Well-connected residuated lattices and residually finite residuated lattices

Authors: Yang Jiang, Wang Juntao, Xia Changchun, Xia Changchun

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Abstract

The aim of this paper is to study well-connected residuated lattices and residually finite residuated lattices. So far, well-connected residuated lattices have been not only a main tool for studying RLsi but also subdirectly irreducible representation objects of residuated lattices. In this paper, we investigate both of the above two aspects using some different methods. Finally, we introduce residually finite residuated lattices and characterize them from algebraic, logical, and topological perspectives, respectively.

Full Text

Preamble

Jiang Yanga, **Juntao Wang**b, **Changchun Xia**c,∗

aSchool of Mathematics, Northwest University, Xi'an, 710127, China
bSchool of Science, Xi'an Shiyou University, Xi'an, 710065, China
cSchool of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710129, China

Abstract

This paper aims to study well-connected residuated lattices and residually finite residuated lattices. It is well known that well-connected residuated lattices are not only a main tool for studying RL_{si} but also serve as subdirectly irreducible representation objects of residuated lattices. Based on these facts, we investigate both aspects using different methods. Meanwhile, we introduce residually finite residuated lattices and characterize them from algebraic, logical, and topological perspectives.

Keywords: Well-connected residuated lattice; Residually finite residuated lattice; Hopfian residuated lattice; Filter

1. Introduction

Monoidal Logic (ML), introduced by U. Höhle [16], is a logic whose algebraic counterpart is the class of residuated lattices. This logic is built up from four primitive connectives &, $\rightarrow$, $\wedge$, $\vee$ and the truth constant 0. The difference with Basic Fuzzy logic (BL) is that in ML, $\wedge$ and $\vee$ are not definable from the others and thus need to be introduced as primitive connectives. Commutative residuated lattices were first introduced by M. Ward and R.P. Dilworth [28] as generalizations of ideal lattices of rings. Non-commutative residuated lattices, sometimes called pseudo-residuated lattices, biresiduated lattices, or generalized residuated lattices, are the algebraic counterpart of substructural logics—that is, logics which lack some of the three structural rules, namely contraction, weakening, and exchange. Comprehensive studies on residuated lattices were developed by H. Ono [19, 20], T. Kowalski [19, 20], P. Jipsen [17], C. Tsinakis [18, 26], and N. Galatos [12, 13].

There are two main traditions in defining these structures: algebraic and logical. The algebraic tradition defines residuated lattices as structures with only one constant in the type (namely, the unit of the monoid), whereas the logical tradition makes use of four constants (the top, the bottom, the unit, and a certain zero). Within the logical tradition, the term "residuated lattices" has sometimes been used to denote an even narrower class of algebras: namely, those where the monoid reduct is commutative and its unit and zero coincide, respectively, with the top element and the bottom element of the lattice. Fortunately, the logical tradition has also provided alternative nomenclature. Because of their connection with extensions of what is known in logic as full Lambek calculus, these algebras have been called FL-algebras. Further, among the extensions of full Lambek calculus, there are important ones that arise by adding one or more of the three so-called structural rules of exchange, weakening, or contraction. A notation devised by H. Ono refers to these logics as FL_X, with X being an appropriate subset of $\{e, w, c\}$. N. Galatos, P. Jipsen, T. Kowalski, and H. Ono (see [12] Sect. 3.4) list some extensive classes of residuated structures (mostly residuated lattices and FL-algebras) that appear in algebra and logic.

Substructural logics are non-classical logics that are weaker than classical logic, in the sense that they may lack one or more of the structural rules of contraction, weakening, and exchange in their Gentzen-style axiomatization. These logics encompass a large number of non-classical logics related to computer science (linear logic [14]), linguistics (Lambek Calculus [15]), philosophy (relevant logics [1]), and many-valued reasoning [16].

According to Birkhoff's theorem, each variety is completely determined by its class of subdirectly irreducible members. It is well known that each subdirectly irreducible Heyting algebra is well-connected [10]. In [22], L.L. Maksimova gave an algebraic characterization of the disjunction property for superintuitionistic logics using well-connected Heyting algebras. Following the work of N. Galatos, P. Jipsen, T. Kowalski, and H. Ono [12], in Section 3 we investigate well-connected residuated lattices.

Residual finiteness originates from group theory and appears in algebraic logic [21], topological algebra [24], and modal logic [8]. A logic is finitely approximable if it is characterized by a class of finite algebras (see [8] for more details). As is well known, the congruence lattices of residuated lattices have nice algebraic properties. Note that the residual finiteness of an algebra is closely related to its congruence lattice. Therefore, in Section 4 we study residually finite residuated lattices.

2. Preliminaries

In this section, we summarize some definitions and results about residuated lattices.

Definition 2.1. [16] An algebraic structure $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ of type (2, 2, 2, 2, 0, 0) is called a residuated lattice if it satisfies the following conditions: (1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice; (2) $(L, \otimes, 1)$ is a commutative monoid; (3) $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$, for all $x, y, z \in L$, where $\leq$ is the partial order of the lattice $(L, \wedge, \vee, 0, 1)$.

Throughout this paper, we will slightly abuse notation by using L to denote both the residuated lattice and its underlying set when there is no risk of confusion.

For the convenience of readers, we provide some basic properties of residuated lattices in the following proposition.

Proposition 2.2. [27] In any residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, the following properties hold for any $x, y, z \in L$:

(R1) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$;
(R2) $x \leq y$ if and only if $x \rightarrow y = 1$;
(R3) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$, $z \rightarrow x \leq z \rightarrow y$ and $x \otimes z \leq y \otimes z$;
(R4) $x \otimes (x \rightarrow y) \leq y$;
(R5) $x \otimes y \leq x \wedge y$, $x \leq y \rightarrow x$;
(R6) $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z = y \rightarrow (x \rightarrow z)$;
(R7) $x \vee (y \otimes z) \geq (x \vee y) \otimes (x \vee z)$, hence $x^m \vee y^n \geq (x \vee y)^{mn}$.

Definition 2.3. [27] Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. A filter is a nonempty set $F \subseteq L$ such that for each $x, y \in L$: (i) $x, y \in F$ implies $x \otimes y \in F$; (ii) if $x \in F$ and $x \leq y$, then $y \in F$.

Note that in a residuated lattice L, a filter F of L is equivalent to a deductive system, meaning F satisfies: (i) $1 \in F$, and (ii) $x, x \rightarrow y \in F$ implies $y \in F$. We denote by $F(L)$ the set of all filters of a residuated lattice L.

With any filter F of L we can associate a congruence relation θ_F on L by defining $(x, y) \in \theta_F$ if and only if $x \rightarrow y, y \rightarrow x \in F$.

For any $x \in L$, let x/F be the equivalence class x/θ_F. If we denote by L/F the quotient set L/θ_F, then L/F becomes a residuated lattice with operations induced from those of L. A filter F of L is called prime if $x \vee y \in F$ implies $x \in F$ or $y \in F$ for any $x, y \in L$. We denote the set of prime filters of a residuated lattice L by $P(L)$.

Theorem 2.4. [16] Let L be a residuated lattice. For every element $a \in L$ with $a \neq 1$, there is a prime filter P in L with $a \notin P$.

Theorem 2.5. [16] Let L be a residuated lattice. Then $\bigcap P(L) = \{1\}$.

Let L be a residuated lattice. By $F(L)$ ($Con(L)$), we mean the set of all filters (congruences) of L. There is a close correspondence between congruences and filters of residuated lattices. For each congruence θ on residuated lattice L, let $[1]_\theta = \{x \in L : (1, x) \in \theta\}$. Then $[1]_\theta$ is a filter of L, called the filter determined by a congruence θ. Conversely, for each filter F, $\theta_F = \{(x, y) \in L \times L : (x \to y) \otimes (y \to x) \in F\}$ is a congruence, called the congruence determined by a filter F. Moreover, the following result holds.

Theorem 2.6. [12] For every residuated lattice L, we have $F(L) \cong Con(L)$.

Proposition 2.7. [29] A residuated lattice L is a subdirect product of a family L_i of residuated lattices if and only if there is a family $\{F_i\}_{i \in I}$ of filters of L such that: (i) $L_i \cong L/F_i$ for each $i \in I$; (ii) $\bigcap_{i \in I} F_i = \{1\}$.

3. Well-connected residuated lattices

A variety is a class of algebras closed under taking homomorphic images, subalgebras, and arbitrary direct products. Since the intersection of a nonempty family of varieties is again a variety, we may consider the variety generated by any class C of algebras, denoted $V(C)$. One of Birkhoff's most celebrated theorems states that classes of algebras defined by identities are precisely varieties. It is well known that the class RL of residuated lattices is a variety [3]. Moreover, RL is a congruence distributive variety. This follows from the fact that each residuated lattice has a lattice reduct. Thus we can show the congruence distributivity for RL by the same majority term as for lattices (see [7] Theorem 12.3).

An algebra A is (congruence) permutable if for all $\theta, \phi \in Con(A)$, $\theta \circ \phi = \phi \circ \theta$. A variety is permutable if every member is permutable.

Proposition 3.1. [3] The variety RL is permutable.

Recall that an algebra is arithmetical if and only if it is both congruence distributive and permutable. A variety is arithmetical if every member is arithmetical, and a variety has the congruence extension property (CEP) if for every algebra A in the variety, for any subalgebra B of A and for any congruence θ on B, there exists a congruence θ' on A such that $\theta' \cap B^2 = \theta$.

Proposition 3.2. [19] The variety RL is arithmetical and has CEP.

Recall that an element a of a lattice L is said to be join-irreducible if $x \vee y = a$ implies $x = a$ or $y = a$. Dually, an element a of L is called meet-irreducible if $a = x \wedge y$ implies $a = x$ or $a = y$. The element a is completely meet-irreducible if $a \neq 1$ and whenever $a = \bigwedge_{i \in I} b_i$, there is a $j \in I$ such that $a = b_j$. We shall say that the join $\bigvee_{i=1}^{m} x_i$ is irredundant if for every k, $\bigvee_{i \neq k} x_i < x_1 \vee \cdots \vee x_{k-1} \vee x_{k+1} \vee \cdots \vee x_m$. Roughly speaking, a join is irredundant if the removal of any term results in something smaller. If $\bigvee_{i \neq k} x_i = \bigvee_{i=1}^{m} x_i$, then we say that the element x_k is redundant.

For distributive lattices with the descending chain condition, we have the following strengthening of the Kurosh-Ore Theorem.

Theorem 3.3. [6] If L is a distributive lattice that satisfies the descending chain condition, then every element of $L \ \ \{0\}$ can be expressed uniquely as an irredundant join of join-irreducibles.

Definition 3.4. A filter F of a residuated lattice L is said to be join-irreducible if F is a join-irreducible element of the lattice $(F(L), \subseteq)$. Dually, F is called meet-irreducible if it is a meet-irreducible element of the lattice $(F(L), \subseteq)$.

The following result is a rather direct application of a more general result (see [25] Theorem 1.9). We include the complete proof for the reader's convenience.

Theorem 3.5. Let L be a residuated lattice. Then the meet-irreducible filters of L are precisely the prime filters.

Proof. Let P be a meet-irreducible filter of L and $x \vee y \in P$. If $x \notin P$ and $y \notin P$, then we claim that $P = (x \vee P) \cap (y \vee P)$, where $x \vee P = \langle x \rangle \vee P = \{z : x^n \otimes a \leq z \text{ for } a \in P \ \&\ n \in \mathbb{Z}^+\}$. It suffices to show that $(x \vee P) \cap (y \vee P) \subseteq P$, since the other inclusion is obvious. Let $z \in (x \vee P) \cap (y \vee P)$. Then there exist $a \in P$ and $n, m \in \mathbb{Z}^+$ such that $x^n \otimes a \leq z$ and $y^m \otimes a \leq z$. By Proposition 2.2 (R7), it follows that $(x \vee y)^{nm} \otimes a \leq z$. Therefore, we have $z \in P$, which contradicts the meet-irreducibility of P.

Conversely, suppose that P is a prime filter of L. Since $(F(L), \subseteq)$ is a distributive lattice, it suffices to show that P is a meet-prime element of $(F(L), \subseteq)$. If $F \cap G \subseteq P$ with $F \nsubseteq P$ and $G \nsubseteq P$, then there are $x \in F$ and $y \in G$ such that $x \notin P, y \notin P$. Thus, by the primality of P, we have $x \vee y \in F \cap G$ and $x \vee y \notin P$. This shows that $F \cap G \nsubseteq P$, which is a contradiction. Therefore, $F \subseteq P$ or $G \subseteq P$, which completes the proof.

In the following, we establish a property of prime filters of residuated lattices similar to those of commutative rings [2].

Corollary 3.6. Let $F_1, \ldots, F_n$ be filters of a residuated lattice L and let P be a prime filter containing $\bigcap_{i=1}^{n} F_i$. Then $P \supseteq F_i$ for some i. If $P = \bigcap_{i=1}^{n} F_i$, then $P = F_i$ for some i.

We now give another proof of Theorem 2.5.

Corollary 3.7. Let L be a residuated lattice. Then $\bigcap P(L) = \{1\}$.

Proof. Let $CMF(L)$ denote the set of all completely meet-irreducible filters of L. Clearly, $\{1\} = \bigcap CMF(L)$ since $(F(L), \subseteq)$ is an algebraic lattice. By Theorem 3.5, $\{1\} = \bigcap P(L)$ since every completely meet-irreducible filter is a meet-irreducible filter and every filter is the meet of a set of completely meet-irreducible filters ([23] Theorem 2.19).

If a residuated lattice L satisfies the descending chain condition (DCC), we have the following theorem similar to Theorem 3.3.

Theorem 3.8. If L is a residuated lattice that satisfies the descending chain condition, then every element of $F(L)$ $\{1\}$ can be expressed uniquely as an irredundant join of join-irreducible filters.

Proof. This follows directly from Proposition 3.2 and Theorem 3.3.

Proposition 3.9. [12] A residuated lattice L is subdirectly irreducible if and only if it satisfies the condition: there exists an element $a < 1$ such that for all $x < 1$, $x^n \leq a$ for some positive integer n.

Proposition 3.10. Let L be a residuated lattice. If L has an element $a \neq 1$ such that $x \leq a$ for each $1 \neq x \in L$, and a is not a nilpotent element (i.e., L $\{1\}$ has a maximum non-nilpotent element), then L is subdirectly irreducible.

Proof. Let $a \in L$ satisfy the indicated condition. Then the filter $F = [a)$ is obviously the smallest of all filters distinct from the unit filter [1). By Theorem 2.7, there is a smallest nontrivial congruence of L. Therefore, by ([7] Theorem 8.4), L is subdirectly irreducible.

Proposition 3.11. Let L be a non-trivial residuated lattice. If L is subdirectly irreducible, then the smallest non-unit filter is a principal filter.

Proof. Let L be subdirectly irreducible. By ([7] Theorem 8.4) and Theorem 2.7, there exists a filter F which is the smallest of all filters distinct from [1). This filter F is principal, i.e., $F = [a)$ for some $a \in L$. Indeed, for any $1 \neq a \in F$, it follows that $[a) \subseteq F$ and thus \$[a) = F.

Corollary 3.12. Let L be a non-trivial residuated lattice. L is subdirectly irreducible if and only if there is a smallest non-unit principal filter in L.

Proof. The "only if" part follows from the above proposition. Now suppose the condition holds, i.e., $[a)$ is the smallest non-unit principal filter of L. For any $x \neq 1$ belonging to L where x is not a nilpotent element, we have $[a) \subseteq [x)$. This proves that the condition of Proposition 3.10 holds, which completes the proof.

In what follows, we observe that 1 is a join-irreducible element if and only if the unit filter $\{1\}$ is a prime filter.

Proposition 3.13. Let L be a residuated lattice. Then 1 is a join-irreducible element if and only if $\{1\}$ is a prime filter.

Proof. The proof is straightforward.

Proposition 3.14. Let L be a residuated lattice. If P is a prime filter of L, then the top element $1/P = P$ of the quotient residuated lattice L/P is a join-irreducible element.

Proof. Suppose that $x/P \vee y/P = 1/P$. Then we have $x/P \vee y/P = (x \vee y)/P = 1/P$. Thus, since $1 \to (x \vee y) \in P$ and $1 \in P$, it follows that $x \vee y \in P$. Note that P is a prime filter, so we have $x \in P$ or $y \in P$. Therefore, we get $x/P = 1/P$ or $y/P = 1/P$.

Proposition 3.15. Let P be a prime filter of a residuated lattice L. If L is finite, then the quotient residuated lattice L/P is subdirectly irreducible.

Proof. By hypothesis and Theorem 3.5, the prime filter P is completely meet-irreducible in $(F(L), \subseteq)$. Thus, by ([4] Theorem 3.23), it follows that L/P is subdirectly irreducible.

Definition 3.16. [12] A residuated lattice L is said to be well-connected if 1 is a join-irreducible element of L. The class of all well-connected residuated lattices will be denoted by $WcRL$.

Note that each linear residuated lattice is well-connected, and by Proposition 3.14, the quotient residuated lattice associated with a prime filter is a well-connected residuated lattice. Also, if a well-connected residuated lattice satisfies prelinearity (i.e., $(x \to y) \vee (y \to x) = 1$), then it is linear. Therefore, well-connectedness is a generalization of linearity. Recall that an MTL-algebra is a residuated lattice satisfying prelinearity. Thus, an MTL-algebra is linear if and only if it is well-connected.

Theorem 3.17. Up to isomorphism, well-connected residuated lattices are precisely the quotient residuated lattices via prime filters.

Proof. By the above argument, we have that L/P is a well-connected residuated lattice for a prime filter P of L. Conversely, suppose that L is a well-connected residuated lattice. By Proposition 3.13 and the definition of well-connected, we get that $\{1\}$ is a prime filter. Since $L \cong L/\{1\}$, the proof is complete.

Recall that for a homomorphism $h : A \to B$ of two residuated lattices A, B, we shall use the symbol $Ker(h)$ to denote $Ker(h) = \{x \in A \mid h(x) = 1\}$, the kernel of h. In the next lemma, we summarize, for further reference, some easy relations between filters and kernels of homomorphisms.

Lemma 3.18. Let A, B be residuated lattices, and $h : A \to B$ a homomorphism. Then the following properties hold: (1) for each filter F of B, the set $h^{-1}(F) = \{x \in A \mid h(x) \in F\}$ is a filter of A; thus, in particular, $Ker(h) \in F(A)$; (2) $h(x) \leq h(y)$ if and only if $x \to y \in Ker(h)$; (3) h is injective if and only if $Ker(h) = \{1\}$; (4) $Ker(h) \neq A$ if and only if B is nontrivial; (5) $Ker(h)$ is a prime filter if and only if B is nontrivial and the image $h(A)$, as a subalgebra of B, is a well-connected residuated lattice.

Proof. The proof is straightforward.

Proposition 3.19. If L is a well-connected MTL-algebra, then all proper filters of L are prime filters.

Proof. Suppose that F is a proper filter of L. Then L is a linear residuated lattice, and thus L/F is also a linear residuated lattice. Since $h_F : L \to L/F$ ($x \mapsto x/F$) is a surjective homomorphism and a linear residuated lattice is well-connected, by Lemma 3.18, $Ker(h_F) = F$ must be a prime filter.

As an immediate consequence, we also have that all proper filters of a linear residuated lattice are prime filters.

Proposition 3.20. Let L be a finite residuated lattice. If L is well-connected, then it is subdirectly irreducible.

Proof. Let L be a well-connected residuated lattice. Note that $L/\{1\} \cong L$; thus, by Proposition 3.15, L is subdirectly irreducible.

In the following, we give another proof of the well-known result in [12].

Theorem 3.21. Let L be a residuated lattice. If L is subdirectly irreducible, then it is well-connected.

Proof. By Theorem 2.5, we have $\bigcap P(L) = \{1\}$. Since L is subdirectly irreducible, $\{1\}$ is completely meet-irreducible; it follows that $\{1\} \in P(L)$. Therefore, $\{1\}$ is a prime filter. Thus, by the definition of well-connected residuated lattices, L is a well-connected residuated lattice.

Corollary 3.22. $RL_{si} \subseteq WcRL$.

Corollary 3.23. Let L be a finite residuated lattice. Then L is well-connected if and only if it is subdirectly irreducible.

Proof. This follows immediately from Proposition 3.20 and Theorem 3.21.

Remark 3.24. It is easily seen that every linear residuated lattice is well-connected. Hence, by Corollary 3.23, every finite linear residuated lattice is subdirectly irreducible. Thus, we have given a complete characterization of finite subdirectly irreducible members of the variety RL.

Let $JF(L)$ denote the set of join-irreducible filters of a residuated lattice L.

Proposition 3.25. If L is a well-connected residuated lattice that satisfies the descending chain condition, then $(JF(L), \supseteq)$ is cofinal in $(F(L), \supseteq)$.

Proof. This follows immediately from ([29] Lemma 5.14).

In what follows, we give a result similar to Chang's Subdirect Representation Theorem [9] to prove that an equation holds in all residuated lattices if it is sufficient to check that it holds in all well-connected residuated lattices.

Theorem 3.26. (Subdirect Representation Theorem) Every nontrivial residuated lattice is a subdirect product of well-connected residuated lattices.

Proof. Let L be a residuated lattice. By Theorem 2.5, we have $\bigcap P(L) = \{1\}$. Notice that by Proposition 2.7 and the fact that for each prime filter P of L, the quotient residuated lattice L/P is well-connected, this is precisely the assertion of the theorem.

In view of the above fact, we obtain the following result, which is important in studying the variety RL. Note that the class $WcRL$ is not a variety, since it is not closed under the direct product of well-connected residuated lattices.

Corollary 3.27. $RL = V(WcRL)$.

Lemma 3.28. [16] Every proper filter of a residuated lattice is an intersection of prime filters.

Given an algebra A and a nonempty family $(\theta_i)_{i\in I}$ of congruences on A, there is a natural injective homomorphism $A/\bigcap_{i\in I}\theta_i \to \prod_{i\in I} A/\theta_i$.

Proposition 3.29. Every nontrivial quotient residuated lattice can be embedded into the direct product of well-connected residuated lattices.

Proof. This follows from Lemma 3.28 and the above fact.

It is well known that in an MTL-algebra L, if P is a prime filter of L, then L/P is a linear MTL-algebra. Thus, by the above Subdirect Representation Theorem, we deduce the following well-known result about the completeness of MTL with respect to linearly ordered MTL-algebras given by F. Esteva and L. Godo [11].

Corollary 3.30. Every nontrivial MTL-algebra can be embedded into the direct product of linear MTL-algebras.

Let $\{A_i : i \in I\}$ be an indexed family of algebras of the same type. By an ultrafilter on I, we mean an ultrafilter of $P(I)$, viewed as a Boolean algebra. Let U be an ultrafilter on I. For elements a and b of the direct product $A = \prod_{i\in I} A_i$, we define their equalizer as $\{i \in I : a(i) = b(i)\}$. We define a relation $\eta_U = \{(a, b) \in A^2 : equalizer(a, b) \in U\}$. It is straightforward to verify that η_U is a congruence relation on A.

Theorem 3.31. Let $L = \varprojlim_{i\in I} L_i$ be a profinite residuated lattice. Then for every prime filter P on L, there is a filter F of $\prod_{i\in I} L_i$ such that $F \cap L \subseteq P$.

Proof. According to Theorem 3.5, P is meet-irreducible in the lattice $(F(L), \subseteq)$. By Theorem 2.7, we have that θ_P is meet-irreducible in the lattice $(Con(L), \subseteq)$. Since $(Con(L), \subseteq)$ is congruence-distributive, it satisfies the conditions of Jónsson's Lemma (e.g., see [4] Lemma 5.9). Hence, there is an ultrafilter U over I such that $\eta_U \cap L \subseteq \theta_P$. Let $F = [1]_{\eta_U}$. Thus, we have $[1]_{\eta_U} = F \cap L \subseteq P$, since $[1]_{\theta_P} = P$.

Let K be a class of algebras. Let $P_u(K)$ denote the class of all algebras isomorphic to ultraproducts of members of K. Also, we denote by K_{si} the class of all subdirectly irreducible members of K.

Theorem 3.32. $RL_{si} \subseteq HSP_u(WcRL)$.

Proof. This follows from Corollary 3.27 and ([4] Theorem 5.10).

4. Residually finite residuated lattices

This section is devoted to the study of residually finite residuated lattices. As their name suggests, residually finite residuated lattices generalize finite residuated lattices. They are defined as residuated lattices whose elements can be distinguished after taking finite quotients.

Definition 4.1. A residuated lattice L is called residually finite if for each element $g \in L$ with $g \neq 1_L$, there exist a finite residuated lattice F and a homomorphism $\varphi : L \to F$ such that $\varphi(g) \neq 1_F$.

Proposition 4.2. Let L be a residuated lattice. Then the following conditions are equivalent: (a) L is residually finite; (b) for all $g, h \in L$ with $g \neq h$, there exist a finite residuated lattice F and a homomorphism $\varphi : L \to F$ such that $\varphi(g) \neq \varphi(h)$.

Proof. The fact that (b) implies (a) is obvious, since (b) gives (a) by taking $h = 1_L$. Conversely, suppose that L is residually finite. Let $g, h \in L$ with $g \neq h$. Then $g \to h \neq 1$ or $h \to g \neq 1$. Without loss of generality, we can assume that $g \to h \neq 1$. Then there exist a finite residuated lattice F and a homomorphism $\varphi : L \to F$ such that $\varphi(g \to h) \neq 1$. Since $\varphi(g \to h) = \varphi(g) \to \varphi(h) \neq 1$, it follows that $\varphi(g) \neq \varphi(h)$. Therefore, (a) implies (b).

Example 4.3. Every finite residuated lattice is residually finite.

Given a residuated lattice L, the intersection of all filters of finite index of L is called the residual filter (or profinite kernel) of L.

Proposition 4.4. Let L be a residuated lattice and let N denote the residual filter of L. Then L is residually finite if and only if $N = \{1\}$.

Proof. Let L be residually finite. For $a \in L$ with $a \neq 1$, there exist a finite residuated lattice F_a and a homomorphism $\varphi_a : L \to F_a$ such that $\varphi_a(a) \neq 1$. Thus, by the fundamental homomorphism theorem for residuated lattices, we have $L/Ker(\varphi_a) \cong F_a$; hence $Ker(\varphi_a)$ is a finite-index filter for any $a \in L$ $\{1\}$. It follows that $\bigcap_{a \in L \ \{1\}} Ker(\varphi_a) = \{1\}$.

The converse is obvious.

In the following, we study the stability properties of residually finite residuated lattices.

Proposition 4.5. Every subalgebra of a residually finite residuated lattice is residually finite.

Proof. Let L be a residually finite residuated lattice and let H be a subalgebra of L. Let $h \in H$ such that $h \neq 1$. Since L is residually finite, there exist a finite residuated lattice F and a homomorphism $\varphi : L \to F$ such that $\varphi(h) \neq 1_F$.

If $\varphi' : H \to F$ is the restriction of φ to H, we have $\varphi'(h) = \varphi(h) \neq 1_F$. Consequently, H is residually finite.

Proposition 4.6. Let $(L_i)_{i\in I}$ be a family of residually finite residuated lattices. Then their direct product $L = \prod_{i\in I} L_i$ is residually finite.

Proof. Let $g = (g_i)_{i\in I} \in L$ such that $g \neq 1_L$. Then there exists $i_0 \in I$ such that $g_{i_0} \neq 1_{L_{i_0}}$. Since L_{i_0} is residually finite, we can find a finite residuated lattice F and a homomorphism $\varphi : L_{i_0} \to F$ such that $\varphi(g_0) \neq 1_F$. Consider the homomorphism $\varphi' : L \to F$ defined by $\varphi' = \pi \circ \varphi$, where $\pi : L \to L_{i_0}$ is the projection onto L_{i_0}. We have $\varphi'(g) = \varphi(g_0) \neq 1_F$. Consequently, L is residually finite.

Proposition 4.7. The homomorphic image of a residually finite residuated lattice is residually finite.

Proof. This follows from the fundamental homomorphism theorem for residuated lattices.

Theorem 4.8. The class of residually finite residuated lattices is a variety, denoted by $RfRL$.

Proof. This follows from Propositions 4.5, 4.6, and 4.7.

Corollary 4.9. Let L be a residuated lattice. Then the following conditions are equivalent: (a) L is residually finite; (b) L is isomorphic to a subalgebra of the direct product $\prod_{i\in I} L_i$ of a family $(L_i)_{i\in I}$ of finite residuated lattices.

Proof. The fact that (b) implies (a) follows from Proposition 4.3, Proposition 4.5, and Proposition 4.6. Conversely, suppose that L is residually finite. Then, for each $g \in L \ \{1_L\}$, we can find a finite residuated lattice F_g and a homomorphism $\varphi_g : L \to F_g$ such that $\varphi_g(g) \neq 1_{F_g}$. Consider the residuated lattice $H = \prod_{g\in L \ \{1_L\}} F_g$. The homomorphism $\psi : L \to H$ defined by $\psi(x) = (\varphi_g(x))_{g\in L \ \{1_L\}}$ is injective. Therefore, L is isomorphic to a subalgebra of H. This shows that (a) implies (b).

In the following, we deduce that the class of residually finite residuated lattices is closed under taking inverse limits.

Proposition 4.10. If a residuated lattice L is the limit of an inverse system of residually finite residuated lattices, then L is residually finite.

Proof. Let $\{L_i, \varphi_{ij}, I\}$ be an inverse system of residually finite residuated lattices such that $L = \varprojlim_{i\in I} L_i$. By the construction of inverse limits, L is a subalgebra of the residuated lattice $\prod_{i\in I} L_i$. We deduce that L is residually finite by using Proposition 4.5 and Proposition 4.6.

Recall that a residuated lattice L is called profinite if it is isomorphic to the inverse limit of an inverse system of finite residuated lattices. The class of such algebras is denoted by $ProRL$.

An immediate consequence of Proposition 4.10 is the following:

Corollary 4.11. Every profinite residuated lattice is residually finite.

Corollary 4.12. $ProRL \subseteq RfRL$.

Recall that an algebra A is finitely approximable if A is isomorphic to a subalgebra of a product of finite algebras [21]. It follows that A is finitely approximable if and only if A is a subdirect product of its finite homomorphic images.

Proposition 4.13. A residuated lattice is residually finite if and only if it is finitely approximable.

Proof. This follows immediately from Corollary 4.9.

We denote the class of finite residuated lattices by fRL. Thus, we have the following result:

Theorem 4.14. $RfRL = V(fRL)$.

Remark 4.15. In view of the above characterization of $RfRL$, for every $L \in RfRL$ there exists a family $(L_i)_{i\in I}$ of finite residuated lattices such that L can be embedded into $\prod_{i\in I} L_i$. By Theorem 3.31, it follows that if P is a prime filter of L, then there is a filter F of $\prod_{i\in I} L_i$ such that $F \cap L \subseteq P$. Also, we deduce that $RfRL_{si} \subseteq HSP_u(fRL)$.

Proposition 4.16. Let L be a residuated lattice. The canonical map $e_L : L \to \hat{L}$ is a monomorphism if and only if L is finitely approximable.

Proof. If e_L is injective, then it follows from the definition of $\hat{L}$ that L is isomorphic to a subalgebra of a product of finite algebras; thus L is finitely approximable. Conversely, if L is finitely approximable, then L is a subdirect product of the collection $\{L/\theta : \theta \in I\}$ of finite homomorphic images of L. Therefore, $a \neq b$ in L implies that there exists $\theta \in I$ such that $[a]_\theta \neq [b]_\theta$. Thus, the images of a and b in the inverse limit of $\{L/\theta, \varphi_{\theta\phi}, I\}$ are different, and so e_L is injective.

Proposition 4.17. Let L be a residuated lattice and $\hat{L}$ be its profinite completion. Then $\hat{L}$ is complete.

Proof. To show that $\hat{L}$ is complete, suppose $X \subseteq \hat{L}$, and denote each $a \in X$ by $a = (a_\theta)_{\theta\in I}$. For each $\theta \in I$, we set $b_\theta = \bigvee \pi_\theta(X)$, and then set $b = (b_\theta)_{\theta\in I}$. It is then clear that $b = \bigwedge X$ in $\prod_{\theta\in I} L/\theta$. It suffices to show that $b \in \hat{L}$. To see this, if $\psi \subseteq \chi$, then $\varphi_{\psi\chi}(a_\psi) = a_\chi$ for each $a \in X$. Therefore, using the fact that the quotients L/F are finite, $\varphi_{\psi\chi}(b_\psi) = \varphi_{\psi\chi}(\bigwedge \pi_\psi(X)) = \bigwedge \pi_\chi(X) = b_\chi$, and so $b \in \hat{L}$.

Remark 4.18. Suppose that an algebra A has a lattice reduct. Then $\hat{A}$ is complete. Likewise, if every member of a variety has a lattice reduct, then its profinite algebras are always complete.

In what follows, we obtain some results similar to the notion of residually finite algebras.

Lemma 4.19. Let L be a residuated lattice and $a \in L$ with $a \neq 1$. Then there exists a well-connected residuated lattice H and a surjective homomorphism $f : L \to H$ such that $f(a) \neq f(1)$.

Proof. Let a be an element of L with $a \neq 1$. By Theorem 2.4, there is a prime filter P of L such that $a \notin P$. Suppose that $f : L \to L/P$ is the canonical homomorphism. Obviously, f is surjective. According to Proposition 3.14, L/P is a well-connected residuated lattice. Hence, it suffices to show that $f(a) \neq f(1)$. Note that $f(a) = a/P$ and $f(1) = 1/P$, and since $a \notin P$, it follows that $a/P \neq 1/P$.

Theorem 4.20. Let L be a residuated lattice and $a, b \in L$ with $a \neq b$. Then there exists a well-connected residuated lattice H and a surjective homomorphism $f : L \to H$ such that $f(a) \neq f(b)$.

Proof. Note that $a \neq b$ if and only if $a \to b \neq 1$ or $b \to a \neq 1$. Without loss of generality, we can assume $a \to b \neq 1$. Thus, by the above lemma, there is a prime filter P of L such that $a \to b \notin P$, whence we have $a/P \to b/P \neq 1/P$. Therefore, we obtain that $f(a) = a/P \neq b/P = f(b)$.

Definition 4.21. A residuated lattice L is called Hopfian if every surjective endomorphism of L is injective.

Example 4.22. (a) Every finite residuated lattice is Hopfian. (b) Every simple residuated lattice is Hopfian.

Lemma 4.23. Let L be a finitely generated residuated lattice and let H be a finite residuated lattice. Then the set $Hom(L, H)$ is finite.

Proof. The proof is straightforward.

Theorem 4.24. Every finitely generated residually finite residuated lattice is Hopfian.

Proof. Let L be a finitely generated residually finite residuated lattice. Suppose that $\psi : L \to L$ is a surjective endomorphism. Let F be a filter of finite index of L and let $\rho : L \to L/F$ denote the canonical homomorphism. Consider the map $\Phi : Hom(L, L/F) \to Hom(L, L/F)$ defined by $\Phi(u) = u \circ \psi$ for all $u \in Hom(L, L/F)$. The map Φ is injective since ψ is surjective by our hypothesis. As the set $Hom(L, L/F)$ is finite by Lemma 4.23, we deduce that Φ is also surjective. In particular, there exists a homomorphism $u_0 \in Hom(L, L/F)$ such that $\rho = u_0 \circ \psi$. This implies $Ker(\psi) \subseteq Ker(\rho) = F$.

It follows that $Ker(\psi)$ is contained in the intersection of all filters of finite index of L. As L is residually finite, we deduce that $Ker(\psi) = \{1\}$ by Proposition 4.4. Thus ψ is injective. This shows that L is Hopfian.

If P is a property of residuated lattices, we say that a residuated lattice L is virtually P if L contains a subalgebra of finite index (since the variety RL

satisfies CEP, thus if H is a subalgebra of L, then there exists a congruence $\theta \in Con(L)$ such that $H \times H = \theta \cap H$; hence we mean H is a subalgebra of finite index, that is, $[1]_\theta$ is a finite-index filter of L) which satisfies P.

Lemma 4.25. Let H be a subalgebra of finite index of a residuated lattice L and let F be a filter of finite index of H. Then F is a filter of finite index of L.

Proof. Let $h_1, \dots, h_n$ be a complete set of representatives of the cosets of L modulo H and $k_1, \dots, k_p$ be a complete set of representatives of the cosets of H modulo F. Observe that the elements $h_i k_j$, where $1 \leq i \leq n$ and $1 \leq j \leq p$, form a complete set of representatives of cosets of L modulo F. Therefore, $|L/F| < \infty$.

Compared with Proposition 4.5, in the following we give an opposite approach to find residually finite residuated lattices.

Theorem 4.26. Every virtually residually finite residuated lattice is residually finite.

Proof. Let L be a residuated lattice and H a subalgebra of finite index of L. By Lemma 4.25, the intersection of the filters of finite index of L is contained in the intersection of the filters of finite index of H. Since a residuated lattice is residually finite if and only if the intersection of its filters of finite index is reduced to the identity element (Proposition 4.4), we deduce that L is residually finite if H is residually finite.

Theorem 4.27. Let L be an infinite simple residuated lattice. Then L is not residually finite (profinite).

Proof. The only filter of finite index of L is L itself. Therefore, L is not residually finite by Proposition 4.4.

Example 4.28. If we consider $I = [0, 1]$, $\otimes$ to be the usual multiplication of real numbers, and for $x, y \in I$,

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y/x & \text{otherwise.} \end{cases}$$

Then $(I, \max, \min, \otimes, \to)$ is a residuated lattice (called Product structure or Gaines structure). Routine calculation shows that it is an infinite simple residuated lattice; thus $(I, \max, \min, \otimes, \to)$ is neither residually finite nor profinite.

Recall that a variety is residually finite [23] if every subdirectly irreducible member of it is finite. In Example 4.28, since the Product structure is an infinite simple residuated lattice, by the definition of a residually finite variety, we have the following result:

Theorem 4.29. RL is not residually finite.

Theorem 4.30. $RfRL$ is residually finite.

Proof. This follows from Proposition 4.16 and ([5] Proposition 1.5).

Corollary 4.31. Any subvariety of $RfRL$ is residually finite.

We now summarize the characterizations of residually finite residuated lattices as follows:

Theorem 4.32. Let L be a residuated lattice. Then the following conditions are equivalent: (a) L is residually finite; (b) the elements of L can be distinguished after taking finite quotients (Proposition 4.2); (c) $N_L = \{1\}$ (Proposition 4.4); (d) L is finitely approximable (Proposition 4.13); (e) the canonical map $e_L : L \to \hat{L}$ is an injective homomorphism (Proposition 4.16); (f) the finite index topology of L is Hausdorff ([29] Theorem 5.2).

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Note: Figure translations are in progress. See original paper for figures.