Matrix Models of Gentle Algebras and Their Global Dimension

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Full Text

The Matrix Model of Gentle Algebras and Their Global Dimension

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Abstract

This paper employs the matrix model of gentle algebras to characterize simple modules and projective modules over gentle algebras, providing a matrix representation of the projective resolution of simple modules. Consequently, it demonstrates that the global dimension of a gentle algebra can be described by a special class of submatrix sequences induced by its matrix model. Furthermore, the paper shows that these special submatrix sequences correspond to maximal nontrivial forbidden paths on the quiver of the gentle algebra, thereby

establishing that the global dimension of a gentle algebra equals the length of the maximal nontrivial forbidden path on its quiver.

Key words: projective module; projective resolution; homological dimension; matrix representation; quiver representation
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Gentle algebras were introduced by Assem and Skowroński to study tilted algebras of type $\tilde{A}$ and to establish a corresponding derived equivalence classification [?]. Gentle algebras are closely related to many important classes of algebras, such as biserial algebras, special biserial algebras, string algebras, and skew-gentle algebras.

As early as 1985, Wald and Waschbüsch classified biserial algebras using quiver representations [?]. Subsequently, Butler and Ringel further classified indecomposable modules over string algebras via quiver representations in 1987 [?]. The study of morphisms between indecomposable modules over gentle algebras was carried out by Crawley-Boevey [?] and Krause [?], respectively. Consequently, the module category of gentle algebras can be completely characterized through quiver representations. A geometric description of the module category of gentle algebras was provided by Baur and Coelho-Simões [?], who associated indecomposable modules over gentle algebras with permissible (closed) curves on surfaces, and irreducible morphisms between indecomposable modules with pivot elementary moves of curves.

Building upon this work, He, Zhu, and Zhou further characterized the geometric model for skew-gentle algebras [?] and described support τ-tilting modules as surface triangulations. Burban and Drozd gave matrix representations for gentle and skew-gentle algebras [?] and classified indecomposable objects in the category of projective complexes. Thus, gentle algebras can effectively provide computational examples or counterexamples for many algebraic problems.

On the other hand, the study of algebras can be reduced to the study of modules defined over them, with special classes of modules directly reflecting algebraic properties. For instance, homological dimensions of algebras are defined through special modules and used to classify algebras. Among these, global dimension measures the distance from a given algebra to hereditary algebras, while self-injective dimension reflects the Gorenstein homological properties of algebras. Reference [?] provided a surface characterization of the global dimension of gentle algebras using geometric models. In this paper, we will give a matrix characterization of the global dimension of gentle algebras using matrix models.

Conventions. In this paper, we fix the following conventions: k denotes an algebraically closed field. For any finite-dimensional k-algebra A, we always assume that A is a basic algebra. Therefore, by Gabriel's theorem, there exists a unique quiver Q such that $A \cong kQ/I$ for some admissible ideal I. All modules considered in this paper are finitely generated right A-modules over finite-dimensional

k-algebras A, and we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules.

1.1. Gentle Algebras and Their Definitions

This section introduces the quiver representation and matrix representation of gentle algebras. Here, a quiver Q refers to a quadruple (Q_0, Q_1, s, t), where Q_0 and Q_1 are sets (whose elements are called vertices and arrows, respectively), and $s, t : Q_1 \to Q_0$ are maps assigning to each arrow $\alpha \in Q_1$ its source $s(\alpha)$ and target $t(\alpha)$. A path p of length $l \geq 1$ on Q is an ordered sequence of arrows $p = \alpha_1 \alpha_2 \cdots \alpha_l$ such that $s(\alpha_{i+1}) = t(\alpha_i)$ for all $1 \leq i < l$. Note that each arrow (or vertex) can be regarded as a path of length 1 (or 0). The length of p is denoted by $l(p)$. The sets Q_0 and Q_1 also denote the collections of all paths of length 0 and length 1, respectively.

The path algebra kQ of a quiver Q is a k-algebra satisfying the following conditions: - kQ is a k-vector space with basis {all paths in Q}; - For any two paths $p = \alpha_1 \cdots \alpha_l$ and $q = \beta_1 \cdots \beta_m$ in Q, their product is defined by

$$pq = \begin{cases} \alpha_1 \cdots \alpha_l \beta_1 \cdots \beta_m & \text{if } t(\alpha_l) = s(\beta_1), \\ 0 & \text{otherwise.} \end{cases}$$

An ideal I of the path algebra kQ is called an admissible ideal if there exists $m \geq 2$ such that $R_Q^m \subseteq I \subseteq R_Q^2$, where R_Q denotes the ideal of kQ generated by all paths of length at least 1.

Definition 1.2 (Assem-Skowroński [?]; see also [?, Chapter IX, Definition 6.1]). A finite-dimensional algebra $A = kQ/I$ is called a **gentle algebra** if it satisfies the following conditions: - (G1) For each vertex $a \in Q_0$, there are at most two arrows starting at a and at most two arrows ending at a; - (G2) For any arrow $\alpha \in Q_1$, there exists at most one arrow $\beta \in Q_1$ such that $t(\beta) = t(\alpha)$ and $\alpha\beta \in I$ (resp. at most one arrow $\gamma \in Q_1$ such that $s(\gamma) = s(\alpha)$ and $\gamma\alpha \in I$); - (G3) For any arrow $\alpha \in Q_1$, there exists at most one arrow $\beta \in Q_1$ such that $t(\beta) = t(\alpha)$ and $\alpha\beta \notin I$ (resp. at most one arrow $\gamma \in Q_1$ such that $s(\gamma) = s(\alpha)$ and $\gamma\alpha \notin I$); - (G4) The ideal I is generated by paths of length 2.

To characterize the global dimension of gentle algebras via their quivers, we need to introduce the concept of forbidden paths. Forbidden paths were introduced by Avella-Alaminos and Geiß, including both trivial and nontrivial types. The dual concept is permitted paths. Forbidden and permitted paths determine the AG-derived invariants of gentle algebras and are used to study their classification under derived equivalence, see [?, ?]. Here, we only need nontrivial forbidden paths.

A **nontrivial forbidden path** F on a gentle algebra A is a path $F = \alpha_1 \alpha_2 \cdots \alpha_j$ of length $j \geq 1$ such that $\alpha_i \alpha_{i+1} \in I$ for all $1 \leq i < j$. In particular, F is called **maximal** if for any arrow $\alpha \in Q_1$ with $s(\alpha) = s(\alpha_1)$ (resp. $t(\alpha) = t(\alpha_j)$), we always have $\alpha\alpha_1 \notin I$ (resp. $\alpha_j \alpha \notin I$).

We now give the matrix definition of gentle algebras and use matrices to characterize simple modules over gentle algebras (see Corollary 1.8).

Definition 1.3. For any $n \in \mathbb{Z}^+$, define T_n to be the algebra of $n \times n$ lower triangular matrices over the field k, i.e., matrices $X = (x_{ij})$ with $x_{ij} = 0$ for all $i < j$.

Definition 1.4 [**?**]. An algebra A is called a **gentle algebra** if:

$$A \cong T_m / \langle E_{ij} \mid \epsilon(i,j) = 1 \rangle,$$

where T_m is the algebra of $m \times m$ lower triangular matrices over k, and $\epsilon : O(m) \to \{0,1\}$ is a binary map defined on $O(m) = \{(i,j) \mid 1 \leq i \leq j \leq m\}$ satisfying: - For some $j \in O(m)$, there uniquely exists $(i,j) \in O(m)$ such that $\epsilon(i,j) = 1$ and for any $(j,k) \in O(m)$, we have $\epsilon(i,k) = \epsilon(j,k)$; - Meanwhile, for the remaining $j \in O(m)$, we have $\epsilon(i,j) = 0$ for all i.

In particular, T_m is called the **normalization** of the gentle algebra A.

Example 1.5. Consider a gentle algebra A. In the $(2,2)$ sense, A can be written in block diagonal matrix form with blocks corresponding to $(1,3)$, $(2,1)$, etc. Moreover, by Definition 1.2, we have $A \cong kQ/I$. The normalization algebra is $T_{(3,2)} = T_3 \times T_2$, the Cartesian product of T_3 and T_2, which correspond respectively to the two sub-blocks of the block diagonal matrix. According to Definition 1.4, $A \cong T_{(3,2)}/I$, where $I = \langle E_{(1,1)}, E_{(1,2)}, ...\rangle$.

Notation 1.6. For any matrix M in T_m satisfying the following conditions, we use the notation $M(i,j)$ for $1 \leq i \leq j \leq m$ to denote its (i,j)-entry. Furthermore, define $(i,j) \in O(m)$ as the position index.

Lemma 1.7. Let $A \cong T_m/I$ be a gentle algebra, and denote by $\{E_{(i,j)} \mid (i,j) \in O(m)\}$ the matrix units. Then $\{e_{(i,j)} = E_{(i,j)} + I \mid (i,j) \in O(m)\}$ is a complete set of primitive orthogonal idempotents of A.

Proof. First, we verify orthogonality: for any $(i,j),(u,v) \in O(m)$, we have $e_{(i,j)}e_{(u,v)} = \delta_{j,u}e_{(i,v)}$. We consider three cases: (a) $j < u$; (b) $j > u$; and (c) $j = u$. We only prove case (a); cases (b) and (c) are similar.

Take $(i,j),(u,v) \in O(m)$ with $j < u$. Then $e_{(i,j)}e_{(u,v)} = (E_{(i,j)}+I)(E_{(u,v)}+I) = E_{(i,j)}E_{(u,v)} + I$. Note that $E_{(i,j)}E_{(u,v)} = 0$ in T_m, otherwise $E_{(i,j)}E_{(u,v)} = E_{(i,v)}$ would contradict the definition of I since $\epsilon(i,v) = 0$. Similarly, $E_{(u,v)}E_{(i,j)} = 0$. Therefore, by matrix multiplication, $e_{(i,j)}e_{(u,v)} = 0$. The case $j > u$ is proved analogously.

Second, it is clear that each element in $\{E_{(i,j)}\}$ is primitive in T_m. For any element y in $\{e_{(i,j)}\}$, suppose $y = E_{(i,j)} + I$ has an additive decomposition $y = y_1 + y_2$ in A. Note that $E_{(i,j)}$ has a unique additive decomposition in T_m. Since $\epsilon(i,j) = 0$ and $\epsilon(j,j) = 0$ by the definition of gentle algebras, this additive decomposition is not a decomposition in A, i.e., y is primitive in A.

Finally, for any $x \in A$, we have $x = \sum_{(i,j)\in O(m)} e_{(i,j)} x e_{(i,j)}$ by direct matrix multiplication. Hence $\{e_{(i,j)}\}$ is a complete set of primitive orthogonal idempotents of A.

Let $\operatorname{simp} A$ denote the set of all simple modules over A up to isomorphism. Lemma 1.7 has provided a complete set of primitive orthogonal idempotents of A, from which we immediately obtain the following conclusion.

Corollary 1.8. Using the notation from Lemma 1.7, we have:

$$\operatorname{simp} A = \{S_{(i,j)} \mid (i,j) \in O(m)\},$$

where each simple module $S_{(i,j)}$ corresponds to a primitive idempotent $e_{(i,j)}$.

Remark 1.9. From Corollary 1.8, if $(i,j) \in O(m)$, then we have a right A-module isomorphism $S_{(i,j)} \cong e_{(i,j)}A/\operatorname{rad}(e_{(i,j)}A)$. Note that there is a matrix equality $e_{(i,j)}T_m \cong T_{(i,j)}$, where for simplicity we denote by $T_{(i,j)}$ the subspace of T_m consisting of matrices with nonzero entries only in the i-th row and j-th column. It is easy to verify that this matrix is a finitely generated right A-module.

If $(i,j),(r,s) \in O(m)$, then we have the following right A-module isomorphism:

$$e_{(i,j)}A \cong \operatorname{span}_k\{E_{(i,j)}\} \oplus \bigoplus_{(u,v)\in R(i,j)} kE_{(u,v)},$$

where $R(i,j) = \{(u,v) \in O(m) \mid \epsilon(u,v) = 0\}$. Note that although $e_{(i,j)}A$ is a right A-module, $T_{(i,j)}$ itself is not a right A-module. Furthermore, for convenience, we denote by $R_{I,J}$ (where $I,J \subseteq \{1,2,\dots,m\}$) the subset of A consisting of matrices that are zero outside the positions in $I \times J$. Then we have the following k-linear isomorphism:

$$S_{(i,j)} \cong R_{\{i\},\{j\}} \cong k,$$

where $R_{\{i\},\{j\}} = kE_{(i,j)}$ and the isomorphism is as k-vector spaces.

Example 1.10. Take A to be the gentle algebra given in Example 1.5. Then the primitive idempotents correspond to $(1,1)$, $(1,2)$, $(2,2)$, $(1,3)$, and $(2,1)$. Therefore, up to isomorphism, there are 4 indecomposable projective modules over A: $P_{(1,1)}$, $P_{(1,2)} \cong P_{(2,2)}$, $P_{(1,3)}$, and $P_{(2,1)}$. In the matrix representation, elements in the same position in T_m have the same values. Furthermore, we can obtain the matrix descriptions of the 4 simple modules: $S_{(1,1)}$, $S_{(1,2)}$, $S_{(2,2)}$, etc.

1.2. Projective Modules and Projective Resolutions

Let A be a finite-dimensional k-algebra. We know that for any right A-module $M \in \operatorname{mod} A$, there exists an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

where each P_i is projective. Such an exact sequence is called a **projective resolution** of M. In particular, if there exists $n \in \mathbb{N}$ such that $P_{n+1} = 0$, then n is called the length of this projective resolution. A module M may have different projective resolutions. In this paper, we call the projective resolution of M with minimal length its **minimal projective resolution**, or simply its projective resolution. In this case, $P_n \neq 0$ and the exact sequence

$$0 \to \Omega^i(M) \to P_{i-1} \to \cdots \to P_0 \to M \to 0$$

is called the projective cover of M, where each map $P_i \to \ker(\partial_{i-1})$ is a projective cover of $\ker(\partial_{i-1})$. The module $\ker(\partial_{i-1})$ is called the ***i*-th syzygy** of M, denoted by $\Omega^i(M)$.

Definition 1.11 (Projective dimension and global dimension) [**?**]. Let A be a finite-dimensional k-algebra. We say that a module $M \in \operatorname{mod} A$ has projective dimension $\operatorname{proj.dim} M \leq n$ if $\Omega^{n+1}(M) = 0$. In particular, the **global dimension** of A is defined as the supremum of projective dimensions of all finitely generated right A-modules:

$$\operatorname{gl.dim} A = \sup\{\operatorname{proj.dim} M \mid M \in \operatorname{mod} A\}.$$

(Equivalently, $\operatorname{proj.dim} M$ equals the length of a minimal projective resolution of M.)

Remark 1.12. In general, for any ring R, the right (resp. left) projective dimension of a right (resp. left) R-module M is defined as $\operatorname{proj.dim}_R M \leq n$ if M has a right (resp. left) projective resolution of length n. Therefore, the right (resp. left) global dimension of R is defined as $\operatorname{gl.dim} R = \sup\{\operatorname{proj.dim}_R M\}$, where the supremum is taken over all finitely generated right (resp. left) R-modules. In general, $\operatorname{gl.dim} R$ need not equal $\operatorname{gl.dim} R^{op}$. Kaplansky gave the first example of a ring with different left and right global dimensions [**?**]. Jategaonkar proved that for any $1 \leq m < n \leq \infty$, there exists a ring R with $\operatorname{gl.dim} R = m$ and $\operatorname{gl.dim} R^{op} = n$ [**?**]. Fossum, Griffith, and Reiten generalized these results to Abelian categories [**?**, pp.74–75]. When R is a finite-dimensional k-algebra, we have $\operatorname{gl.dim} R = \operatorname{gl.dim} R^{op} < \infty$.

Let $\operatorname{proj} A$ denote the set of all finitely generated projective right A-modules. Using the notation from Lemma 1.7, by [**?**, Corollary 5.17], we have the following corollary.

Corollary 1.13. For a finite-dimensional algebra A, we have:

$$\operatorname{gl.dim} A = \sup\{\operatorname{proj.dim} S \mid S \in \operatorname{simp} A\}.$$

In particular, the global dimension of A can be computed from the projective dimensions of its simple modules.

2. Nonzero Syzygies of Simple Modules and Their Matrix Representations

Note that the global dimension of an algebra can be computed through the projective dimensions of its simple modules, i.e., $\mathrm{gl.dim}\, A = \sup\{\mathrm{proj.dim}\, S \mid S \in \mathrm{simp}\, A\}$. Therefore, we need to characterize the syzygies of simple modules. For convenience, we always assume that A is a gentle algebra with normalization T_m, and the binary map $\epsilon : O(m) \to \{0, 1\}$ is fixed in advance.

2.1. First Syzygies of Simple Modules

The following lemma provides a matrix representation of the projective cover of a simple module, from which we can further obtain a matrix representation of its first syzygy (see Remark 2.2).

Lemma 2.1. For any $(i, j) \in O(m)$, let $e_{(i,j)}$ be the corresponding primitive idempotent. The canonical projection $\hat{p} : e_{(i,j)}A \to S_{(i,j)}$ is defined by mapping any matrix $X \in e_{(i,j)}T_m$ to its diagonal element while sending all off-diagonal entries to 0. Then we have a k-vector space isomorphism:

$$\mathrm{Hom}_A(e_{(i,j)}A, S_{(i,j)}) \cong \mathrm{span}_k\{\hat{p}\}.$$

Proof. If $(i, j) \in O(m)$, then there exists a matrix unit $E_{(i,j)}$ such that $e_{(i,j)} = E_{(i,j)} + I$. First, $e_{(i,j)}A$ is both the projective cover of $S_{(i,j)}$ and the canonical surjection onto $S_{(i,j)}$. By Remark 1.9, we have $e_{(i,j)}A/\,\mathrm{rad}(e_{(i,j)}A) \cong S_{(i,j)}$. From this it is easy to see that $\mathrm{Hom}_A(e_{(i,j)}A, S_{(i,j)})$ is a 1-dimensional k-vector space spanned by $\hat{p}$.

The map $\hat{p}$ preserves the diagonal elements of X and sends all other entries of X to 0, where X is any matrix in $e_{(i,j)}T_m$. Thus there exists a k-linear isomorphism:

$$\mathrm{Hom}_A(e_{(i,j)}A, S_{(i,j)}) \cong \mathrm{span}_k\{\hat{p}\}.$$

On the other hand, $\mathrm{Hom}_A(e_{(i,j)}A, S_{(i,j)})$ is 1-dimensional because $S_{(i,j)}$ is a simple module over a finite-dimensional algebra and is therefore a 1-dimensional k-vector space. Hence the map is a k-linear isomorphism. The case for other idempotents is proved similarly.

Remark 2.2. Using the notation from Lemma 2.1, for a matrix X, denote by $X_{[x,y]}$ the submatrix consisting of rows x through y. Then Lemma 2.1 gives a matrix description of the projective cover $\hat{p} : e_{(i,j)}A \to S_{(i,j)}$.

Furthermore: (i) If $(i, j) \in O(m)$ with $\epsilon(i, j) = 0$, then we have a right A-module isomorphism:

$$\ker(\hat{p}) \cong \bigoplus_{(u,v)\in R(i,j)} e_{(u,v)}A,$$

where $R(i, j) = \{(u, v) \in O(m) \mid \epsilon(u, v) = 0 \text{ and } v = i\}$.

(ii) If $(i,j) \in O(m)$ with $\epsilon(i,j) = 1$, then we have a right A-module isomorphism:

$$\ker(\hat{p}) \cong \bigoplus_{(u,v)\in R'(i,j)} e_{(u,v)}A,$$

where $R'(i,j) = \{(u,v) \in O(m) \mid \epsilon(u,v) = 0 \text{ and } v = i\}$.

Note that the isomorphism in case (ii) further indicates that $\Omega^1(S_{(i,j)})$ is isomorphic as a k-vector space to:

$$\bigoplus_{(u,v)\in R'(i,j)} kE_{(u,v)}.$$

The notation $\oplus$ denotes direct sum as k-vector spaces, and the k-vector space isomorphism in the first case is also a right A-module isomorphism.

Corollary 2.3. Let $M = \bigoplus_{(i,j)\in X} e_{(i,j)}A$ for some $X \subseteq O(m)$. Then M is indecomposable projective if and only if for any $(i,j) \in X$, we have $\epsilon(i,j) = 0$.

Proof. Note that M is a right A-module. By Remark 2.2, it is easy to see that M is a direct summand of some $e_{(u,v)}A$. If for any $(i,j) \in X$ we have $\epsilon(i,j) = 0$, then by Lemma 1.7, each $e_{(i,j)}A$ is an indecomposable projective module, and thus M is indecomposable projective. Conversely, if there exists $(i,j) \in X$ with $\epsilon(i,j) = 1$, then by Lemma 1.7, $e_{(i,j)}A$ is not projective, and therefore by Remark 1.9, M cannot be projective, a contradiction.

2.2. Higher Syzygies of Simple Modules

Similar to Lemma 2.1, the following proposition gives a matrix representation of the projective cover of any syzygy of a simple module, and consequently, analogous to Remark 2.2, it also provides a matrix representation of any syzygy of a simple module.

Proposition 2.4. Let $e_{(i,j)}$ be a primitive idempotent of $A \cong T_m/I$. If $\Omega^n(S_{(i,j)}) \cong \bigoplus_{(u,v)\in X_n} e_{(u,v)}A$ for some $X_n \subseteq O(m)$, then we have a k-vector space isomorphism:

$$\mathrm{Hom}_A\Big(\bigoplus_{(u,v)\in X_n} e_{(u,v)}A, S_{(p,q)}\Big) \cong \mathrm{span}_k\{\hat{p}_{(u,v)}\},$$

where $\hat{p}_{(u,v)}$ is the canonical projection induced by the image space as a subspace of the domain.

Note that $S_{(i,j)}$ is a 1-dimensional k-vector space. Therefore, for any $n \geq 1$, $\mathrm{Hom}_A(\Omega^n(S_{(i,j)}), S_{(p,q)})$ as a k-linear transformation is isomorphic to $\hat{p}$. Thus we have a canonical embedding:

$$\kappa : \mathrm{span}_k\{(u,v) \in X_n\} \to \mathrm{Hom}_A(\Omega^n(S_{(i,j)}), S_{(p,q)}).$$

Since there is a right A-module isomorphism $\Omega^n(S_{(i,j)}) \cong \bigoplus_{(u,v)\in X_n} e_{(u,v)}A$, we obtain a k-vector space isomorphism:

$$\mathrm{Hom}_A(\Omega^n(S_{(i,j)}), S_{(p,q)}) \cong \bigoplus_{(u,v)\in X_n} \mathrm{Hom}_A(e_{(u,v)}A, S_{(p,q)}).$$

Because each $\mathrm{Hom}_A(e_{(u,v)}A, S_{(p,q)})$ is a 1-dimensional k-vector space, κ is a k-linear isomorphism. The proof for the case $\Omega^n(S_{(i,j)}) = 0$ is similar.

Remark 2.5. Using the notation from Proposition 2.4, analogous to Remark 2.2, we have:

$$\Omega^{n+1}(S_{(i,j)}) \cong \ker(\hat{p}) \cong \bigoplus_{(u,v)\in X_{n+1}} e_{(u,v)}A,$$

where $X_{n+1} = \{(u,v) \in O(m) \mid \epsilon(u,v) = 0 \text{ and } v \in \{1,2\}\}$. Moreover, the k-vector space isomorphism in the first case is also a right A-module isomorphism.

Notation 2.6. For any $M \in \mathrm{mod}\, A$, we denote by $\Omega(M)$ the set of all nonzero syzygies of M.

3. Global Dimension of Gentle Algebras

To characterize the global dimension of gentle algebras, we introduce the notion of associated subblocks of the normalization algebra T_m. We say that two subblocks T_r and T_s of T_m are **associated** if there exist $(i,j),(u,v) \in O(m)$ such that $\epsilon(i,j) = \epsilon(u,v) = 0$ and $j = u$. The set $\{(i,j),(u,v)\}$ is called an **association**, and the collection of such pairs forms an **associated subblock sequence**.

Formally, an **associated subblock group** $C = \{(i_k, j_k)\}_{k=1}^t$ (where $t \in \mathbb{Z}^+$ or $t = \infty$) is called **continuous** if for any $1 \le k < t$, the association between (i_k, j_k) and (i_{k+1}, j_{k+1}) satisfies $j_k = i_{k+1}$. In particular, a continuous associated subblock group C_1 is called **maximal** if for any continuous associated subblock group C_2 with $C_1 \subseteq C_2$, we always have $C_1 = C_2$.

Proposition 3.1. For any simple module $S_{(i,j)} \in \mathrm{simp}\, A$, there exists a maximal continuous associated subblock group $C = \{(i_k, j_k)\}_{k=1}^t$ (where $t \in \mathbb{Z}^+$ or $t = \infty$) such that the set of association nodes $\{j_k \mid 1 \le k \le t\}$ is in bijection with the set of nonzero syzygies $\Omega(S_{(i,j)})$.

Proof. Without loss of generality, let $S = S_{(i,j)}$. By Lemma 2.1, we have a k-vector space isomorphism $\mathrm{Hom}_A(e_{(i,j)}A, S) \cong \mathrm{span}_k\{\hat{p}\}$. By Remark 2.2, $\Omega^1(S) \cong \bigoplus_{(u,v)\in R_1} e_{(u,v)}A$ for some $R_1 \subseteq O(m)$. If $R_1 \neq \emptyset$, then by Proposition 2.4 (with $n = 1$), we have a k-vector space isomorphism $\mathrm{Hom}_A(\Omega^1(S), S') \cong \mathrm{span}_k\{\hat{p}\}$ for each simple composition factor S'. The association set is $\{(i_1,j_1),(i_2,j_2)\}$. By Remark 2.5 (with $n = 1$), we have $\Omega^2(S) \cong \bigoplus_{(u,v)\in R_2} e_{(u,v)}A$. By Corollary 2.3, each summand is indecomposable projective. Applying Proposition 2.4 again (with $n = 2$), we obtain

$\Omega^3(S) \cong \bigoplus_{(u,v)\in R_3} e_{(u,v)}A$ with association set $\{(i_2, j_2), (i_3, j_3)\}$. By induction, we can construct a maximal continuous associated subblock group C such that there is a bijection between $\{j_k\}$ and $\Omega(S)$. The result follows from Corollary 1.13 and Proposition 3.1.

Theorem 3.2. Let A be a non-simple gentle algebra, and let $\mathcal{C}$ be the set of all maximal continuous associated subblock groups of its normalization algebra T_m. Then:

$$\operatorname{gl.dim} A = \max\{|C| \mid C \in \mathcal{C}\},$$

where $|C|$ denotes the length of the associated subblock group C.

Proof. This follows directly from Corollary 1.13, Proposition 3.1, and the definition of global dimension.

Let $\mathcal{F}$ be the set of all maximal nontrivial forbidden paths on the quiver of A, and let $\mathcal{M}$ be the set of all maximal continuous associated subblock groups of A. We now give a quiver description of the global dimension of gentle algebras.

Theorem 3.3. Let A be a gentle algebra. Then:

$$\operatorname{gl.dim} A = \max\{\ell(F) \mid F \in \mathcal{F}\},$$

where $\ell(F)$ denotes the length of the forbidden path F.

Proof. To prove this theorem, we only need to establish a bijection $\Phi : \mathcal{F} \to \mathcal{M}$. The proof is divided into parts (a) and (b). In (a), we construct Φ and prove it is injective. In (b), we prove Φ is also surjective.

(a) Let $F = \alpha_1\alpha_2 \cdots \alpha_t$ be a maximal nontrivial forbidden path with $\alpha_i\alpha_{i+1} \in I$ for $1 \leq i < t$. Then there exists a sequence of indices $i_1, i_3, i_5, \ldots$ such that $1 \leq i_k \leq t-1$. In this case, define $\Phi(F) = \{(i_k, i_{k+1})\}_{k=1}^{t}$, which forms an associated subblock group as described. This construction uniquely determines $\Phi(F)$. Thus Φ is injective.

Clearly, for different maximal nontrivial forbidden paths F, the associated subblock groups $\Phi(F)$ constructed above are distinct, as their association sets differ.

(b) Conversely, take any maximal continuous associated subblock group $C = \{(j_k, j_{k+1})\}_{k=1}^{t} \in \mathcal{M}$. By definition, each pair (j_k, j_{k+1}) corresponds to a vertex in Q_0 and an arrow $\alpha_k \in Q_1$ such that $s(\alpha_k) = j_{k-1}$ and $t(\alpha_k) = j_k$, with $\alpha_{k-1}\alpha_k \in I$ and $\alpha_k\alpha_{k+1} \in I$. Thus C corresponds to a nontrivial forbidden path $p = \alpha_1\alpha_2 \cdots \alpha_t$. Since C is maximal, p must also be maximal. Otherwise, if there existed a larger maximal forbidden path $p' \supset p$, then by part (a), $\Phi(p')$ would properly contain C, contradicting the maximality of C. This constructs a forbidden path $F \in \mathcal{F}$ with $\Phi(F) = C$. Hence Φ is surjective.

By Theorem 3.2, the theorem follows.

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Note: Figure translations are in progress. See original paper for figures.