65, 88$ (color-coded as in [Figure 4: see original paper]) are considered. With particle energy $E = 0.995$ and angular momentum $L = 4.6$ fixed, Figure 5: see original paper shows two strongly chaotic orbits, (b) shows two weakly chaotic orbits, (c) shows one weakly chaotic orbit, and (d) shows no chaotic orbits—all are quasi-periodic. Therefore, as the absolute value of negative β decreases, orbits transition from chaotic to regular. When $\beta < 0$, increasing β (i.e., decreasing β^2) weakens chaos because the reduced β^2 diminishes the gravitational-like effect of the Lorentz force. In summary, [Figure 4: see original paper] and [Figure 5: see original paper] demonstrate that increasing the absolute value of β intensifies orbital chaos regardless of sign.

[Figure 6: see original paper] shows Poincaré sections for varying particle energy E (color-coded as in [Figure 4: see original paper]). Panels (a)–(c) fix $\beta = 8.9 \times 10^{-4}$ and $L = 4.6$ while varying $E = 0.99, 0.992, 0.995$. In (a), all three orbits are regular. In (b), Orbit 3 comprises arrow-shaped and triangular orbits connected at a hyperbolic fixed point, showing weak chaos, while Orbits 1 and 2 remain regular. In (c), Orbits 1 and 3 are strongly chaotic, while Orbit 2 forms five islands (regular). Panels (d)–(f) fix $\beta = 1.5 \times 10^{-3}$ and $L = 4.6$ with $E = 0.99, 0.992, 0.995$. In (d), all three saddle-shaped orbits are regular. In (e), Orbits 1 and 2 are clearly chaotic while Orbit 3 is regular. In (f), all orbits are chaotic. Thus, appropriately increasing particle energy E intensifies chaos because higher energy strengthens gravitational effects. Additionally, Figure 6: see original paper,(e) and (c),(f) support the conclusion that increasing positive magnetic parameters enhances chaos.

[Figure 7: see original paper] examines orbital evolution for three different angular momenta L (color-coded as in [Figure 4: see original paper]). Panels (a)–(c) fix $\beta = 8.9 \times 10^{-4}$ and $E = 0.995$ while varying $L = 4.21, 4.6, 4.88$. In (a), all three orbits ($r = 40, 65, 88$) are strongly chaotic. Figure 7: see original paper confirms this through maximum Lyapunov exponents that stabilize at positive values, and (g) shows exponential FLI growth for all three orbits. Figure 7: see original paper (same parameters as Figure 4: see original paper) shows Orbits 1 and 3 as chaotic and Orbit 2 as regular, confirmed by (e) and (h). In (c), Orbit 1 is regular, Orbit 3 is chaotic, and Orbit 2 forms multiple islands (regular), confirmed by (f) and (i). In summary, increasing particle angular momentum L evolves orbits from weak chaos to regularity and from strong chaos to weak chaos. Larger angular momentum suppresses chaos because it acts as a repulsive effect that weakens gravitational attraction.

As noted in [22], FLI is an excellent tool for describing qualitative properties beyond Poincaré sections. [Figure 8: see original paper] shows FLI dependence on parameters, where each point represents one orbit. Panels (a) and (b) fix $\beta = 8.9 \times 10^{-4}$ and $L = 4.6$ while varying E from 0.99 to 1. Each FLI value is taken at integration time $\tau = 10^7$; FLI $= 3$ separates regular from chaotic orbits (FLI > 3 indicates chaos). In (a) with $r = 11$, orbits are regular for $E < 0.9947$ and chaotic for $E > 0.9957$; point 3 shows weak chaos while points 1 and 2 are regular. In (b) with $r = 88$, chaos begins for $E > 0.9928$; point 1 shows strong chaos, point 2 weak chaos, and point 3 regularity. These results align with [Figure 5: see original paper], confirming that increasing particle energy E enhances chaos.

Panels (c) and (d) fix $\beta = 8.9 \times 10^{-4}$ and $E = 0.995$ while varying L from 3.8 to 5. In (c) with $r = 11$, orbits are chaotic for $L < 4.11$ and regular for larger L; point 1 shows chaos and point 2 regularity. In (d) with $r = 88$, orbits become regular for $L > 6.35$; points 1 and 2 show weak chaos. These results align with [Figure 6: see original paper], confirming that increasing angular momentum L reduces chaos.

Panels (e) and (f) fix $L = 4.6$ and $E = 0.995$ while varying β from 0 to 1.6×10^{-3}. Critical β values inducing chaos are 9.01×10^{-4} and 6.92×10^{-4}. These results correspond to [Figure 3: see original paper] and [Figure 4: see original paper], confirming that increasing positive magnetic parameter β enhances chaos.

[Figure 9: see original paper] displays FLI distributions in two-dimensional parameter spaces. Panel (a) shows the FLI distribution in the E-L plane for $r = 11$ and $\beta = 8.9 \times 10^{-4}$. Chaos begins around $E \approx 0.9957$ and intensifies with increasing E up to a point, after which chaos weakens, shown by the color transition from blue to red and back to blue. The chaotic region roughly spans $E = 0.9957$ to $E = 1$, consistent with Figure 8: see original paper. Increasing angular momentum L also reduces chaos, matching Figure 8: see original paper at $E = 0.995$.

Panel (c) shows the E-L parameter space for $r = 88$, more clearly demonstrating that increasing L weakens chaos, consistent with Figure 8: see original paper and (d). Panels (b) and (d) show E-β parameter spaces for $r = 11$ and $r = 88$ respectively, both with $L = 4.6$. Higher energy intensifies chaos by strengthening gravity, but when E approaches 1, orbits become unstable and chaos weakens. Increasing β transitions orbits from regular to chaotic with growing FLI values, indicating stronger chaos. These results match the one-dimensional scans in Figure 8: see original paper and (f).

4 Summary and Outlook

Symplectic algorithms preserve geometric structure, providing reliable numerical results for long-term qualitative evolution studies in astrophysics. Explicit symplectic algorithms offer better computational efficiency than implicit methods. The Hamiltonian for magnetized Schwarzschild spacetime can be decomposed into three, four, or five explicitly integrable parts, enabling explicit symplectic algorithm construction. The optimized fourth-order Partitioned-Runge-

Kutta PRK64 explicit symplectic algorithm combined with three-part Hamiltonian decomposition achieves optimal numerical performance.

Applying this integrator to study charged particle dynamics around a Schwarzschild black hole with an asymptotically uniform magnetic field, we examine how magnetic parameter β, particle energy E, and angular momentum L affect orbital dynamics. The magnetic field is the key factor causing Hamiltonian nonintegrability and chaos. Increasing the absolute value of β transitions particle motion from regular to chaotic and intensifies chaos. Increasing particle energy also strengthens chaos, while larger angular momentum suppresses it.

The algorithm established in this work applies to many complex relativistic spacetime Hamiltonians or time-transformed Hamiltonians, enabling exploration of black hole orbital dynamics and numerical simulation of black hole shadows.

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Note: Figure translations are in progress. See original paper for figures.