Initial Orbit Determination for Small Celestial Bodies Based on Intelligent Optimization Algorithms (Postprint)

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Abstract

Classical initial orbit determination methods include the Laplace method and the Gauss method, as well as their various variations. In addition to these classical methods, based on the characteristics of contemporary optical observation data, scholars have also proposed some other initial orbit determination methods, including the double r (distance of the target from the observer) method and the feasible region method. One implementation of the double r method is to guess the distance of the target from the observer at two moments (usually the beginning and end moments of the orbit determination arc), combine it with the observer's position vector in space, and then solve the corresponding Lambert arc as an initial guess for the target's orbit. Furthermore, using the RMS (Root Mean Square) at other observation moments as an optimization variable can improve the initial guess to determine the initial orbit. The feasible region method, on the other hand, is aimed at a set of initial observation parameters (including right ascension, declination, and their rates), constrains the target's (distance from observer) distance and its rate within a feasible region based on some initial assumptions, and finds a guessed solution that minimizes the observation RMS through a stepwise approximation approach using triangular partitioning. For a series of simulated observation data and actual measurement data, intelligent optimization algorithms (particle swarm algorithm) are applied to these two initial orbit methods, and the results are compared with those of an improved Laplace algorithm. Since the double r method can be used not only for short-arc orbit determination but also for long-arc correlation, correlation results for long-arc data are further presented.

Full Text

Initial Orbit Determination for Small Celestial Bodies Based on Intelligent Optimization Algorithms

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Abstract

Classical methods for initial orbit determination (IOD) include the Laplace method, Gauss method, and their various derivatives. In addition to these classical approaches, scholars have proposed other IOD methods based on the characteristics of modern optical observation data, including the double-r method (where r represents the target-observer distance) and the admissible region method. One implementation of the double-r method involves guessing the target's distance from the observer at two epochs (typically the first and last of the orbit determination arc), which, combined with the observer's position vector in space, allows solving the corresponding Lambert arc as an initial orbit guess. Subsequently, using the RMS (Root Mean Square) of other observation epochs as an optimization variable can improve this initial guess to determine the preliminary orbit. The admissible region method, on the other hand, starts from a set of initial observational parameters (including right ascension, declination, and their rates). Based on certain initial assumptions, it constrains the target's distance from the observer and its rate of change within a feasible region, and gradually approximates the solution that minimizes the observed RMS through triangular partitioning.

For a series of simulated and real observation data, we apply an intelligent optimization algorithm (particle swarm optimization) to these two IOD methods and compare the results with those from an improved Laplace algorithm. Since the double-r method can be used for both short-arc orbit determination and long-arc correlation, we further present correlation results for long-arc data.

Keywords: celestial mechanics, minor planets, methods: numerical

1. Introduction

The orbit determination of near-Earth objects typically involves two steps: initial orbit determination and precise orbit determination. Initial orbit determination utilizes limited optical or radar observational data to provide an initial orbital value for precise orbit determination. Better initial orbit determination results make the precise orbit determination process more likely to converge. Generally, optical observational data includes the target's right ascension and declination. With advances in observation technology, photon-counting detectors have emerged. The raw data obtained from such devices, in addition to angular information, can also provide angular rate information. In many cases, even short arcs can yield second-order angular rate information [?]. Some near-Earth objects also have radar observational data (time delay and Doppler shift), but in this work, we only consider the initial orbit determination problem based on optical observational data of near-Earth objects.

Since Laplace's method was proposed in 1780, classical initial orbit calculation methods have mainly included the Laplace, Gauss, and Escobal (double-r method, where r is the target-observer distance) methods and their derivatives. Both Laplace and Gauss methods derive formulas from three sets of optical observational data, ultimately reducing to solving an eighth-order equation. Charlier [?] noted that the Laplace method sometimes yields non-unique solutions, and Gronchi pointed out that the same problem exists for the Gauss method [?]. An improved Laplace method [?] avoids solving the eighth-order equation and instead uses a numerical iteration approach to solve a system of nonlinear equations expressed in "linear" form. In 1965, Escobal [?] proposed the double-r method, which was later improved by Gooding [?] and Briggs et al. [?]. The essence of the double-r method is that knowing the radial information (target-observer distance) for at least two observation points allows orbit determination. The method proposed by Briggs et al. [?] utilizes orbital integration equations, while Gooding [?] employs the solution of Lambert arcs. In addition, there are various improvement methods for different situations, such as Zeinalov [?], who proposed a solution without time information under the assumption of circular orbits.

In modern times, due to improved observation technology, numerous unclassified extremely short arcs have emerged. Direct application of classical initial orbit determination methods to a single extremely short arc is often ineffective and cannot yield valid orbital information. Based on the characteristics of current optical observation data (with relatively accurate angular rate information), Milani et al. [?] proposed the admissible region method for minor planets. Tommei et al. later applied this method to initial orbit determination for near-Earth space objects and extended it to radar-type observational data [?]. For optical data, this method uses a basic observation set (TSA—Too Short Arc) that includes not only the two angles but also their rates. For a TSA, the admissible region can be determined based on certain constraints. By correlating the admissible regions of two TSAs, an initial orbit can be further determined [?].

Scholars have proposed different constraints for various application scenarios; for example, Maruskin et al. [?] constrained the perigee and apogee, while DeMars et al. [?] provided constraints on semi-major axis and eccentricity based on specific conditions. It is worth mentioning that another initial orbit determination scheme based on two TSAs directly uses energy and angular momentum integrals for solving [?, ?].

Intelligent computing, also called "soft computing," is inspired by natural (biological) laws and mimics problem-solving algorithms according to their principles. Intelligent optimization algorithms effectively solve optimization problems under multivariable and multi-constraint conditions, with commonly used methods including simulated annealing, genetic algorithms, and particle swarm optimization. Some scholars have integrated intelligent optimization algorithms into initial orbit calculations. Wang Xueying et al. combined particle swarm optimization with the admissible region method, eliminating triangulation of the admissible region and directly using the particle swarm algorithm for global optimization of space-based short-arc observation data, improving computational efficiency [?]. Ansalone et al. combined genetic algorithms with the double-r method to estimate orbital parameters of space targets from extremely short-arc space-based observation data, using the line-of-sight distances at the first and last moments of the observation arc as optimization parameters and determining initial orbital parameters by solving the Lambert problem [?]. Hinagawa et al. used genetic algorithms for orbit determination of Geostationary Orbit (GEO) targets from short-arc ground-based optical data, dividing orbital elements into two groups as optimization variables for separate solving [?]. Li Xinran et al. also used genetic algorithms to optimize the extremely short-arc orbit determination problem for space targets, similarly choosing orbital elements rather than line-of-sight distance as optimization parameters [?, ?, ?, ?]. Additionally, Li Xinran et al. used evolutionary algorithms for initial orbit calculation of near-Earth asteroids, also selecting initial orbital elements as optimization parameters and proposing that for objects with large eccentricity, optimal solutions need to be sought regionally to some extent [?].

This paper examines the initial orbit determination of near-Earth minor planets, focusing on the near-Earth asteroid Toutatis (number 4179) with relatively large eccentricity (approximately 0.6). We employ two methods—the admissible region method and the double-r method—using particle swarm optimization for global optimization of the variables, and simultaneously provide a method for estimating the covariance matrix. Considering different arc scenarios, we calculate arcs of different durations and observation numbers, compare the advantages and disadvantages of the improved Laplace method, admissible region method, and double-r method, and analyze how different arc lengths and observation numbers affect the accuracy of different methods. Experimental results show that arc duration has a greater impact on the improved Laplace method and double-r method, while observation number has a greater impact on the admissible region method. Based on this, we finally provide recommendations on which method to use for initial orbit calculation for arcs of different lengths.

Section 2 briefly describes initial orbit calculation methods from classical to modern—Gauss method, improved Laplace method, double-r method, and admissible region method. Section 3 describes the double-r method and admissible region method combined with intelligent optimization algorithms and presents covariance matrix calculations for all three methods, including the improved Laplace method. Section 4 shows the data processing methods used in this paper and the results of initial orbit calculations for different arcs using the three methods. Finally, we discuss and summarize the experimental results.

2.1.1 Gauss Method

[Figure 1: see original paper] shows the relative geometric configuration of the central body Sun (S), Earth (E), and minor body (A), where $\mathbf{r}$ represents the position vector of the minor body relative to the Sun; represents the position vector of the minor body relative to the observation station; $\mathbf{R} = (X, Y, Z)^T$ represents the position vector of the observation station relative to the Sun; ϕ represents the angle between vectors $\mathbf{r}$ and $\mathbf{R}$; and θ represents the angle between vectors $\mathbf{R}$ and . In the International Celestial Reference System (ICRS), the measurement geometry satisfies:

$$\mathbf{r} = \quad + \mathbf{R}.$$

Considering three sets of observational data $\{(\alpha_1, \delta_1), (\alpha_2, \delta_2), (\alpha_3, \delta_3)\}$ corresponding to observation times t_i $(i = 1, 2, 3)$, where α_i and δ_i represent the right ascension and declination of the i-th set of observational data, and with the corresponding station position vectors $\mathbf{R}_i$ $(i = 1, 2, 3)$ in the solar system heliocentric coordinate system, denoting their magnitudes as R_i $(i = 1, 2, 3)$, and the position vectors of the minor body in the heliocentric coordinate system as $\mathbf{r}_i$ $(i = 1, 2, 3)$. Gronchi et al. provide the specific derivation of the Gauss method [**?**], which will not be repeated here. We directly present the resulting eighth-order equation:

$$\mathbf{r}_2^2 + 2A_2(\;_2 \cdot \mathbf{R}_2)/\rho_2 + R_2^2 - [A_2^2 B_2 [A_2 + (\;_2 \cdot \mathbf{R}_2)/\rho_2] r_3^2 - B_2$$

where A_2 and B_2 are coefficients combined from known conditions; r_2 represents the magnitude of the minor body's position vector $\mathbf{r}_2$ at the second observation time; and $_2$ represents the relative position vector from the observation station to the target at the second observation time. The solution can be obtained using Newton's iteration method.

Since the Laplace method also ultimately yields an eighth-order equation of the form shown above, the equation is usually written in the following more general form:

$$P(r_2) = C^2 r_2^8 - q^2(C^2 + 2C\gamma\cos\theta_2 + \gamma^2) r_2^6 - 2q^5(C\cos\theta_2 + \gamma) r_2^3 - q^8,$$

where $P(r_2)$ represents a function of r_2, C and γ are constants from the derivation process, $q = R_2$, and θ_2 is the angle between $_2$ and $\mathbf{R}_2$.

Assuming $\gamma = 1$, we have:

$$P(r_2) = (r_2 - q) P_1(r_2),$$

$$P_1(r_2) = C^2 r_2^6 - (r_2 + q)[q^5 - (2C\cos\theta_2 + 1) q^2 r_2^3 + q r_2 + q^2],$$

$$P(0) = -q^8 < 0, \quad r_2 \to +\infty \Rightarrow P(r_2) = +\infty.$$

According to Descartes' rule of signs, this equation must have three positive roots. Due to physical properties, the root $r_2 = q$ should be removed. The other two roots divide the plane into several regions based on the zero circle and limiting curve. [Figure 2: see original paper] shows how the zero circle and limiting curve divide the plane into different solution regions: regions outside the zero circle indicate two roots, regions between the zero circle and limiting curve indicate one root, and regions inside the limiting curve indicate two roots. Gronchi et al. [**?**] provide the specific derivation for this part; here we simply use this diagram to illustrate that multiple solutions can occur when solving initial orbit problems.

2.1.2 Improved Laplace Method

Without considering perturbations, the motion equation of a minor body satisfies:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}, \quad t_0 : \mathbf{r}_0 = \mathbf{r}(t_0), \dot{\mathbf{r}}_0 = \dot{\mathbf{r}}(t_0),$$

where $\mu = GM$ represents the gravitational constant of the central body, G is the universal gravitational constant, M is the mass of the central body, $\mathbf{r}_0 = (x_0, y_0, z_0)^T$ is the position vector of the target body at the initial time, $\ddot{\mathbf{r}}$ represents the second derivative of the position vector $\mathbf{r}$ (i.e., the target's acceleration), and $\dot{\mathbf{r}}(t)$ represents the velocity of the target body at time t.

Under the premise of a short arc, the solution satisfying the initial conditions can be expanded as a power series in the time interval $\Delta t = t - t_0$:

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{r}_0^{[1]}\Delta t + \mathbf{r}_0^{[2]}\Delta t^2 + \cdots + \mathbf{r}_0^{[k]}\Delta t^k + \cdots,$$

where $\mathbf{r}_0^{[k]}$ is the k-th order derivative of $\mathbf{r}(t)$ evaluated at t_0, i.e.:

$$\mathbf{r}_0^{[k]} = \left(\frac{d^k\mathbf{r}}{dt^k}\right)_{t=t_0}.$$

At this point, all higher-order derivatives $\mathbf{r}_0^{[k]}$ $(k \geq 2)$ in equation (5) can be given by the motion equation in (4), i.e.:

$$\mathbf{r}_0^{[k]} = \mathbf{r}_0^{[k]}(t_0, \mathbf{r}_0, \dot{\mathbf{r}}_0).$$

Simultaneously, $\mathbf{r}_0^{[1]} = \dot{\mathbf{r}}_0$, which can be expanded as:

$$\mathbf{r}(t) = F^*(\mathbf{r}_0, \dot{\mathbf{r}}_0, \Delta t)\mathbf{r}_0 + G^*(\mathbf{r}_0, \dot{\mathbf{r}}_0, \Delta t)\dot{\mathbf{r}}_0,$$

where F^* and G^* can be calculated by equating (6) with (5). The specific forms are:

$$\begin{cases} F^* = 1 - \frac{1}{2}u_0\tau^2 + \frac{1}{2}u_0q_0\tau^3 - \frac{1}{24}(u_0p_0 - 15u_0^2)\tau^4 + \cdots, \\ G^* = \tau - \frac{1}{6}u_0\tau^3 + \frac{1}{4}u_0q_0\tau^4 + \cdots, \end{cases}$$

where $u_0 = 1/r_0^3$, r_0 represents the magnitude of $\mathbf{r}_0$, $p_0 = (\mathbf{r}_0 \cdot \dot{\mathbf{r}}_0)/r_0^2$, and $q_0 = v_0^2$, with all parameters with subscript 0 representing values at the initial time t_0.

Combining the measurement geometry relation (1) with (6) and simultaneously crossing with the unit vector ˆ of   yields:

$$\hat{\ } \times F^*(\mathbf{r}_0, \dot{\mathbf{r}}_0, \Delta t)\mathbf{r}_0 + G^*(\mathbf{r}_0, \dot{\mathbf{r}}_0, \Delta t)\dot{\mathbf{r}}_0 = \hat{\ } \times \mathbf{R}.$$

For three sets of observational data (α_i, δ_i), the geometric relationship is:

$${}_i = \rho_i \begin{pmatrix} \cos\alpha_i \cos\delta_i \\ \sin\alpha_i \cos\delta_i \\ \sin\delta_i \end{pmatrix} = \rho_i \hat{\ }_i.$$

From the station information corresponding to each set of observational data $\mathbf{R}_i = (X_i, Y_i, Z_i)^T$, we obtain:

$$\begin{cases}(F_i^*\lambda_i)x_0 + (G_i^*\lambda_i)\dot{x}_0 - (F_i^*\nu_i)z_0 - (G_i^*\nu_i)\dot{z}_0 = \nu_{iX}i - \lambda_{iZ}i,\\ (F_i^*\mu_i)y_0 + (G_i^*\mu_i)\dot{y}_0 - (F_i^*\nu_i)z_0 - (G_i^*\nu_i)\dot{z}_0 = \nu_{iY}i - \mu_{iZ}i,\\ (F_i^*\mu_i)x_0 + (G_i^*\mu_i)\dot{x}_0 - (F_i^*\lambda_i)y_0 - (G_i^*\lambda_i)\dot{y}_0 = \mu_{iX}i - \lambda_{iY}i,\end{cases}$$

where F_i^* and G_i^* represent F^* and G^* corresponding to the i-th observation. It can be seen that only two of the above equations are independent. To solve for $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$, at least three sets of observational data are required. The result calculated from equation (8) is denoted as $(\mathbf{r}_0^*, \dot{\mathbf{r}}_0^*)$. Using (8), we iteratively apply the following formulas:

$$\mathbf{r}_0^{(m)} = \mathbf{r}_0^{(m-1)} + \frac{(\mathbf{r}_0^* - \mathbf{r}_0^{(m-1)})}{1+\epsilon}, \quad \dot{\mathbf{r}}_0^{(m)} = \dot{\mathbf{r}}_0^{(m-1)} + \frac{(\dot{\mathbf{r}}_0^* - \dot{\mathbf{r}}_0^{(m-1)})}{1+\epsilon},$$

where $\mathbf{r}_0^{(m)}$ and $\dot{\mathbf{r}}_0^{(m)}$ represent the values from the m-th iteration, and normalization is performed with:

$$\epsilon = |\mathbf{r}_0^{(m)} - \mathbf{r}_0^{(m-1)}|^2 + |\dot{\mathbf{r}}_0^{(m)} - \dot{\mathbf{r}}_0^{(m-1)}|^2.$$

The iteration continues until ϵ meets a certain precision requirement. In practical work, this method demonstrates good convergence.

2.2 Double-r Method

The fundamental difficulty in determining initial orbits from angular measurements is the lack of radial information (distance and velocity). The basic idea of the double-r method is that knowing the radial information for at least two observation points allows orbit determination. This approach can be used for both extremely short arcs (Too Short Arc) and arcs with long time intervals (even multiple revolutions). Typically, at least three angular measurement points are required, generally assuming the radial distances for the first and last angular measurement points to derive and correct the radial distances for intermediate observation times. Overall, the double-r method has two forms: one combined with the Lambert problem and one combined with orbital motion integration. We briefly describe both methods below.

2.2.1 Double-r Method Combined with Orbit Integration We briefly describe the method proposed by Briggs et al. [**?**], which solves the so-called Gibbs problem and is also known as the BSE (Binary-Single Evolution) method. Assuming three sets of observational data, based on the coplanarity property of the minor body's heliocentric position vectors $\mathbf{r}_i$ $(i = 1, 2, 3)$ and the orbital equation, expressions for the six orbital elements can be derived (not detailed here). The mean anomalies M_i $(i = 1, 2, 3)$ satisfy:

$$M_i = nt_i + \omega,$$

where n represents the mean motion and ω represents the argument of periapsis. Letting $\omega = nT_i$, we have $t_i = -T_i + M_i/n$, which allows calculating t_{13} and t_{23}. Defining $\Delta t_{ij} = t_i - t_j$ $(i, j = 1, 2, 3)$, the difference with observed times can be taken. Thus, the problem reduces to finding ρ_1 and ρ_2 such that the objective function satisfies:

$$\Delta_k(\rho_1, \rho_2) = 0; \quad k = 1, 2.$$

This can be solved iteratively using the Newton-Raphson formula. $\Delta_1(\rho_1, \rho_2)$ and $\Delta_2(\rho_1, \rho_2)$ describe two curves in the $\rho_1 - \rho_2$ plane, and their intersection point is the solution to the problem. Briggs et al. [?] described the initial value selection and iteration termination conditions in their article, and also proposed corrections for perturbed cases, which will not be discussed further here.

2.2.2 Double-r Method Combined with the Lambert Problem We briefly describe the method based on Gooding [?]. Simply put, the Gibbs problem solves for ρ_1 and ρ_2 iteratively to satisfy constraints. In Gooding's algorithm, the radial distances ρ_1 and ρ_3 at two points are still solved iteratively to satisfy two constraints, but their definitions differ.

After guessing values for ρ_1 and ρ_3, the Lambert problem can be solved to obtain the orbit, and the position at the intermediate time t_2 can be obtained as P_c. In [Figure 3: see original paper], the dashed curve represents the true orbit, P' represents the target's true position, and the dashed line represents the true direction at time t_2. P_1, P_2, and P_3 represent guessed positions. A plane is defined that passes through P_c and is perpendicular to the true direction of the target position at t_2, with the intersection point of this plane and the direction serving as the origin of a coordinate system. Let the projection of P_c on this coordinate system be f and g, with the coordinate system established such that $f > 0$ and $g = 0$. According to this coordinate system setup, when P_c is the target's true position, $f = 0$. Gooding [?] uses the Newton-Raphson formula for iterative correction. Using x and y to replace ρ_1 and ρ_3:

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix},$$

where the partial derivatives of f and g with respect to x and y are given by numerical difference methods. Gooding's method has two advantages over Briggs' method: first, Gooding's method has a wider convergence domain, as solving the Lambert problem yields an appropriate solution regardless of the initial value provided; second, if the solution itself is uncertain, classical methods and the BSE method cannot converge. This paper will make simple improvements to this method to adapt it to intelligent optimization algorithms.

2.3 Admissible Region Method

Thanks to improved observation equipment and large-scale sky survey programs, the current optical observation mode has changed. Most of the time, telescopes point to fixed regions of the sky for short exposures of a few images, then point to different sky regions for observation. The discovery of near-Earth objects typically uses the method of comparing several images of the same sky region, but generally, new large-scale survey equipment cannot conduct follow-up tracking observations of targets and can only obtain data from one night. Such a short duration is insufficient for initial orbit determination using classical methods. This short observation arc is also called an extremely short arc. For several short-exposure images, interpolation from the images typically yields $(\alpha, \delta, \dot{\alpha}, \dot{\delta})$ at a reference epoch, i.e., the right ascension and declination of the moving target in the station-centered celestial coordinate system and their rates. The data corresponding to a TSA is also called an attributable. Milani et al. [**?**] proposed this term to indicate whether this TSA can be correlated with known arcs, and based on this, analyzed the information contained in a TSA (or attributable) and proposed the concept of the admissible region [**?**], which we briefly introduce here.

For the information contained in a TSA, the following relationships exist in the heliocentric coordinate system:

$$\dot{\mathbf{r}} = \dot{} + \dot{\mathbf{R}},$$

$$\dot{} = \dot{\rho}\hat{} + \rho(\hat{}_{\alpha}\dot{\alpha} + \hat{}_{\beta}\dot{\beta}),$$

where:

$$\hat{}_{\alpha} = \begin{pmatrix} -\sin\alpha\cos\delta \\ \cos\alpha\cos\delta \\ 0 \end{pmatrix}, \quad \hat{}_{\beta} = \begin{pmatrix} -\cos\alpha\sin\delta \\ -\sin\alpha\sin\delta \\ \cos\delta \end{pmatrix}.$$

To obtain the orbital information of the observed target, radial information is also needed, i.e., the range ρ and range rate $\dot{\rho}$. The admissible region method uses physical information about the target for region partitioning. Based on the properties of near-Earth minor planets, we define the admissible region and assume that near-Earth minor planets will not be captured by Earth:

1. **Negative heliocentric energy (bound orbit)**

$$R_A = \{(\mathbf{r}, \dot{\mathbf{r}}) \mid E_{\text{Sun}} = \frac{\dot{r}^2}{2} - \frac{\mu_S}{r} < 0\};$$

2. **Not captured by Earth**

$$R_B = \{(\mathbf{r}, \dot{\mathbf{r}}) \mid E_{\text{Earth}} = \frac{\dot{r}_E^2}{2} - \frac{\mu_E}{r_E} > 0\};$$

3. **Distance from Earth exceeds Earth's Hill sphere**

$$R_C = \{(\mathbf{r}, \dot{\mathbf{r}}) \mid r_E > a_E(\mu_E/3)^{1/3}\};$$

4. **Distance from Earth greater than Earth's radius**

$$R_D = \{(\mathbf{r}, \dot{\mathbf{r}}) \mid r_E > R_E\},$$

where $\mathbf{r}$ and $\dot{\mathbf{r}}$ represent the position and velocity vectors of the minor body in the solar system heliocentric coordinate system, E_{Sun} represents the energy of the minor body in this coordinate system, $\mathbf{r}_E$ and $\dot{\mathbf{r}}_E$ represent the position and velocity vectors of the minor body in the geocentric celestial coordinate system, and r_E and $\dot{r}_E$ represent the magnitudes of these two physical quantities, respectively. E_{Earth} represents the energy of the minor body in this coordinate system. R_E represents Earth's radius, $\mu_E = GM_E$ and $\mu_S = GM_S$, where M_E and M_S represent the masses of Earth and the Sun, respectively, and a_E represents the semi-major axis of Earth's orbit.

Therefore, the admissible region is:

$$R = R_A \cap R_B \cap R_C \cap \overline{R_D}.$$

Finally, the admissible region is triangulated. Each vertex of the triangular region can provide information $(\alpha, \delta, \dot{\alpha}, \dot{\delta}, \rho, \dot{\rho})$. If the near-Earth minor body is at this vertex, this information can provide the state vector and orbital elements of the near-Earth minor body. If another TSA of this minor body is available, correlation can be performed to generate an initial orbit estimate [**?**].

3.1 Improvement of Double-r Method Used in This Paper

Assume there are N_R ($N_R \geq 3$) sets of observational data (α_m^o, δ_m^o), $m = 1, 2, \cdots, N_R$, where the superscript o indicates observed values. Assuming the station-centered distances ρ_1 and ρ_{N_R} for the first and N_R-th observational data are known, the heliocentric position vectors of the minor body can be obtained (see [Figure 1: see original paper]), allowing the Lambert problem to be solved and theoretical observational values at intermediate observation times to be calculated as (α_m^c, δ_m^c), $m = 1, 2, \cdots, N_R$, where the superscript c indicates calculated values. The observation residual vector at each epoch is recorded as:

$$\mathbf{y}_m = [\cos\delta_m^o(\alpha_m^c - \alpha_m^o), (\delta_m^c - \delta_m^o)]^T.$$

All residual vectors are recorded as:

$$\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{N_R})^T.$$

The variable system is denoted as $\mathbf{X}$, which in this problem is:

$$\mathbf{X} = (\rho_1, \rho_{N_R})^T.$$

The initial orbit determination problem using the double-r algorithm is to find appropriate $\mathbf{X}$ that minimizes:

$$\min J(\mathbf{X}) = \min\{\mathbf{Y}^T\mathbf{Y}\}.$$

This is equivalent to solving:

$$\frac{\partial J}{\partial \mathbf{X}} = 0.$$

Let:

$$\mathbf{B} = (b_1, b_2, \dots, b_{N_R})^T, \quad b_m = \frac{\partial \mathbf{y}_m}{\partial \mathbf{X}}.$$

Equation (13) can be simplified as:

$$\sum_m \mathbf{y}_m^T \mathbf{B} = 0.$$

The covariance matrix of the estimated parameters is:

$$\mathbf{P}_X = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{P}_Y\mathbf{B}(\mathbf{B}^T\mathbf{B})^{-T}.$$

Considering that N_R may be large, in actual calculations we can avoid direct computation of large matrices. When calculating $\mathbf{P}_Y$, in addition to considering the variance of observations, we must also consider the contribution of angular errors from the first and N_R-th observation epochs, i.e.:

$$\mathbf{P}_Y = \mathbf{P}_Y^{(1)} + \mathbf{P}_Y^{(2)},$$

where:

$$\mathbf{P}_Y^{(1)} = \mathrm{diag}(\sigma_{\alpha_m}^2 \cos^2 \delta_m^o, \sigma_{\delta_m}^2),$$

with $\sigma_{\alpha_m}^2$ and $\sigma_{\delta_m}^2$ representing the variances of right ascension and declination for the m-th group, respectively. Meanwhile:

$$\mathbf{P}_Y^{(2)} = \frac{\partial \mathbf{Y}}{\partial(\alpha_1^o, \delta_1^o, \alpha_{N_R}^o, \delta_{N_R}^o)} \mathbf{P}_{\alpha_1^o, \delta_1^o, \alpha_{N_R}^o, \delta_{N_R}^o} \frac{\partial \mathbf{Y}^T}{\partial(\alpha_1^o, \delta_1^o, \alpha_{N_R}^o, \delta_{N_R}^o)}.$$

Let the solution of equation (15) be $\mathbf{X}^*$, and introduce $\Delta\mathbf{X} = \mathbf{X} - \mathbf{X}^*$. Then:

$$\frac{\partial^2 J}{\partial \mathbf{X}^2} = \mathbf{B}^T\mathbf{B} + \mathbf{Y}^T \frac{\partial \mathbf{B}}{\partial \mathbf{X}}.$$

A reasonable assumption is that when equation (15) is satisfied, the observation residuals $\mathbf{Y} \to 0$, so:

$$\frac{\partial^2 J}{\partial \mathbf{X}^2} \approx \mathbf{B}^T\mathbf{B}.$$

Considering this approximation, the iteration formula from (15) becomes:

$$\Delta\mathbf{X} = -\left(\frac{\partial^2 J}{\partial \mathbf{X}^2}\right)^{-1} \frac{\partial J}{\partial \mathbf{X}} = -(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{Y}.$$

To obtain the covariance matrix of the position and velocity at the initial time, we need further calculation. Let:

$$\tilde{\mathbf{X}} = (\mathbf{X}, \alpha_1^o, \delta_1^o, \alpha_{N_R}^o, \delta_{N_R}^o)^T = (\rho_1, \rho_{N_R}, \alpha_1^o, \delta_1^o, \alpha_{N_R}^o, \delta_{N_R}^o)^T.$$

Then:

$$\mathbf{P}_{\tilde{X}} = \begin{pmatrix} \mathbf{P}_X & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\alpha_1^o, \delta_1^o, \alpha_{N_R}^o, \delta_{N_R}^o} \end{pmatrix}.$$

The covariance matrix of position and velocity at the initial time is:

$$\mathbf{P}_1 = \ \mathbf{P}_{\tilde{X}} \ ^T,$$

where the last three rows of matrix   can be obtained through dynamic mapping. Obviously, the condition for equation (16) to hold is $\mathbf{Y} \to 0$, which requires a good initial guess for $\mathbf{X}$, but this cannot be simply obtained in general cases. In this work, we use the particle swarm algorithm with global optimization properties to solve this problem. The entire double-r algorithm for initial orbit determination consists of two steps: first, using the particle swarm algorithm to solve equation (12) to find the global optimal solution, and then using equation (16) for gradual iteration to obtain the local optimal solution.

3.2 Improvement of Admissible Region Method Used in This Paper

Assume there are N_K ($N_K \geq 2$) sets of TSA data. The cost function to be optimized can still be expressed in the form of equation (12), but the observation residual at each epoch becomes:

$$\mathbf{y}_d = [\cos\delta_d^o(\alpha_d^c - \alpha_d^o - \dot{\alpha}_d^o), (\delta_d^c - \delta_d^o - \dot{\delta}_d^o)]^T, \quad d \geq 2.$$

The entire solution process is similar to the double-r method described above: first, use the intelligent optimization algorithm for global optimization within the admissible region to find appropriate $\mathbf{X}$ that minimizes the residuals in equation (12), and then use equation (16) for further iterative solution in the local region. The difference from the double-r method lies only in the different form of matrix $\mathbf{B}$. To calculate the covariance matrix of the estimated parameters, we introduce the covariance matrix $\mathbf{P}_Y = E[\mathbf{Y}\mathbf{Y}^T]$, and the covariance matrix of the estimated parameters can still be calculated using equation (17). Similarly, for the covariance matrix $\mathbf{P}_{y_d}$ at each epoch, in addition to considering the variance of the observation itself, we must also consider the error contribution from the observations of the first TSA, i.e., the form is still as shown in equation (18), where:

$$\mathbf{P}_Y^{(1)} = \mathrm{diag}(\sigma_{\alpha_d}^2 \cos^2\delta_d^o, \sigma_{\delta_d}^2, \sigma_{\dot{\alpha}d}^2 \cos^2\delta_d^o, \sigma_{\dot{\delta}d}^2)$$

represents the variance information of the observation itself, while:

$$\mathbf{P}_Y^{(2)} = \frac{\partial \mathbf{Y}}{\partial(\alpha_1^o, \delta_1^o, \dot{\alpha}_1^o, \dot{\delta}_1^o)} \mathbf{P}_{\alpha_1^o, \delta_1^o, \dot{\alpha}1^o, \dot{\delta}_1^o} \frac{\partial \mathbf{Y}^T}{\partial(\alpha_1^o, \delta_1^o, \dot{\alpha}_1^o, \dot{\delta}_1^o)}$$

represents the variance contribution mapped from the first TSA's variance information to subsequent observation epochs. The matrix in the above equation can be simply solved through dynamic mapping at the initial epoch.

Similar to the double-r method, equation (17) only provides the covariance information of the estimated parameters. To obtain the covariance information of the position and velocity at the initial time, further solution is required. The calculation formula is still as shown in equation (19), but:

$$\tilde{\mathbf{X}} = (\mathbf{X}, \alpha_1^o, \delta_1^o, \dot{\alpha}_1^o, \dot{\delta}_1^o)^T = (\rho_1, \dot{\rho}_1, \alpha_1^o, \delta_1^o, \dot{\alpha}_1^o, \dot{\delta}_1^o)^T.$$

The matrix $\mathbf{P}_{\tilde{X}}$ is:

$$\mathbf{P}_{\tilde{X}} = \begin{pmatrix} \mathbf{P}_X & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\alpha_1^o, \delta_1^o, \dot{\alpha}1^o, \dot{\delta}_1^o} \end{pmatrix}.$$

Thus, the Jacobian matrix in equation (19) can be obtained through the geometric relationship in equation (11).

3.3 Covariance Matrix Calculation for Improved Laplace Algorithm

The subsequent iteration formula is similar to those for the admissible region method and double-r method, but with corresponding changes to matrices **B** and . Define:

$$\gamma_m = -\frac{\partial(\lambda_m, \mu_m, \nu_m)}{\partial(\alpha_m, \delta_m)}.$$

All residual vectors are:

$$\mathbf{Y} = (\gamma_1, \gamma_2, \dots, \gamma_{N_R})^T.$$

In this method, the estimated parameters should be modified to:

$$\mathbf{X} = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)^T.$$

Substituting into equation (17) allows calculation of the covariance matrix of the estimated parameters.

3.4 Particle Swarm Algorithm

In the work presented here, for finding the parameters required by each method—such as ρ_1 and ρ_{N_R} in the double-r method, and ρ_1 and $\dot{\rho}_1$ in the admissible region method—we use the particle swarm algorithm for direct search to minimize the objective function. We briefly introduce the particle swarm algorithm and its computational steps. The particle swarm algorithm originates from studies of collective bird behavior: individual birds are attracted to their habitat and remember their own position closest to the habitat while sharing this information with neighbors. Each particle moves according to its own position and velocity, and corrects its velocity based on information about optimal values. The velocity correction considers weight and acceleration factor modifications. The specific computational process for particle swarm optimization is as follows [**?**]:

Step 1: Assume there are h-dimensional variables to be optimized, with each dimension's range being $[a_h, b_h]$. Set the initial population size *sizepop*, i.e., the number of particles. Ensure each dimension is within the set numerical range, generate *sizepop* initial particles, each of which can calculate an initial fitness value (in this problem, the observation residuals). Additionally, according to the maximum flight velocity limit, generate the corresponding initial velocity for each particle.

Step 2: For the l-th iteration process, i.e., the l-th generation population, update the weight, acceleration factor, velocity, and particles sequentially according to formulas. It should be noted that after velocity update, if it exceeds the velocity limit range, the maximum velocity is taken if too large, and the minimum velocity if too small. Once a particle is updated beyond the initial set range, a random number is found within the particle constraint range as its value, and the velocity of this particle is updated to 0. The fitness is recalculated, and then individual and group extrema are updated, completing this iteration.

Step 3: Record the optimal value and particle position for each generation. Repeat Steps 2 and 3 until l reaches the set maximum iteration number.

Step 4: Based on the recorded optimal values for each generation, find the optimal value and corresponding particle, and the solution ends.

For the calculation of this problem, the optimization variables are the values to be guessed, such as ρ_1 and ρ_n, ρ_1 and $\dot{\rho}_1$. The fitness calculation is the calculation of the objective function, while the range of optimization variables, population size, and iteration number need to be selected based on experience and actual conditions.

4.1 Data Preparation

The data used in this paper refer to real observation data and station data from the Minor Planet Center (MPC) of the International Astronomical Union (IAU). Using the orbital information of minor planet Toutatis (4179) provided by NEODyS (Near Earth Objects Dynamic Site) maintained by ESA (European Space Agency) as the initial value for integration, short arcs are predicted to give the corresponding right ascension, declination, and their rates at observation times, with random errors added. The errors in right ascension and declination do not exceed $1''$, and the errors in their rates do not exceed $1'' \cdot h^{-1}$.

4.1.1 Basic Information of Toutatis (04179) provides basic information about Toutatis (04179), including semi-major axis, eccentricity, corresponding Modified Julian Date, and period.

4.1.2 Angular Rate Error Estimation Taking right ascension as an example, assuming a linear method is used to give the observed angle:

$$\dot{\alpha}\Delta t = \frac{\alpha_2 - \alpha_1}{t_2 - t_1} - \ddot{\alpha}\Delta t.$$

Using first-order term approximation, the error is:

$$\delta\dot{\alpha} = \frac{\delta\alpha_2 - \delta\alpha_1}{\Delta t} - \frac{(\alpha_2 - \alpha_1)}{(\Delta t)^2}\delta(\Delta t) + \ddot{\alpha}\Delta t.$$

Therefore, the covariance is:

$$E[\delta\dot{\alpha}\delta\dot{\alpha}] = E\left[\frac{\delta\alpha_2 - \delta\alpha_1}{\Delta t} - \frac{(\alpha_2 - \alpha_1)}{(\Delta t)^2}\delta(\Delta t) + \ddot{\alpha}\Delta t\right]^2.$$

Assuming each observation epoch is very accurate (i.e., $\delta(\Delta t)$ in the above equation is a small quantity, which is an acceptable assumption under the current time unit system), and assuming $E(\delta\alpha_2) = E(\delta\alpha_1) = 0$, then:

$$E[\delta\dot{\alpha}\delta\dot{\alpha}] \approx \frac{1}{(\Delta t)^2}[E(\delta\alpha_2\delta\alpha_2) + E(\delta\alpha_1\delta\alpha_1)] + \ddot{\alpha}^2(\Delta t)^2.$$

In actual calculations, the residuals in right ascension and declination are $1''$. To ensure the accuracy of right ascension and declination rates, the time difference Δt between two observations is assumed to be about 1.4 hours. The accelerations in right ascension and declination are both small quantities, approximately $4'' \cdot h^{-2}$ for right ascension, which is negligible compared to the magnitude of the first term. Therefore, the residuals in right ascension and declination rates can be estimated to be about $1'' \cdot h^{-1}$. In this work, this value is used as the random residual added.

4.1.3 Station Velocity Calculation Since the coordinate system used for observations is the geocentric celestial coordinate system, while station positions are in the Earth-fixed coordinate system, we consider the transformation between the geocentric celestial coordinate system and the Earth-fixed coordinate system. This paper uses programs provided by SOFA[1] for coordinate conversion. Additionally, when calculating right ascension and declination rates, station velocity information is needed. The station position expression is:

$$\hat{\mathbf{R}}_t = (HG)^T\hat{\mathbf{r}}_t,$$

where $\hat{\mathbf{R}}_t$ represents the station position vector in the geocentric mean equinox of epoch, $\hat{\mathbf{r}}_t$ represents the station position vector in the Earth-fixed system, and $(HG)^T$ represents the transformation matrix between these two coordinate systems, including Earth rotation, polar motion, precession, and nutation. To obtain station velocity, the above equation needs to be differentiated. However, deriving the derivative of the (HG) matrix from theoretical formulas is complex. Therefore, we adopt a numerical method—the fourth-order Lagrange interpolation formula—to directly interpolate the calculated angular measurement data and differentiate the interpolation formula to give the rate of angular measurement data. The fourth-order Lagrange interpolation formula is:

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x}{(x_3-x_0)(x_3}$$

4.1.4 Light Aberration Calculation When performing orbit integration, we use the RKF78 integrator. Due to the short time length, only the Sun's gravitational force is considered in the force model. For artificial satellites, due to the short distance from the station, light aberration can be ignored in initial orbit determination. However, for near-Earth minor bodies, the distance from the station can be large, even reaching 1 AU. Based on observational data at time t_i, when the minor body-station distance is ρ_i, the time correction is:

$$\delta t_i = \rho_i / c.$$

Assuming ρ_i is 1 AU, $\delta t_i \approx 500$ s. If the minor body's motion velocity is 20 $\mathrm{km \cdot s^{-1}}$, the resulting observation angle error is about 10^{-5} rad, equivalent to $2''$. Therefore, light aberration correction is needed. The actual time correction formula should be:

$$\delta t_{\mathrm{true}} = \frac{\|\mathbf{r}_i(t_i - \delta t_{\mathrm{true}}) - \mathbf{R}_i(t_i)\|}{c}.$$

The above equation should be solved iteratively, but:

$$\delta t_i - \delta t_{\mathrm{true}} \approx \frac{\|\mathbf{r}_i(t_i - \delta t_{\mathrm{true}}) - \mathbf{r}_i(t_i)\|}{c} \approx \frac{v_i(t_i)\delta t_{\mathrm{true}}}{c} \approx O(10^{-2}).$$

The angle error caused by this time difference is about $0.0001''$, which is negligible.

4.1.5 Other Notes When generating populations in the particle swarm algorithm, constraints on optimization variables are needed, along with setting iteration numbers and population sizes. For the admissible region method, we set the population size to 1000, iteration number to 300, range of line-of-sight distance to $(a_E, 2$ AU$)$, and range of line-of-sight distance rate to $(-0.00593$ AU$\cdot$

$s^{-1}, 0.00593$ AU $\cdot$ s^{-1}). For the double-r method, we set the population size to 1000, iteration number to 300, and range of line-of-sight distance to $(a_E, 2$ AU).

The arc data used in this paper are shown in the appendix. To compare with true orbital values, we retain the actual observation times and station information from the real observation data, while the observation information (right ascension, declination, and their rates) is given by integrating the established orbit plus random errors.

4.2 Test Results

Considering the observation time as t and time difference as Δt, taking x_0, x_1, x_2, x_3 as $t - 2\Delta t, t - \Delta t, t + \Delta t, t + 2\Delta t$ respectively, differentiating equation (25) yields:

$$f'(x) = \frac{8y_3 - 8y_2 - y_1 + y_0}{12\Delta t}.$$

In this section, we present the calculation results of three methods—improved Laplace method, double-r method, and admissible region method—for different arc scenarios. In this paper, arcs with observation duration less than one day and fewer than 5 observations are defined as extremely short arcs, arcs longer than 150 days as long arcs, and the remaining arcs as normal arcs. In this work, to prevent random errors from accidentally affecting the judgment of results, we add new random errors to each set of original arc data multiple times to generate different arc data for calculation and comparison of final results. Additionally, to prevent differences caused by mechanical models from affecting the comparison with the double-r method and improved Laplace method, the admissible region method integrates under the two-body problem.

4.2.1 Discussion on the Second Iteration Step In this work, when using the admissible region method or double-r method, the theoretical solution process should be completed in two steps. The first step uses the intelligent optimization algorithm to find the global optimal solution as much as possible, and the second step further optimizes the global optimal solution using an iterative method (see equation (16)) to expect further optimization (i.e., to make observation residuals smaller). However, our actual numerical verification shows that in many cases, using the second step of local iteration actually increases residuals. The reason for this phenomenon is that in the derivation of the second step iteration formula, the assumption $\mathbf{Y} \to 0$ is made, removing the second derivative term $\mathbf{Y}^T \partial \mathbf{B}/\partial \mathbf{X}$, which is common in precise orbit determination. However, in extremely short-arc orbit determination, due to insufficient information to constrain the orbit, the assumption $\mathbf{Y} \to 0$ may no longer hold, i.e., the assumption in the process from equation (15) to equation (16) that $\mathbf{Y}^T \partial \mathbf{B}/\partial \mathbf{X} \ll \mathbf{B}^T \mathbf{B}$ no longer holds. Based on this phenomenon, in the case of extremely short-arc orbit determination, we actually only performed the first step and used its result as the final orbit determination output.

4.2.2 Extremely Short Arc Test Results [Figure 4: see original paper] and [Figure 5: see original paper] show the calculation results for an extremely short arc with a duration of about 2 hours and 4 observation data points. Here, 20 sets of test data are generated (each set contains 4 observation values, but with different random errors applied). The horizontal axis represents the test data group number, and the vertical axis represents the difference between the calculated semi-major axis and the true semi-major axis or the residual RMS for each test data group. Cases where the double-r method calculation yields no result are not shown in the figures. In the 20 tests, the improved Laplace method fails in all cases. Numerical verification shows that the iterative solution jumps between two points, and using the Gauss method to directly solve the eighth-order equation can confirm that this phenomenon is caused by the aforementioned multiple solution problem.

Under the combination of extremely short arcs and the admissible region method, the geometric condition of the arc itself may lead to orbits with extremely large semi-major axes or eccentricities approaching 1 after optimization. Such orbits are still within the admissible region described in Section 2.3 but obviously do not match reality. In this work, we artificially added the constraint $a < 5.2$ AU. Even so, in many test cases of the admissible region method, the calculated semi-major axis a approaches 5.2 AU, but its residuals (see [Figure 5: see original paper]) remain within $1''$. For extremely short arcs, the calculation results of initial orbit determination largely depend on orbital geometric constraints, and more constraints need to be added to obtain more reasonable results. Overall, for extremely short arcs, the improved Laplace method is basically not applicable, while the admissible region method and double-r method can mostly provide orbit determination results, but as shown in [Figure 4: see original paper], these results have low credibility. If more accurate orbits need to be calculated, more constraints must be added.

4.2.3 Normal Length Arc Test Results In [Figure 6: see original paper] and [Figure 7: see original paper], we consider Arc 2 with a duration of about 5 days and 4 observations. Compared with Arc 1, only the arc duration is increased. Multiple sets of data are generated to compare the calculated semi-major axis and residuals.

[Figure 7: see original paper] shows that the residuals of the admissible region method are significantly higher than the other two methods, but most are still less than $1''$. [Figure 6: see original paper] shows that the calculation results from the improved Laplace method and double-r method are very consistent with the true semi-major axis, and the results from these two methods are very similar. Comparing these results with Arc 1, when the observation arc duration increases, the improved Laplace method and double-r method can provide good results, indicating that when using these two methods for initial orbit calculation, the arc cannot be too short. However, the results of the admissible region method are similar to Arc 1, and its calculation results have no obvious relationship with arc length. For the admissible region method, when the observation number is the same but the arc length increases, the results are not satisfactory. We hypothesize two possible reasons: first, the errors in right ascension and declination rates $(\dot{\alpha}^c, \dot{\beta}^c)$ themselves are large; second, the constraints are insufficient, and the admissible region method is more affected by observation number than by observation arc length. To verify these hypotheses, we conducted two tests: first, during the global search of the particle swarm optimization algorithm, we did not constrain the right ascension and declination rates, using only the first set of right ascension and declination rates as input. The test found no obvious change in results, thus excluding the first hypothesis. Second, we increased the observation number while maintaining an arc length of about 5 days, changing the observation number to 15 (Arc 3) for calculation. [Figure 8: see original paper] and [Figure 9: see original paper] show the calculation results for Arc 3. Comparing with the results in [Figure 6: see original paper] and [Figure 7: see original paper], it is not difficult to find that the results of the admissible region method are significantly improved at this time. On the other hand, the double-r method shows a phenomenon where the semi-major axis is close to the true value but residuals are large, while the improved Laplace method still performs the best among the three.

Next, we consider Arc 4 with a duration of about 6 days and 22 observations. [Figure 10: see original paper] and [Figure 11: see original paper] provide multiple calculation results for Arc 4. For Arc 4, both the improved Laplace method and admissible region method can provide good results. Compared with Arc 2, the calculation results of the admissible region method have improved. However, the residual results of the double-r method are not ideal, but its semi-major axis results are similar to Arc 2. The reason for this result from the double-r method will be analyzed later.

Finally, we consider Arc 5 with a duration of 21 days and 62 observation data points. In [Figure 12: see original paper] and [Figure 13: see original paper], multiple sets of simulated observation data are generated for Arc 5, and the semi-major axis and residuals from orbit determination are compared with previous arc results. As shown in [Figure 13: see original paper], the calculation results of the improved Laplace method are very stable, with the semi-major axis basically fluctuating within a small range, indicating that when there is more observation data, more constraints are available, and calculation results can converge to a fixed region. However, the difference between the semi-major axis and the true value ([Figure 12: see original paper]) is larger compared with previous arcs. Upon investigation, this set of data starts on January 3, 1989, and 8 days earlier, on December 26, 1988, the body had a close encounter with Earth. Selecting 62 observation sample points starting from January 4, 1989, with a total duration of about 19.6 days (called Arc 6), the initial orbit calculation results are shown in [Figure 14: see original paper] and [Figure 15: see original paper]. At this time, the semi-major axis from the improved Laplace method differs little from the true value, and residuals remain below 1″.

Additionally, for Arcs 1-5 described above, as arc duration and observation number increase, the difference between the semi-major axis calculated by the three methods and the true semi-major axis becomes smaller, showing that more observation constraints yield more realistic orbital information. However, residual results do not show this phenomenon, and even increase for some arcs and methods, indicating that in initial orbit calculation, residual results are only a reference item and do not play a decisive role.

Regarding the calculation results of the double-r method for Arcs 4, 5, and 6, although the residuals are not ideal, the semi-major axis is close to the true value. We provide a schematic diagram of the convergence regions for semi-major axis and residuals of the double-r method in [Figure 16: see original paper] and [Figure 17: see original paper]. [Figure 16: see original paper] describes the function $\delta a(\rho_1, \rho_2)$ for the difference between the calculated semi-major axis and the true semi-major axis (in AU), while [Figure 17: see original paper] describes the residual function $RMS(\rho_1, \rho_2)$ (in arcseconds). The problem is to find points that simultaneously satisfy minimum in the above two figures. As shown in [Figure 17: see original paper], although the region with residuals less than 80″ is displayed, its convergence domain only exists on extremely thin straight lines. In contrast, [Figure 16: see original paper] shows a very wide convergence region for the semi-major axis. When the objective function is residuals, it is difficult to find the global minimum point, resulting in large residuals, but the corresponding semi-major axis at its position is close to the true value.

When the arc is long, the applicability of the improved Laplace method based on time series expansion is challenged because the series may not converge. However, the double-r method remains applicable. We conducted several numerical verification experiments, with results shown in . It can be seen that for Arcs 7 and 8, the calculated orbits are very similar to the true orbital elements, but the results for Arc 9 are worse than the previous two arcs (possibly because the time is too long, and orbit integration under the two-body problem differs significantly from that under the N-body problem).

Generally, orbit correlation can be calculated based on the defined Mahalanobis distance, which is essentially a "clustering" problem. Milani et al. [**?**] mentioned the application of the admissible region method in orbit correlation. Gronchi et al. [**?**] also performed orbit correlation work while using orbit integration for initial orbit determination, also utilizing the admissible region method. In addition, many scholars have conducted orbit correlation work for Earth-orbiting targets based on the idea of admissible regions. The idea of the double-r method proposed by Gooding [**?**] can also be applied to orbit correlation, i.e., placing observation data with long time intervals (number greater than 2) together for initial orbit calculation. If the residuals are greater than the threshold, they do not belong to the same observation target arc.

5. Conclusion

This paper examines the initial orbit determination of the large-eccentricity body Toutatis, selecting extremely short arcs with observation duration less than one day, normal-length arcs, and long arcs with duration exceeding 150 days. Three methods (improved Laplace method, double-r method, and admissible region method) are used for initial orbit calculation. The results show that for extremely short arcs, especially with few observation data, both the admissible region method and double-r method can provide results with small residuals, while the improved Laplace method usually cannot provide calculation results. If realistic orbits are needed, more constraints must be added to further determine the orbit. For normal arcs, our research shows that the calculation results of the double-r method and improved Laplace method are relatively more dependent on observation arc length rather than observation number, while the results of the admissible region method, when observation arc length is the same, are more dependent on observation number. When observation number is large, the improved Laplace method shows more prominent result accuracy and calculation speed. For long arcs, the double-r method is feasible and can be applied to orbit correlation of near-Earth object observation data.

For different arc durations and observation numbers, different methods can be adopted for initial orbit determination. For extremely short arcs, choose the double-r method or admissible region method; for normal-length to several-day arcs, choose the improved Laplace method. If other options are selected, choose the admissible region method when observation number is large, and the double-r method when observation number is small; for long arcs, the double-r method can be chosen.

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Note: Figure translations are in progress. See original paper for figures.