

An old problem of Erdős: a graph without two cycles of the same length postprint

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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph on n vertices in which any two cycles are of different lengths. Let $f^*(n)$ be the maximum number of edges in a simple graph on n vertices in which any two cycles are of different lengths. Let M_n be the set of simple graphs on n vertices in which any two cycles are of different lengths and with the edges of $f^*(n)$. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$. In this paper, it is proved that for n sufficiently large, $mc(n) \leq \frac{15}{16}n$. We make the following conjecture:

$$\lim_{n \rightarrow \infty} \frac{mc(n)}{n} = 0.$$

Full Text

Preamble

An old problem of Erdős: a graph without two cycles of the same length
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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph on n vertices in which any two cycles are of different lengths. Let $f^*(n)$ be the maximum number of edges in a simple graph on n vertices in which any two cycles are of different lengths. Let M_n be the set of

simple graphs on n vertices with $f^*(n)$ edges in which any two cycles are of different lengths. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$. In this paper, it is proved that for n sufficiently large, $mc(n) \leq 16n$. We make the following conjecture: **Conjecture.** $mc(n) \leq \frac{15}{16}n$.

Introduction

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, P. Erdős raised the problem of determining $f(n)$ (see [?], p.247, Problem 11). Let $f^*(n)$ be the maximum number of edges in a simple graph on n vertices in which any two cycles are of different lengths. Let M_n be the set of simple graphs on n vertices with $f^*(n)$ edges in which any two cycles are of different lengths. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$, $sc(n)$ be the second-largest cycle length for all $G \in M_n$, and $tc(n)$ be the third-largest cycle length for all $G \in M_n$. A natural question concerns the values of $mc(n)$, $sc(n)$, and $tc(n)$. Let $mcn(n)$ be the maximum number of cycles for all $G \in M_n$. Another natural question concerns the value of $mcn(n)$. Let $b(n)$ be the maximum number of 2-connected blocks for all $G \in M_n$. Yet another natural question concerns the value of $b(n)$.

Shi [?] proved the following fundamental result:

Theorem 1 (Shi [?]). $f(n) \geq n + \lfloor (\sqrt{8n - 23} + 1)/2 \rfloor$ for $n \geq 3$, and $f(n) = f^*(n) + 1 + 3$ for $n \geq 3$.

Lai [?] established an upper bound:

Theorem 2 (Lai [?]). For $n \geq e^{2m}(2m+3)/4$, $f(n) < n - n \ln(4n/(2m+3)) + 2n + \log_2(n+6)$.

Chen, Lehel, Jacobson and Shreve [?] gave a quick proof of this result. Jia [?], Lai [?, ?, ?, ?, ?, ?, ?, ?], Shi [?, ?, ?, ?, ?, ?], Shi, Tang, Tang, Gong, Xu [?], and Shi, Xu, Chen, Wang [?] obtained additional related results.

Lai [?] proved the following asymptotic bound:

Theorem 3 (Lai [?]). $\liminf \frac{f(n)}{n} \geq 1$.

Lai [?] conjectured a stronger lower bound:

Conjecture 4 (Lai [?]). $\liminf \frac{f(n)}{\sqrt{n}} \geq s$.

Boros, Caro, Füredi and Yuster [?] proved a precise asymptotic result:

Theorem 5 (Boros, Caro, Füredi and Yuster [?]). $f(n) = n + 1.98\sqrt{n}(1 + o(1))$.

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length. In 1988, Shi [?] proved:

Theorem 6 (Shi [?]). For every integer $n \geq 3$, $f_2(n) \geq 2(\sqrt{8n})$.

In 1998, G. Chen, J. Lehel, M. S. Jacobson, and W. E. Shreve [?] proved:

Theorem 7 (Chen, Lehel, Jacobson and Shreve [?]). $f_2(n) = n + o(\sqrt{n})$.

In 2001, E. Boros, Y. Caro, Z. Füredi and R. Yuster [?] improved this lower bound significantly:

Theorem 8 (Boros, Caro, Füredi and Yuster [?]). $f_2(n) \geq n + \sqrt{n} - 20$.

They also conjectured:

Conjecture 9 (Boros, Caro, Füredi and Yuster [?]). $\lim_{n \rightarrow \infty} \frac{f_2(n) - n}{\sqrt{n}} = 1$.

It is easy to see that this conjecture implies the (difficult) upper bound in the Erdős-Turán Theorem [?, ?] (see [?]).

Markström [?] raised the following problem:

Problem 10 (Markström [?]). Determine the maximum number of edges in a Hamiltonian graph on n vertices with no repeated cycle lengths.

Let $g(n)$ be the maximum number of edges in an n -vertex Hamiltonian graph with no repeated cycle lengths. J. Lee and C. Timmons [?] proved:

Theorem 11 (J. Lee, C. Timmons [?]). If q is a power of a prime and $n = q^2 + q + 1$, then $g(n) \geq n + \sqrt{2n} + O(1)$. A simple counting argument shows that $g(n) < n + \sqrt{2n} + 1$.

Let MH_n be the set of Hamiltonian graphs on n vertices with $g(n)$ edges in which any two cycles are of different lengths. Let $mcn_H(n)$ be the maximum number of cycles for all $G \in MH_n$. A natural question concerns the value of $mcn_H(n)$.

J. Ma and T. Yang [?] proved:

Theorem 12 (Ma, Yang [?]). Any n -vertex 2-connected graph with no two cycles of the same length contains at most $n + \sqrt{n} + o(\sqrt{n})$ edges.

Let $f_2(n, k)$ be the maximum number of edges in a graph G on n vertices in which no two cycles have the same length and G consists of k 2-connected blocks. A natural question concerns the maximum number of edges $f_2(n, k)$. It is clear that $f_2(n, 1) = f_2(n)$. By Theorem 5, it follows that $f_2(n, k) \leq f(n) \leq n + 1.98\sqrt{n}(1 + o(1))$.

H. Lin, M. Zhai, and Y. Zhao [?] proved:

Theorem 13 (Lin, Zhai, Zhao [?]). Let G be a graph of order n . If $\rho(G) > \rho(K_{1,1})$, then G contains two cycles of the same length unless $G \cong K_{1,1}$.

They also posed the following problem:

Problem 14 (Lin, Zhai, Zhao [?]). What is the maximum spectral radius among all 2-connected n -vertex graphs without two cycles of the same length?

Y. Shi [?] proved:

Theorem 15 (Shi [?]). $mc(n) \leq \lfloor (\sqrt{8n} + 15)/2 \rfloor + 1$.

C. Lai [?] proved:

Theorem 16 (Lai [?]). $mc(n) \geq 1$ for $n \geq \sum_{i=1}^5 i$.

Survey papers on this problem can be found in Tian [?], Zhang [?], and Lai and Liu [?]. The progress on all 50 problems in [?] can be found in Locke [?].

Let $v(G)$ denote the number of vertices and $\varepsilon(G)$ denote the number of edges. In this paper, we prove:

Theorem 17. For n sufficiently large, $mc(n) \leq \frac{15}{16}n$.

2 Proof of Theorem 17

Proof. Suppose, for contradiction, that $mc(n) > \frac{15}{16}n$ for n sufficiently large. Then there exists a simple graph G on n vertices with $f^*(n)$ edges in which any two cycles have different lengths, and the maximum cycle length of G is $mc(n)$. Let G_1 be the block containing the cycle of length $mc(n)$.

It is clear that $v(G_1) > \frac{15}{16}n$. By the result of Ma and Yang [?], we have $\varepsilon(G_1) \leq v(G_1) + \sqrt{v(G_1)} + o(\sqrt{v(G_1)})$. By the result of Boros, Caro, Füredi and Yuster [?], we have:

$$\begin{aligned} \varepsilon(G) &\leq \varepsilon(G_1) + (n - v(G_1)) + 1 + 1.98\sqrt{n - v(G_1)}(1 + o(1)) \\ &\leq v(G_1) + \sqrt{v(G_1)} + o(\sqrt{v(G_1)}) + n - v(G_1) + 1 + 1.98\sqrt{\frac{1}{16}n}(1 + o(1)) \\ &= n + 1 + \sqrt{n} + o(\sqrt{n}) + 1.98 \cdot \frac{1}{4}\sqrt{n}(1 + o(1)) \\ &= n + 1 + \sqrt{n} + o(\sqrt{n}) + 0.495\sqrt{n}(1 + o(1)). \end{aligned}$$

However, by the result of Shi [?] and Lai [?], we have $\varepsilon(G) = f^*(n) = n + \frac{3}{2}\sqrt{2n} + o(\sqrt{n})$ for n sufficiently large. Note that $n + 1 + \sqrt{n} + o(\sqrt{n}) + 0.495\sqrt{n}(1 + o(1)) < n + \frac{3}{2}\sqrt{2n}$ for n sufficiently large. This contradiction completes the proof.

It is clear that $mc(n) \geq mc(n)$. By Theorem 3, it follows that $mc(n) = n + o(n)$.

We make the following conjecture:

Conjecture. $mc(n) \leq \frac{15}{16}n$.

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