Review of Atomic Clock Frequency Stability Evaluation Algorithms: Postprint

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Abstract

Atomic clocks measure time through stable frequency standards, but the output signals of frequency standards are affected by noise from different sources; therefore, the phase deviation between atomic clocks can be regarded as a continuous random process obeying a power law $h\alpha f^{\alpha}$. This random process is not stationary; generally, high-order differencing is employed to make the sequence stationary, and commonly used time-domain frequency stability evaluation methods adopt this approach. Power-law noise is actually derived from differentiating and integrating white noise processes; noises with different power laws correspond to different stochastic differential equations, and in fact, the simulation of power-law noise is also implemented based on white noise. First, identification methods for different power-law noises are introduced, and differential equations corresponding to common power-law noises are provided in conjunction with literature; then, the relationships between different frequency stability evaluation methods and their corresponding transfer functions are introduced, and calculation methods for their confidence intervals are briefly summarized. The work of this paper facilitates the construction of random models for atomic clocks and frequency stability evaluation methods.

Full Text

A Compilation of the Theoretical Fundamentals of Atomic Clock Frequency Stability Evaluation Methods

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Abstract

Atomic clocks measure time through stable frequency standards, but the output signals of these standards are affected by noise from various sources. Consequently, the phase deviation between atomic clocks can be modeled as a continuous random process following a power-law spectral density of the form $h_\alpha f^\alpha$. This random process is non-stationary, and the conventional approach for time-domain frequency stability assessment employs high-order differencing to achieve stationarity. Power-law noise is essentially derived through differentiation and integration of white noise processes, with different power-law exponents corresponding to different stochastic differential equations. In practice, power-law noise simulation is also implemented based on this principle. This paper first introduces methods for identifying different types of power-law noise and presents, in conjunction with relevant literature, the corresponding stochastic differential equations for common power-law noise processes. It then discusses the relationships among various frequency stability assessment methods and their transfer functions, and briefly summarizes the calculation methods for their confidence intervals. This work facilitates the construction of stochastic models for atomic clocks and the development of frequency stability assessment methods.

Keywords: atomic clocks; power-law spectral noise; equivalent degrees of freedom; confidence intervals; unbiased estimates

1 Introduction

Atomic clocks measure time via stable frequency standards, but the output signals of these oscillators are inevitably affected by electronic component noise. Different levels of white noise accumulate over time, manifesting as power-law characteristics in the power spectral density of clock phase deviations, which contain distinct periodic features. While the autocorrelation function (ACF) and power spectral density (PSD) contain complete information about a continuous random process, extracting useful quantitative information from them is challenging. Allan variance and Hadamard variance provide quantitative descriptions of atomic clock frequency stability, and Hutsell established the mathematical relationship between these variances and the process noise coefficients in Kalman filter models.

Allan variance can identify frequency white noise, frequency flicker noise, and frequency random walk noise (with power-law exponents of 0, -1, and -2 for frequency data, respectively) and can be used to calculate noise level coefficients. However, it diverges for frequency flicker walk noise (power-law exponent -3) and frequency random run noise (power-law exponent -4), which unfortunately affect rubidium atomic clocks. In contrast, Hadamard variance based on third-order differences converges for these low-frequency noise types. When smoothing time is long, the confidence level of Hadamard variance estimates decreases. To address this, researchers proposed a total variance estimation method that

extends data through specific mapping to improve confidence. Similar to the overlapping Allan variance, the overlapping Hadamard variance can be defined, which maximally utilizes all possible triple differences in the sample and thus makes full use of the available data, thereby improving estimation confidence.

Unlike standard Hadamard variance estimation, the overlapping version employs correlated samples, so its estimates do not follow a chi-square distribution with integer degrees of freedom but instead empirically approximate a chi-square distribution with non-integer equivalent degrees of freedom ν. For variance estimation, larger equivalent degrees of freedom yield smaller confidence intervals at the same significance level, resulting in more accurate estimates. With the establishment of the BeiDou-3 global system, research on high-performance spaceborne atomic clock time deviation calculation, prediction, and evaluation has been conducted, with the prerequisite being the construction of fundamental mathematical models for these atomic clocks. Section 2 of this paper first introduces the most widely accepted atomic clock model, which is arguably the simplest and most effective model, incorporating both deterministic and stochastic components. The stochastic component is a non-stationary power-law spectral model. For atomic clock frequency, it is generally believed to be affected by the superposition of five integer power-law noise processes (exponents 2 to -2), with subsequent rubidium atomic clock studies adding two additional power-law noise processes (-3, -4). Section 2 also introduces methods for identifying different power-law noise types. Power-law processes can be obtained from white noise processes through differentiation and integration. For example, random walk is a power-law process with exponent -2; as an independent increment process, its differentiation yields white noise. Section 3 primarily discusses power-law noise simulation and its corresponding stochastic differential equations, focusing mainly on time-domain analysis. Frequency stability assessment essentially evaluates the energy of power-law noise. Allan variance and Hadamard variance are fundamentally consistent, achieving stationarity through differencing in the time domain. From an overview perspective, Section 4 presents a unified description of these two frequency stability assessment methods and their relationship, while also organizing their frequency-domain transfer functions. For a Gaussian process (white noise process), the most important statistics are the mean and variance functions; the mean is often assumed zero, and the variance primarily reflects noise energy. It is well known that variance estimators generally follow a chi-square distribution, so the confidence intervals for Allan or Hadamard variance can similarly be determined through the chi-square distribution. Section 5 introduces the determination of equivalent degrees of freedom and confidence intervals for frequency stability assessment methods.

2 Atomic Clock Model and Power-Law Noise Identification

Atomic clocks measure time through stable frequency standards by comparing phase differences between clocks and deriving physical time differences based on the reference frequency. However, atomic clock frequency standards are

susceptible to noise from various sources, with temperature and radiation simultaneously affecting frequency stability. A widely accepted and relatively simple electronic oscillator model for atomic clock frequency standards is:

$$V(t) = (V_0 + \varepsilon(t)) \sin[2\pi\nu_0 + \phi(t)]$$

where V_0 and ν_0 are the amplitude and frequency, respectively; $\varepsilon(t)$ is amplitude noise, which can be neglected for precision atomic clock oscillators; and $\phi(t)$ is a non-stationary continuous power-law random process satisfying $|\dot{\phi}(t)| \ll 2\pi\nu_0$. Since $|\dot{\phi}(t)| \ll 2\pi\nu_0$, the noise' s effect on frequency is far lower than the atomic clock' s reference frequency, making it more readily observable in phase and frequency comparison sequences after certain accumulation periods.

Generally, an atomic clock frequency standard is considered a highly stable frequency source, with output frequency remaining within a small interval around the reference frequency ν_0. Time measurement is achieved through phase comparison: when phase changes from $t_0 \cdot 2\pi\nu_0$ to $t_0 \cdot 2\pi\nu_0 + \Delta t \cdot 2\pi\nu_0 = (t_0 + \Delta t) \cdot 2\pi\nu_0$, time increases by Δt seconds. However, an actual frequency standard inevitably has a systematic offset from its nominal frequency, the initial phase is rarely zero, and frequency aging and drift phenomena may occur. Consequently, measured time deviations are affected by deterministic components including initial phase deviation x_0, initial frequency deviation y_0, and linear frequency drift D, as well as random components $\phi(t)$. The instantaneous time (phase) deviation $x(t)$ is defined as:

$$x(t) = D_{\text{system}} + \frac{\phi(t)}{2\pi\nu_0} = x_0 + y_{0t} + Dt^2 + \varphi_x(t)$$

where D_{system} represents phase deviation caused by deterministic processes (initial phase deviation x_0, initial frequency deviation y_0, and linear frequency drift D), and $\phi(t)$ represents phase deviation caused by random processes with power spectrum $S_x(f) \cdot 2\pi\nu_0$, where $S_x(f)$ denotes the power spectral density.

In natural environments, atomic clock oscillator output frequency cannot be constant: (1) the frequency standard output itself contains white noise; (2) environmental factors and frequency standard aging cause incremental processes containing deterministic components, which are also contaminated by white noise and introduce frequency random walk noise into the frequency standard output. Therefore, the original atomic clock random model only included noise superposition with power-law exponents from 2 to -2, where exponent 2 represents white noise naturally present in phase comparison, exponent 0 represents white noise in the frequency standard output itself, and exponent -2 represents the aforementioned frequency random walk noise. Exponents 1 and -1 correspond to flicker noise caused by non-ideal electronic components and circuit loops. Subsequent rubidium atomic clock studies added two more power-law

noise processes (-3, -4). The phase deviation $x(t)$ between atomic clocks is essentially the integral of frequency deviation, yielding:

$$S_y(f) = (2\pi f)^2 \cdot S_x(f) = \sum_{\alpha=-4}^{2} h_\alpha f^\alpha$$

where $S_x(f)$ is the PSD of $\varphi_x(t)$, $S_y(f)$ is the PSD of $\varphi_y(t)$ with $\varphi_y(t) = \dot{\varphi}_x(t)$; for frequency f we have $0 \leq f \leq f_h$, where f_h is the upper cutoff frequency; α is the power-law exponent; and h_α is the intensity coefficient for noise with exponent α. The values $\alpha = 2, 1, 0, \cdots, -4$ correspond to white phase modulation noise (WPM), flicker phase modulation noise (FPM), white frequency modulation noise (WFM), flicker frequency modulation noise (FFM), random walk frequency modulation noise (RWFM), flicker walk frequency modulation noise (FWFM), and random run frequency modulation noise (RRFM). The first five noises constitute the original atomic clock random model, while the latter two were added based on rubidium atomic clock research. Equation (3) is the well-known atomic clock noise power-law model.

Regarding the identification of different power-law noise types, the most direct approach is to estimate the noise power spectrum and analyze it in the frequency domain. However, this method is not easily implemented for two main reasons: (1) actual noise is often a superposition of multiple power-law noises; (2) accurate power spectrum estimation is itself challenging. Time-domain analysis methods are generally employed for noise identification, and here we introduce a method based on the time-domain autocorrelation function: the Lag 1 autocorrelation method.

The Lag k autocorrelation function (ACF) is defined as:

$$\rho_k = \frac{E[(z_t - \mu)(z_{t+k} - \mu)]}{\sigma_z^2}$$

where z_t is any time series (which could be discrete samples of instantaneous phase deviation $x(t)$ or instantaneous frequency deviation $y(t) = \dot{x}(t)$), μ is the mean, and σ_z^2 is the variance. This represents the autocorrelation of the sequence shifted by k positions. When $k = 1$, we obtain the Lag 1 autocorrelation function, with statistic r_k defined as its estimate:

$$r_k = \frac{\sum_{t=1}^{N-k}(z_t - \bar{z})(z_{t+k} - \bar{z})}{\sum_{t=1}^{N}(z_t - \bar{z})^2}$$

where $\bar{z}$ is the series mean and N is the series length.

Power-law noise with exponent 2 is essentially a random process obtained by differentiating white noise once. For uniformly sampled time series of this process,

adjacent random variables exhibit linear correlation with a Lag 1 autocorrelation value of $-1/2$, derivable from the Wiener-Khinchin theorem. Power-law noise with exponent 1 is similarly obtained through more complex differentiation (fractional differentiation) of white noise, with a Lag 1 autocorrelation value of $-1/3$. Power-law noise with exponent 0 is simply white noise, where adjacent random variables in a uniformly sampled time series are linearly independent, yielding a Lag 1 autocorrelation value of zero. Figure 1 [Figure 1: see original paper] shows autocorrelation plots for 20 sample functions of simulated random processes with power-law exponents 2, 1, and 0 (noise intensity coefficient of 1, sampling interval of 1 s), with simulated Lag 1 autocorrelation values consistent with theoretical analysis.

If z_t is a frequency deviation sequence, an estimated r_1 near $-1/2$ indicates WPM, near $-1/3$ indicates FPM, and near 0 indicates WFM. These three noise types are stationary for frequency deviation sequences and can be directly identified via Lag 1 autocorrelation. For non-stationary time series, differencing is first required to achieve stationarity before using Lag 1 autocorrelation for noise type identification. Let $\delta = r_1/(1 + r_1)$. Generally, if $\delta < 0.25$, the time series z_t can be considered stationary with Lag 1 autocorrelation $\rho_1 = \delta/(1-\delta)$. For a non-stationary time series z_t: first perform multiple first-order backward differences and calculate δ until $\delta < 0.25$, at which point the series becomes stationary. Note that each differencing operation yields a new series; for the new series, the original noise' s power-law exponent increases by 2. Record the number of differences d to recover the original noise exponent. The differenced series effectively transforms noise with exponents -1 to -4 into noise with exponents 2 to 0, enabling identification via Lag 1 ACF. Combined with the differencing count d, the original noise exponent can be determined. Let $p = -\text{round}(2\delta) - 2d$ (where round(*) denotes the nearest integer). The noise power-law exponent α can be determined from p: for frequency data, $\alpha = p$; for phase data, $\alpha = p + 2$.

Other common power-law noise identification methods include the B1 bias function method. For the Lag 1 autocorrelation method, reference [30] provides further extensions and presents a power-law noise determination algorithm based on the overlapping Hadamard variance.

3 Power-Law Noise Simulation and Stochastic Differential Equations

The power spectral density $S_x(\omega)$ of a white noise process is constant across the entire spectrum. Integrating and differentiating white noise processes can generate colored noise processes with power-law spectra. Colored noise with power-law exponents 2, 0, -2, and -4 can be directly obtained through differentiation or integration of white noise:

$$\begin{cases} \varphi_0 = \omega \\ \frac{d\varphi_{-2}}{dt} = \omega \\ \frac{d^2\varphi_{-4}}{dt^2} = \omega \end{cases}$$

where φ_i represents colored noise with exponent i, and ω is the white noise process. Equation (6) provides the stochastic differential equations for colored noise with exponents 2, 0, -2, and -4, enabling simulation of these noise types from white noise. In fact, the state equations for time scales based on Kalman filtering are built upon this foundation. Equation (7) presents the standard three-variable Kalman filter time scale differential equations describing the system state:

$$\begin{cases} \dot{x}_1 = x_2 + \omega_1 \\ \dot{x}_2 = x_3 + \omega_2 \\ \dot{x}_3 = \omega_3 \end{cases}$$

where x_1, x_2, and x_3 are the phase state, frequency state, and frequency drift state, respectively; ω_1, ω_2, and ω_3 are three independent white noise processes that generate FFM, RWFM, and RRFM.

However, simulating colored noise with exponents 1, -1, and -3 has long been challenging. The main simulation approaches are: (1) ARMA model-based simulation; (2) fractional calculus-based simulation; and (3) wavelet analysis-based simulation. All three methods require different finite processes to approximate colored noise with exponents 1, -1, and -3. Reference [27] investigates the mechanisms underlying such noise generation and identifies the physical causes. This paper only introduces, based on literature, the definition of colored noise with exponent -1 and a noise generation method using four Markov processes for approximation. Considering that noise with exponent -2 is Brownian motion, modified fractional Brownian motion can be defined for any exponent α:

$$B_f(t) = \frac{1}{\Gamma(\alpha/2)} \left[\int_{-\infty}^{0} (|t-\tau|^{\alpha/2-1} - |\tau|^{\alpha/2-1})\omega(\tau)d\tau + \int_{0}^{t} |t-\tau|^{\alpha/2-1}\omega(\tau)d\tau \right]$$

This defines power-law noise for arbitrary α. In fact, noise with exponent -1 is mathematically defined through fractional differentiation, i.e., using $\alpha = 1$ in formula (8). However, simulation of fractional Brownian motion cannot be achieved using formula (8) alone. This paper introduces a simulation method for exponent -1 noise based on superposition of multiple independent Markov processes.

A Markov frequency process can be represented through a Gaussian process via the differential equation:

$$\frac{d\varphi_m}{dt} + R\varphi_m = \omega$$

where φ_m is the Markov frequency process, R is a constant, and ω is white noise. Selecting different constants R to generate four independent Markov processes and superimposing them can approximately produce a colored noise process with exponent -1. Reference [34] also constructed a Kalman time scale including FFM noise, handling FFM noise by approximating it with four independent Markov processes, with R designed as a state parameter in the filter to fit and approximate the actual FFM present in atomic clocks.

Figure 2 [Figure 2: see original paper] shows time series plots of 20 simulated sample functions for four random processes: white noise (exponent 0), fractional Brownian motion (exponent -1), Brownian motion (exponent -2), and random run motion (exponent -4). This figure provides intuitive understanding of different power-law noise characteristics in the time domain.

Note: (a) white noise process; (b) fractional Brownian motion; (c) Brownian motion; (d) random run motion. Simulated noise intensity coefficient is 1, sample function sampling interval is 1 s.

4 Unified Form and Transfer Functions of Frequency Stability Assessment Methods

Common atomic clock frequency stability assessment methods include Allan variance and Hadamard variance. Overlapping Allan variance and overlapping Hadamard variance were developed to address low confidence when data is limited. Modified Allan variance and modified Hadamard variance were introduced because the original variances cannot distinguish between noise with exponents 2 and 1, while total Allan variance and total Hadamard variance were proposed to improve confidence at long smoothing times. These assessment methods are all based on Allan and Hadamard variance. This section discusses the definitions, relationships, and transfer functions of these two variances, and presents a unified form to describe them.

Allan variance was the earliest method proposed for frequency stability analysis. Its essence remains a variance, but since oscillator frequency sequences are affected by random walk noise and are non-stationary, differencing is required to achieve stationarity. Frequency stability actually evaluates the random error of an oscillator' s output frequency—a concept similar to but fundamentally different from variance analysis in measurement adjustment, due to different underlying stochastic models. The random model for frequency stability assessment is a non-stationary process representing the superposition of multiple non-stationary processes, requiring overall error analysis of this multi-noise superimposed non-stationary process. In contrast, error analysis in measurement adjustment deals with stationary white noise. Of course, connections exist: when we model WFM, RWFM, and RRFM and introduce state space, error

analysis in state space transforms into error analysis in measurement adjustment. This transformation essentially converts "multi-noise superimposed random error analysis based on smoothing time variation" into "error analysis of state variables in state space." However, independent analytical expressions for individual power-law noise types are difficult to obtain. Although differencing changes the physical meaning of data and the noise power spectrum transforms from $S(f)$ to $(2\pi f)^2 \cdot S(f)$, this does not affect the overall analysis of random errors in frequency standard output.

Allan variance was specifically designed for frequency stability analysis. For frequency data, since only WFM (one of the main noise sources) has variance that does not vary with time, Allan variance was designed so that its expectation matches that of WFM' s standard variance. Allan variance $\sigma_y^2(\tau)$ is defined as:

$$\sigma_y^2(\tau) \equiv \left\langle (\bar{y}_{k+1} - \bar{y}_k)^2 \right\rangle, \quad \bar{y}_k = \frac{1}{\tau} \int_{t_k}^{t_k+\tau} y(t)dt$$

where $y(t)$ is the instantaneous frequency deviation and τ is the smoothing time. Since the original definition is for continuous processes requiring integration over $t = [0, +\infty]$, $\langle\rangle$ represents averaging over infinite time, which can be understood as the mathematical expectation. If oscillator frequency is affected by random run noise, first-order differencing cannot achieve stationarity, naturally suggesting the use of higher-order differencing. In fact, Hadamard variance is constructed based on Hadamard transforms, and reference [4] modified it for better suitability in time-domain analysis. According to the definition in reference [4], variances constructed from second-order and higher differences of frequency data (or third-order and higher for phase data) are called Hadamard variance. However, for atomic clock frequency stability analysis, differences beyond second order for frequency data are unnecessary; the commonly used Hadamard variance refers specifically to variance based on second-order differences of frequency data. Hadamard variance is defined as:

$$\sigma_H^2(2, \tau) \equiv \left\langle (\bar{y}_{k+2} - 2\bar{y}_{k+1} + \bar{y}_k)^2 \right\rangle, \quad \bar{y}_k = \frac{1}{\tau} \int_{t_k}^{t_k+\tau} y(t)dt$$

Equation (11) can be extended to the N-th order difference Hadamard variance:

$$\sigma_H^2(N, \tau) \equiv \left\langle \frac{(\bar{y}_{k+N} - N \cdot \bar{y}_{k+N-1} + \cdots \pm \bar{y}_k)^2}{(N-1) \cdot N^2 + 2} \right\rangle, \quad \bar{y}_k = \frac{1}{\tau} \int_{t_k}^{t_k+\tau} y(t)dt$$

When $N = 1$, this becomes Allan variance; when $N > 2$, it is Hadamard variance. Note that definition (12) differs slightly from the original definition in reference [4]: the differencing scheme is somewhat different, and the original

definition did not include the denominator term. Considering that differencing amplifies noise and to ensure the variance' s expectation matches that of WFM' s standard variance, the denominator in (12) is necessary. The most commonly used Hadamard variance shown in (11) was improved by Riley and Howe, who also provided its transfer function. Additionally, the extension in (12) was derived by the author based on literature review, though unfortunately a unified transfer function form for $N > 2$ could not be provided.

From the definitions of Allan and Hadamard variance, both require processing through differencing and smoothing operators. Subsequent improvements to these variances essentially add more smoothing operations. Smoothing operators can be viewed as low-pass filters, while differencing operators act as high-pass filters; their combination forms a band-pass filter whose resonant frequency decreases as smoothing time τ increases. Since variance itself evaluates noise energy, increasing smoothing time τ causes different power-law noise types to sequentially become dominant, enabling successive analysis of WFM, FFM, RWFM, FWFM, and RRFM noise energies. Equations (13) and (14) provide the transfer functions for Allan and Hadamard variance, where $H(f)$ and $H_{Ha}(f)$ are the transfer functions for Allan and Hadamard variance, respectively, and τ is the smoothing time.

$$|H(f)|^2 = \frac{2\sin^4(\pi\tau f)}{(\pi\tau f)^2}$$

$$|H_{Ha}(f)|^2 = \frac{24\sin^6(\pi\tau f)}{(\pi\tau f)^2}$$

5 Variance Distribution and Determination of Equivalent Degrees of Freedom

Allan variance and Hadamard variance are primary tools for frequency stability assessment, but their essence remains variance estimation. Variance estimates generally follow a chi-square distribution, so error analysis of variance estimates should be based on the chi-square distribution rather than the normal distribution. This section introduces error bar calculation for variance estimates based on the chi-square distribution.

For an unbiased estimate V of variance σ^2 for a stationary process ($\sigma^2 = E[V]$), the equivalent degrees of freedom (EDF) is:

$$\nu = \nu_{\mathrm{EDF}} = \frac{2(E[V])^2}{\mathrm{Var}(V)}$$

The random variable $(\nu/\sigma^2)V$ approximately follows a chi-square distribution χ^2_ν, so a given significance level ρ can determine the confidence interval for

variance estimate σ^2 as $\nu V/x_2 \leq \sigma^2 \leq \nu V/x_1$ (with $x_1 < x_2$). Reference [14] provides calculation methods for $E[V]$ and $\mathrm{Var}(V)$:

$$E[V] = \sigma^2 = s_z(0)$$

$$\mathrm{Var}(V) = \frac{2}{m^2} \sum_{n_1,n_2=1}^{m} s_z^2[(n_2 - n_1)\tau]$$

where τ is the smoothing time. The autocorrelation function $s_z(t)$ is expressed through the generalized autocovariance (GACV) $s_\varphi(t)$:

$$s_z(t) = (\Delta_\tau \Delta_{-\tau})^d s_\varphi(t)$$

where Δ is the first-order backward difference operator, $d = 2$ for Allan variance, and $d = 3$ for Hadamard variance. For different power-law noises, $s_\varphi(t)$ differs. Table 1 provides the generalized autocorrelation functions for power-law exponents 2, 1, 0, -1, -2, -3, -4.

Table 1 Generalized autocorrelation functions $s_\varphi(t;\alpha)$ for different power-law noises

α	$s_\varphi(t;\alpha)$
2	$-2\lvert t\rvert$
1	$-\lvert t\rvert^3$
0	$-t^4 \ln \lvert t\rvert$
-1	$-\lvert t\rvert^5$
-2	$t^2 \ln \lvert t\rvert$
-3	$-3t^6 \ln \lvert t\rvert$
-4	$-4t^8 \ln \lvert t\rvert$

Using the method introduced in Section 2 to identify the dominant power-law noise type at smoothing time τ, and referring to Table 1 with equations (16) and (17), the equivalent degrees of freedom ν for variance estimation can be calculated. The variance distribution function is then:

$$\frac{\nu \cdot s^2}{\chi^2_\nu} = \sigma^2$$

where s^2 is the sample variance, σ^2 is the true variance, and ν is the equivalent degrees of freedom. Given significance level ρ, the two-sided confidence interval is:

$$s^2 \cdot \frac{\chi^2(\rho,\nu)}{\nu} < \sigma^2 < s^2 \cdot \frac{\chi^2(1-\rho,\nu)}{\nu}$$

This method is equally applicable to overlapping Allan variance and overlapping Hadamard variance. Based on the algorithms introduced in this section, error bars for Allan and Hadamard variance estimates can be calculated.

6 Conclusion

This paper provides a brief introduction to the mathematical principles underlying atomic clock frequency stability analysis methods, specifically: establishment of atomic clock models and identification of power-law spectral noise; simulation of power-law noise and its stochastic differential equations; unified description forms and transfer functions of common frequency stability assessment methods; and calculation of confidence intervals for these methods. The paper focuses on the origins and mathematical principles of frequency stability assessment algorithms. Through this review, readers can gain a systematic and relatively deep understanding of atomic clock stochastic models and frequency stability assessment algorithms: (1) what power-law noise is; (2) why the original atomic clock random model only involved five power-law noise types; (3) how to identify power-law noise; (4) how to generate power-law noise from white noise; (5) the intrinsic relationships among different frequency stability assessment methods; and (6) how to calculate confidence intervals for Allan and Hadamard variance estimates.

Notably, this paper also presents a unified form encompassing Allan variance, Hadamard variance, and higher-order difference variances. Based on the theories presented, a complete assessment of atomic clock frequency stability should include: (1) identifying which random noises primarily affect an atomic clock oscillator' s output frequency; and (2) determining the energy levels of these dominant noise types. For phase or frequency data, the complete assessment process should be: (1) calculate Allan variance, Hadamard variance, and modified Allan variance at different smoothing times, and compute confidence intervals; (2) calculate Lag 1 autocorrelation or B1 bias functions to identify dominant noise types at different smoothing times; (3) complete the analysis of dominant noise types and noise energy levels based on these results.

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Note: Figure translations are in progress. See original paper for figures.