Application of Explicit Symplectic-like Algorithm to Post-Newtonian Orbits of Exoplanets in Binary Systems: Postprint

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Abstract

The long-term dynamical evolution of exoplanets necessitates reliable numerical computational methods. Symplectic algorithms, characterized by their energy-preserving and structure-preserving properties, constitute the optimal integration tool for investigating the long-term evolution of Hamiltonian systems. In the post-Newtonian Hamiltonian system of doubly rotating exoplanets, where coordinates and momenta are inseparable, explicit symplectic algorithms cannot be directly applied. The construction of explicit symplectic-like algorithms via phase space extension avoids introducing artificial dissipation while retaining the advantage of energy preservation. This study primarily examines the numerical performance of the fourth-order midpoint-permutation phase-space-extended explicit symplectic-like algorithm in post-Newtonian orbits of doubly rotating exoplanets. The results demonstrate that for prograde non-nearly-coplanar orbits, the algorithmic accuracy of the midpoint-permutation phase-space-extended explicit symplectic-like algorithm differs by an order of magnitude from that of the RKF8(9) algorithm; for retrograde and near-circular orbits, its accuracy is comparable to that of RKF8(9); and for orbits with eccentricity less than 0.9, it exhibits superior accuracy relative to same-order comparison algorithms. Furthermore, the algorithm demonstrates favorable stability, with computational efficiency approximately three times that of RKF8(9).

Full Text

Preamble

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Application of Explicit Symplectic-like Algorithms in Post-Newtonian Orbits of Double-rotating Exoplanet Systems

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Abstract: Long-term dynamical evolution of exoplanets requires reliable numerical computational methods. Symplectic algorithms possess energy- and structure-preserving properties, making them the optimal integration tool for studying the long-term evolution of Hamiltonian systems. In post-Newtonian Hamiltonian systems of double-rotating exoplanets, coordinates and momenta are inseparable, preventing direct application of explicit symplectic integrators. Constructing explicit symplectic-like algorithms through phase space extension avoids introducing artificial dissipation while maintaining energy conservation advantages. This work primarily investigates the numerical performance of fourth-order explicit symplectic-like algorithms with midpoint permutations in extended phase space for post-Newtonian orbits of double-rotating exoplanet systems. Results demonstrate that for prograde non-near-coplanar orbits, the algorithm's accuracy is one order of magnitude lower than that of RKF8(9); for retrograde and near-circular orbits, its accuracy is comparable to RKF8(9); and for orbits with eccentricity less than 0.9, it exhibits higher accuracy than same-order comparison algorithms. Additionally, the algorithm demonstrates excellent stability with computational efficiency approximately three times that of RKF8(9).

Keywords: exoplanet; implicit midpoint method; extended phase space symplectic-like algorithm
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Introduction

The study of exoplanets achieved a breakthrough in 1995 when Mayor and Queloz [**?**] announced the discovery of the first exoplanet orbiting a Sun-like star (51 Peg b), representing the first observed hot Jupiter. Hot Jupiters are gas giants with masses in the range $0.3M_J \leq m(m \sin i) \leq 13M_J$ (where M_J denotes Jupiter mass and i represents the angle between the orbital plane normal and the observer's line of sight) and orbital periods $P < 10$ days. As of December 8, 2021, 4,576 confirmed exoplanets have been discovered, including 492 hot Jupiters. Observational data reveal that exoplanets exhibit distinctly different orbital characteristics compared to planets within our solar system. The discoveries of hot Jupiters, hot Earths, and high-eccentricity exoplanets all challenge traditional planet formation theories. Through the Rossiter-McLaughlin effect [?, ?, ?], some hot Jupiters have been found to have large angles between their orbital plane normals and their host stars' rotation axes [?, ?, ?, ?, ?, ?, ?, ?], with some even exhibiting retrograde motion [?, ?, ?, ?, ?, ?, ?].

Current mechanisms for explaining the formation of high-inclination hot Jupiters operate within the Newtonian mechanics framework. However, under the influence of inclined orbits from neighboring planets or companions, post-Newtonian spin-orbit coupling terms can induce precession of the host star's rotation. Higher-order post-Newtonian approximations primarily include post-Newtonian Lagrangian approximation equations, self-consistent equations, and post-Newtonian Hamiltonians in ADM coordinates [?, ?, ?, ?, ?, ?, ?, ?]. The physical equivalence between post-Newtonian Hamiltonians in ADM coordinates and post-Newtonian Lagrangian forms at the same order has been verified [?, ?, ?]. Nevertheless, these two post-Newtonian forms are not completely equivalent in orbital dynamical behavior [?, ?, ?, ?, ?]. When transforming one form into another through Legendre transformation, higher-order post-Newtonian terms are truncated. For weak gravitational fields like the solar system, this difference is negligible, but for strong gravitational fields such as compact objects, it may significantly impact the dynamics of both forms. Two types of equations of motion can solve conservative Lagrangian dynamical systems: truncated approximate equations of motion and non-truncated self-consistent equations of motion, which also exhibit differences [?, ?, ?, ?]. These differences may lead to different integrability and non-integrability, or order and chaos, between the two post-Newtonian approximations. Wu and Xie [?] proposed canonical conjugate spin variables that play a crucial role in determining the integrability or non-integrability of post-Newtonian Hamiltonian systems for spinning compact binaries, thereby demonstrating that post-Newtonian single-spin compact binary systems are integrable.

Both in Newtonian gravity and general relativity, long-term evolution of exoplanets with large eccentricities or inclinations (the angle between the planetary orbital plane normal and the host star's rotation axis) requires reliable numerical computational methods. Early methods such as Taylor series [?] and Runge-Kutta methods [?] proved unreliable for long-term orbital integration. Ruth [?] and Feng Kang [?] independently proposed symplectic algorithms in 1983 and 1984. Based on fundamental principles of Hamiltonian mechanics, these algorithms preserve the original symplectic structure of Hamiltonian systems in discretized difference equations, ensuring long-term stability. This breakthrough ushered numerical computational methods into a new era. Within the solar system, the Wisdom-Holman symplectic algorithm [?] is most commonly used, decomposing the Hamiltonian system into principal and perturbation parts for computation. Laskar and Robutel developed higher-order symplectic algorithms for creating long-term ephemerides [?]. Ruth's explicit symplectic algorithm [?] applies only to Hamiltonian systems with separable coordinates and momenta, offering high computational efficiency and stability.

For Hamiltonian systems with inseparable variables, explicit symplectic algorithms cannot be applied directly; instead, implicit symplectic algorithms or explicit-implicit hybrid symplectic algorithms [?, ?, ?, ?, ?, ?] may be used, or phase space extension methods can be employed to construct explicit algorithms [?, ?, ?, ?, ?, ?, ?, ?, ?]. Feng Kang's implicit symplectic algorithm based on the implicit midpoint method [?] applies to any Hamiltonian system but requires multiple iterations to achieve high precision, resulting in significantly lower computational efficiency than explicit symplectic algorithms and making it generally not the first choice.

The explicit extended phase space symplectic-like algorithm was first proposed by Pihajoki [?], enabling application of explicit symplectic algorithms to Hamiltonian systems with inseparable variables through momentum permutations. However, this approach breaks the symplectic structure of Hamiltonian systems and therefore does not qualify as a strict symplectic algorithm, hence the term "symplectic-like algorithm." Tao [?] improved upon this by dividing the Hamiltonian system into three parts, preserving the algorithm's symplectic structure. Subsequently, Liu et al. [?, ?] found better permutation methods based on Pihajoki's work, combining Yoshida's [?] high-order algorithm construction to propose high-order coordinate-momentum permutation extended phase space explicit symplectic-like algorithms. This algorithm achieves high precision and stability through two permutations (coordinate permutation and momentum permutation), but fails to maintain energy stability in some chaotic orbits of post-Newtonian compact binary systems [?, ?]. To address this issue, Luo et al. [?, ?] proposed the midpoint permutation extended phase space explicit symplectic-like algorithm, which achieves high precision and stability with only a single midpoint permutation. Wu et al. [?] proposed optimized algorithms based on extended phase space, showing improved precision compared to non-optimized versions. Li and Wu [?] combined Mikkola and Palmer's [?] approach with various permutation methods to propose the extended phase space logarithmic Hamiltonian explicit symmetric algorithm, which demonstrates good computational precision and efficiency for high-eccentricity and chaotic orbits. Pan et al. [?] constructed a self-consistent post-Newtonian Lagrangian equation-based extended phase space semi-implicit symplectic-like algorithm, also showing good energy conservation. Recently, Chinese scholars have decomposed black hole spacetimes into multiple parts with explicit analytical solutions, then combined them into explicit symplectic algorithms, solving the international challenge of constructing explicit symplectic algorithms for relativistic black hole spacetimes [?, ?, ?, ?, ?, ?, ?, ?].

This paper investigates the application of midpoint permutation extended phase space explicit symplectic-like algorithms in post-Newtonian orbits of double-rotating exoplanet systems. In the first-order post-Newtonian double-rotating two-body problem Hamiltonian system, variables are inseparable and spin variables are expressed in non-canonical forms, lacking global symplectic structure and thus limiting further application of symplectic algorithms. We employ Wu and Xie's method [?] to regularize spin variables by introducing quasi-cylindrical coordinates, endowing the system's phase space with complete symplectic structure. Numerical simulations compare fourth-order implicit midpoint method (IM4), fourth-order coordinate-momentum permutation extended phase space explicit symplectic-like algorithm (S4), fourth-order midpoint permutation extended phase space explicit symplectic-like algorithm (A4), and RKF8(9) algorithm in first-order post-Newtonian double-rotating two-body problems. We discuss computational precision and efficiency of the four algorithms across three types of exoplanet orbits (high-inclination, low-eccentricity, and high-eccentricity orbits), present energy error versus planetary orbital inclination and eccentricity for each algorithm, discuss relationships between energy errors and orbital parameters, and finally compare stellar radial velocities obtained through different numerical algorithms.

2.1 Physical Model

After simplifying the first-order post-Newtonian double-rotating two-body problem to the center-of-mass reference frame, the Hamiltonian can be written as [?, ?, ?, ?, ?, ?]:

$$H = H_N + \epsilon H_1 - \frac{GM\mu}{r}$$

where ϵ is a small parameter in perturbation theory, $\epsilon = 1/c^2$, and c represents the speed of light.

H_1 can be expanded as:

$$H_1 = H_{1PN} + H_{SO}$$

with the first-order post-Newtonian orbital Hamiltonian:

$$H_{1PN} = \frac{\mu}{2}\left[\frac{(3\nu - 1)}{G^2M^2} + \frac{(3+\nu)\mu^2}{r^2} + \frac{\nu(\mathbf{n}\cdot\mathbf{p})^2}{r}\right]$$

and the 1.5PN spin-orbit coupling term:

$$H_{SO} = \frac{\mathbf{J}_1 + \mathbf{J}_2}{r^3}\cdot(\mathbf{r}\times\mathbf{p})$$

In these formulas, G is the gravitational constant, m_1 and m_2 represent the masses of the two bodies, where m_1 denotes the secondary body's mass and m_2 denotes the primary body's mass. $M = m_1 + m_2$ is the total mass, $\mu = m_1m_2/M$, $\nu = \mu/M$, $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$, with $\mathbf{p}_1$ and $\mathbf{p}_2$ being the momenta of bodies 1 and 2 in the center-of-mass coordinate system. $\mathbf{r} = r\mathbf{n}$ is the vector from body 2 to body 1, and $\mathbf{n}$ represents the unit radial vector.

The spin variables $\mathbf{J}_i$ appear as non-canonical variables in equation (5), rendering the Hamiltonian system locally symplectic but lacking complete symplectic structure, which complicates numerical simulation with symplectic algorithms. Wu and Xie [**?**] proposed representing spin variables through a quasi-cylindrical coordinate system, transforming them into canonical coordinates (θ_i) and canonical momenta (ξ_i) using conserved spin quantities. This spin variable regularization reduces the original Hamiltonian system from 12-dimensional space to 10-dimensional phase space, endowing the entire Hamiltonian system with complete symplectic structure.

Expressing spin variables with $(\rho_i, \theta_i, \xi_i)$:

$$\mathbf{J}_i = J_i \begin{pmatrix} \rho_i \cos\theta_i \\ \rho_i \sin\theta_i \\ \xi_i \end{pmatrix}$$

where $\hat{\mathbf{J}}_i$ represents the unit spin vector, i.e., $|\hat{\mathbf{J}}_i| = 1$, expressed as:

$$\hat{\mathbf{J}}_i = \begin{pmatrix} \rho_i \cos\theta_i \\ \rho_i \sin\theta_i \\ \kappa_i \xi_i \end{pmatrix}$$

with $\kappa_i = 1/(\chi_i m_i^2)$, and:

$$\rho_i = \sqrt{1 - (\kappa_i \xi_i)^2}$$

Substituting equations (7)-(9) into equation (6) and performing transformations yields canonical conjugate spin variables:

$$\mathbf{J}_i = J_i \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \\ \kappa_i \xi_i \end{pmatrix} \quad (i = 1, 2)$$

where J_i is the magnitude from equation (6), i.e., $|\mathbf{J}_i|$.

Thus, the first-order post-Newtonian double-rotating two-body problem Hamiltonian with canonical spin variables in the center-of-mass coordinate system is expressed as:

$$H_{1PN} = \frac{\mu}{2}\left[\frac{(3\nu - 1)}{G^2 M^2} + \frac{(3+\nu)\mu^2}{r^2} + \frac{\nu(xp_x + yp_y + zp_z)^2}{r}\right]$$

$$H_{SO} = \frac{J_1 \cos\theta_1}{r^3}(yp_z - zp_y) + \frac{J_1 \sin\theta_1}{r^3}(zp_x - xp_z) + \frac{J_2 \cos\theta_2}{r^3}(xp_y - yp_x) + \frac{J_2 \sin\theta_2}{r^3}(zp_x - xp_z)$$

$$H = H(\mathbf{r}, \mathbf{p}, \mathbf{J}_1(\theta_1, \xi_1), \mathbf{J}_2(\theta_2, \xi_2))$$

Assuming all bodies are spheres rotating about their spin axes and following Barker [**?**, **?**, **?**, **?**] and Wex [**?**], spin variables $\mathbf{J}_i$ can be approximated as the rotational angular momentum of bodies rotating about fixed axes:

$$\mathbf{J}_i = I_i \omega_i$$

where ω_i is the rotational angular velocity of body i and I_i is its moment of inertia. The moments of inertia for stars and planets depend on their respective masses and radii. The stellar moment of inertia is expressed as [**?**]:

$$I \approx m\beta^2 R^2$$

with β ranging between 0.175 and 0.385. The planetary moment of inertia is expressed as [**?**]:

$$I \approx \alpha m R^2$$

where $\alpha = 0.2$ for gas giants and $\alpha = 0.25$ for solid planets.

2.2 Algorithm Description and Construction

Coordinates and momenta in equations (11)-(14) are inseparable, preventing direct application of explicit symplectic algorithms. Both implicit symplectic algorithms and phase space extension symplectic-like algorithms offer suitable alternatives for Hamiltonian systems with inseparable variables. The primary algorithms discussed below are the implicit midpoint method, coordinate-momentum permutation phase space extension symplectic-like algorithm, and midpoint permutation phase space extension symplectic-like algorithm, with corresponding constructions for the test model.

2.2.1 Implicit Midpoint Method

The implicit midpoint method's utility in Hamiltonian systems with inseparable variables has been demonstrated in numerous examples. Solvable directly using Newton's iteration method or the Seidel iteration method, the difference discretization scheme from step n to step $(n+1)$ is written as:

$$\text{IM2}: \begin{cases} \mathbf{q}_{n+1} = \mathbf{q}_n + h\frac{\partial H}{\partial \mathbf{p}}\Big|_{\frac{\mathbf{q}_{n+1}+\mathbf{q}_n}{2}, \frac{\mathbf{p}_{n+1}+\mathbf{p}_n}{2}, \frac{\theta_{1,n+1}+\theta_{1,n}}{2}, \frac{\theta_{2,n+1}+\theta_{2,n}}{2}, \frac{\xi_{1,n+1}+\xi_{1,n}}{2}, \frac{\xi_{2,n+1}+\xi_{2,n}}{2}} \\ \theta_{1,n+1} = \theta_{1,n} + h\frac{\partial H}{\partial \xi_1}\Big|_{\frac{\mathbf{q}_{n+1}+\mathbf{q}_n}{2}, \frac{\mathbf{p}_{n+1}+\mathbf{p}_n}{2}, \frac{\theta_{1,n+1}+\theta_{1,n}}{2}, \frac{\theta_{2,n+1}+\theta_{2,n}}{2}, \frac{\xi_{1,n+1}+\xi_{1,n}}{2}, \frac{\xi_{2,n+1}+\xi_{2,n}}{2}} \\ \theta_{2,n+1} = \theta_{2,n} + h\frac{\partial H}{\partial \xi_2}\Big|_{\frac{\mathbf{q}_{n+1}+\mathbf{q}_n}{2}, \frac{\mathbf{p}_{n+1}+\mathbf{p}_n}{2}, \frac{\theta_{1,n+1}+\theta_{1,n}}{2}, \frac{\theta_{2,n+1}+\theta_{2,n}}{2}, \frac{\xi_{1,n+1}+\xi_{1,n}}{2}, \frac{\xi_{2,n+1}+\xi_{2,n}}{2}} \\ \mathbf{p}_{n+1} = \mathbf{p}_n - h\frac{\partial H}{\partial \mathbf{q}}\Big|_{\frac{\mathbf{q}_{n+1}+\mathbf{q}_n}{2}, \frac{\mathbf{p}_{n+1}+\mathbf{p}_n}{2}, \frac{\theta_{1,n+1}+\theta_{1,n}}{2}, \frac{\theta_{2,n+1}+\theta_{2,n}}{2}, \frac{\xi_{1,n+1}+\xi_{1,n}}{2}, \frac{\xi_{2,n+1}+\xi_{2,n}}{2}} \\ \xi_{1,n+1} = \xi_{1,n} - h\frac{\partial H}{\partial \theta_1}\Big|_{\frac{\mathbf{q}_{n+1}+\mathbf{q}_n}{2}, \frac{\mathbf{p}_{n+1}+\mathbf{p}_n}{2}, \frac{\theta_{1,n+1}+\theta_{1,n}}{2}, \frac{\theta_{2,n+1}+\theta_{2,n}}{2}, \frac{\xi_{1,n+1}+\xi_{1,n}}{2}, \frac{\xi_{2,n+1}+\xi_{2,n}}{2}} \\ \xi_{2,n+1} = \xi_{2,n} - h\frac{\partial H}{\partial \theta_2}\Big|_{\frac{\mathbf{q}_{n+1}+\mathbf{q}_n}{2}, \frac{\mathbf{p}_{n+1}+\mathbf{p}_n}{2}, \frac{\theta_{1,n+1}+\theta_{1,n}}{2}, \frac{\theta_{2,n+1}+\theta_{2,n}}{2}, \frac{\xi_{1,n+1}+\xi_{1,n}}{2}, \frac{\xi_{2,n+1}+\xi_{2,n}}{2}} \end{cases}$$

Yoshida [?] proposed a construction method for higher-order algorithms, primarily building a $(2n+2)$-order algorithm through symmetric composition of three $2n$-order algorithms with appropriate step-size coefficients:

$$S_{2n+2}(h) = S_{2n}(\lambda_1 h)S_{2n}(\lambda_2 h)S_{2n}(\lambda_1 h)$$

with coefficients:

$$\lambda_1 = \frac{1}{2 - 2^{1/(2n+1)}}, \quad \lambda_2 = -\frac{2^{1/(2n+1)}}{2 - 2^{1/(2n+1)}}$$

Using Yoshida's method [?], the fourth-order implicit midpoint method IM4 can be expressed as:

$$\text{IM4}(h) = \text{IM2}(\lambda_1 h)\,\text{IM2}(\lambda_2 h)\,\text{IM2}(\lambda_1 h)$$

with coefficients:

$$\lambda_1 = \frac{1}{2 - 2^{1/3}}, \quad \lambda_2 = -\frac{2^{1/3}}{2 - 2^{1/3}}$$

2.2.2 Phase Space Extension Symplectic-like Algorithm

Pihajoki [?] proposed the phase space extension symplectic-like algorithm in 2015. By extending the original Hamiltonian through phase space expansion, we duplicate the original variables to obtain $(\mathbf{e}_q, e_{\theta_1}, e_{\theta_2}, \mathbf{e}_p, e_{\xi_1}, e_{\xi_2})$, doubling the phase space variables such that $(\mathbf{q}, \mathbf{p}, \theta_1, \theta_2, \xi_1, \xi_2) \rightarrow$

$(\mathbf{q}, \mathbf{e}_q, \mathbf{p}, \mathbf{e}_p, \theta_1, e_{\theta_1}, \theta_2, e_{\theta_2}, \xi_1, e_{\xi_1}, \xi_2, e_{\xi_2})$. This yields two completely identical Hamiltonians $H_1(\mathbf{q}, \theta_1, \theta_2, \mathbf{e}_p, e_{\xi_1}, e_{\xi_2})$ and $H_2(\mathbf{e}_q, e_{\theta_1}, e_{\theta_2}, \mathbf{p}, \xi_1, \xi_2)$, identical to the original Hamiltonian H, which compose a new Hamiltonian:

$$\tilde{H}(\mathbf{q}, \mathbf{e}_q, \theta_i, e_{\theta_i}, \mathbf{p}, \mathbf{e}_p, \xi_i, e_{\xi_i}) = H_1(\mathbf{q}, \theta_i, \mathbf{e}_p, e_{\xi_i}) + H_2(\mathbf{e}_q, e_{\theta_i}, \mathbf{p}, \xi_i), \quad (i = 1, 2)$$

Hamiltonians H_1 and H_2 are coordinate-momentum separable Hamiltonian systems, independent and integrable, allowing separate analytical solutions. The canonical equations for H_1 are:

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H_1}{\partial \mathbf{p}} = 0, \quad \frac{d\theta_1}{dt} = \frac{\partial H_1}{\partial \xi_1} = 0, \quad \frac{d\theta_2}{dt} = \frac{\partial H_1}{\partial \xi_2} = 0$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H_1}{\partial \mathbf{q}}, \quad \frac{d\xi_1}{dt} = -\frac{\partial H_1}{\partial \theta_1}, \quad \frac{d\xi_2}{dt} = -\frac{\partial H_1}{\partial \theta_2}$$

The canonical equations for H_2 are:

$$\frac{d\mathbf{e}_q}{dt} = \frac{\partial H_2}{\partial \mathbf{e}_p}, \quad \frac{de_{\theta_1}}{dt} = \frac{\partial H_2}{\partial e_{\xi_1}}, \quad \frac{de_{\theta_2}}{dt} = \frac{\partial H_2}{\partial e_{\xi_2}}$$

$$\frac{d\mathbf{e}_p}{dt} = -\frac{\partial H_2}{\partial \mathbf{e}_q} = 0, \quad \frac{de_{\xi_1}}{dt} = -\frac{\partial H_2}{\partial e_{\theta_1}} = 0, \quad \frac{de_{\xi_2}}{dt} = -\frac{\partial H_2}{\partial e_{\theta_2}} = 0$$

Second-order symplectic algorithms can be applied to Hamiltonian $\tilde{H}$, where h is the step size:

$$S_2(h) = H_2\left(\frac{h}{2}\right) H_1(h) H_2\left(\frac{h}{2}\right)$$

At the initial moment, original coordinates and momenta are exactly equal to extended coordinates and momenta. However, due to coupling effects between original and extended variables during integration, they gradually become unequal over time. Pihajoki [**?**] corrected this by introducing permutation variables M_i:

$$\tilde{S}_2(h) = M_2 H_2\left(\frac{h}{2}\right) M_1 H_1(h) M_1 H_2\left(\frac{h}{2}\right) M_2$$

where M_1 and M_2 represent momentum permutations $\mathbf{p} \leftrightarrow \tilde{\mathbf{p}}$.

2.2.3 Continuous Coordinate-Momentum Permutation Phase Space Extension Symplectic-like Algorithm

Building upon Pihajoki's work [?], Liu et al. [?, ?] found better permutation methods and proposed the fourth-order continuous coordinate-momentum permutation extended phase space explicit symplectic-like algorithm:

$$S_4 = S_2(\lambda_1 h)S_2(\lambda_2 h)S_2(\lambda_3 h)M_1 \times S_2(\lambda_3 h)S_2(\lambda_2 h)S_2(\lambda_1 h)M_2$$

where M_1 denotes momentum permutation $\mathbf{p} \leftrightarrow \mathbf{e}_p, \xi_1 \leftrightarrow e_{\xi_1}, \xi_2 \leftrightarrow e_{\xi_2}$, M_2 denotes coordinate permutation $\mathbf{q} \leftrightarrow \mathbf{e}_q, \theta_1 \leftrightarrow e_{\theta_1}, \theta_2 \leftrightarrow e_{\theta_2}$, and coefficients are $\lambda_1 = \lambda_2 = 1/[2(2-2^{1/3})], \lambda_3 = 1 - 2\lambda_1$.

The continuous coordinate-momentum permutation extended phase space explicit symplectic-like algorithm requires two triple integrations to achieve high precision. While it has been successfully applied to inseparable Hamiltonian problems such as post-Newtonian Hamiltonian problems and rotational Hamiltonian problems, it fails to maintain energy stability in some chaotic orbits of rotating compact binary post-Newtonian Hamiltonian systems [?, ?].

2.2.4 Midpoint Permutation Phase Space Extension Symplectic-like Algorithm

Luo et al. [?, ?] modified Liu et al.'s approach [?, ?] and proposed the midpoint permutation extended phase space explicit symplectic-like algorithm. This algorithm requires only a single midpoint permutation rather than two permutations to achieve excellent precision and stability. During integration, the midpoint permutation maintains equality between original and extended coordinates and momenta throughout. The fourth-order midpoint permutation extended phase space symplectic-like algorithm A_4 is expressed as:

$$A_4 = M \otimes S_2(\lambda_3 h)S_2(\lambda_2 h)S_2(\lambda_1 h)$$

with coefficients $\lambda_1 = \lambda_2 = 1/(2-2^{1/3}), \lambda_3 = 1-2\lambda_1$. M represents the midpoint permutation:

$$\begin{cases} \mathbf{q} \to \mathbf{q} + \mathbf{e}_q, & \mathbf{e}_q \to \mathbf{q} + \mathbf{e}_q \\ \theta_1 \to \theta_1 + e_{\theta_1}, & e_{\theta_1} \to \theta_1 + e_{\theta_1} \\ \theta_2 \to \theta_2 + e_{\theta_2}, & e_{\theta_2} \to \theta_2 + e_{\theta_2} \\ \mathbf{p} \to \mathbf{p} + \mathbf{e}_p, & \mathbf{e}_p \to \mathbf{p} + \mathbf{e}_p \\ \xi_1 \to \xi_1 + e_{\xi_1}, & e_{\xi_1} \to \xi_1 + e_{\xi_1} \\ \xi_2 \to \xi_2 + e_{\xi_2}, & e_{\xi_2} \to \xi_2 + e_{\xi_2} \end{cases}$$

3 Application in First-order Post-Newtonian Double-rotating Two-body Problem

This section discusses numerical performance of various algorithms in the first-order post-Newtonian double-rotating two-body problem. We select three types of exoplanet orbits (high-inclination, low-eccentricity, and high-eccentricity orbits) for numerical simulation. Fourth-order implicit midpoint method (IM4), fourth-order coordinate-momentum permutation extended phase space explicit symplectic-like algorithm (S4), fourth-order midpoint permutation extended phase space explicit symplectic-like algorithm (A4), and RKF8(9) algorithm are employed for numerical simulation across the three orbit types. RKF8(9) serves as the reference solution for comparative analysis. We scan and present relationships between energy errors and planetary orbital inclination/eccentricity for all four algorithms, discuss these relationships, and finally compare stellar radial velocities obtained from different algorithms.

3.1 Initial Coordinates and Momenta

Initial coordinates and momenta can be obtained from given initial orbital elements. In the two-body problem, there are five invariant orbital elements: semi-major axis a, eccentricity e, orbital inclination i, longitude of ascending node Ω, and argument of periapsis ω. The remaining orbital element, mean anomaly M, varies linearly with time [?]. Mean anomaly M and eccentric anomaly E satisfy Kepler's equation:

$$M = E - e\sin E$$

The Keplerian solution for the two-body problem is:

$$\mathbf{q} = a(\cos E - e)\mathbf{P} + a\sqrt{1-e^2}\sin E\,\mathbf{Q}$$

$$\mathbf{p} = -\frac{\mu a n}{r}\sin E\,\mathbf{P} + \frac{\mu a n\sqrt{1-e^2}}{r}\cos E\,\mathbf{Q}$$

where $n = \sqrt{GM/a^3}$ is the mean angular rate and r is the distance between the two bodies:

$$r = a(1 - e\cos E)$$

The Laplace vector $\mathbf{P}$ is:

$$\mathbf{P} = \begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i \\ \sin\omega\sin i \end{pmatrix}$$

The Laplace vector $\mathbf{Q}$ is:

$$\mathbf{Q} = \begin{pmatrix} -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i \\ -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i \\ \cos\omega\sin i \end{pmatrix}$$

3.2.1 High-inclination Orbit

Orbit 1 is a high-inclination orbit with parameters listed in Table 1 , where M_J denotes Jupiter mass, $M_\odot$ solar mass, $R_\oplus$ Earth radius, and $R_\odot$ solar radius. Using XO-3 b [?] as an example, XO-3 b is a high-inclination hot Jupiter with a projected angle of $37.3^\circ \pm 3.7^\circ$ between planetary orbital angular momentum and host star rotation axis.

Initial orbital elements are set as $a = 0.04539$ AU, $e = 0.05$, $i = 37^\circ$, $\Omega = 0^\circ$, $\omega = 0^\circ$, $M = 0^\circ$, $\theta_1 = 90^\circ$, $\theta_2 = 95^\circ$, with initial orbital period $T = 0.0081$ yr. The computational step size h is set to 1/600 of the initial orbital period, and integration time $t = 10^7 T$. Figure 1 [Figure 1: see original paper] presents energy error comparisons for IM4, A4, S4, and RKF8(9) through absolute energy error $\Delta H = H(t) - H(0)$, where $H(t)$ and $H(0)$ represent Hamiltonian values at times t and 0, respectively. Results show algorithm accuracy from lowest to highest as IM4, S4, A4, RKF8(9). Extended phase space symplectic-like algorithms A4 and S4 exhibit long-term stability, with A4 approximately one order of magnitude more accurate than S4. S4 and IM4 show similar precision during early integration, but IM4's energy error curve begins to shift upward after $t = 10^6 T$ due to excessive iteration count and increased round-off error. RKF8(9) achieves the highest precision but shows unstable energy error with long-term growth, gradually approaching A4's performance; at $t = 10^7 T$, its energy error is approximately one order of magnitude higher than A4's.

3.2.2 Low-eccentricity Orbit

Orbit 2 is a low-eccentricity orbit with parameters listed in Table 2 . Using 51 Peg b [?] as an example, 51 Peg b is a hot Jupiter with eccentricity only 0.0042.

Initial orbital elements are set as $a = 0.05235$ AU, $e = 0.0042$, $i = 0^\circ$, $\Omega = 0^\circ$, $\omega = 0^\circ$, $M = 0^\circ$, $\theta_1 = \theta_2 = 0^\circ$, with initial orbital period $T = 0.0169$ yr. Step size h is 1/400 of the initial orbital period, and integration time $t = 10^7 T$. Figure 2 [Figure 2: see original paper] shows energy error comparisons through $\Delta H = H(t) - H(0)$. Algorithm accuracy ranks from lowest to highest as S4, IM4, A4, RKF8(9). Extended phase space algorithms A4 and S4 maintain stable energy errors without offset, with A4 approximately two orders of magnitude more accurate than S4. IM4's energy error curve begins shifting upward after $t = 10^4 T$. RKF8(9) achieves highest precision but exhibits poor stability with linear error growth and significant offset, ultimately matching A4's energy error within the integration timeframe.

3.2.3 High-eccentricity Orbit

Orbit 3 is a high-eccentricity orbit with parameters listed in Table 3 . Using Kepler-1656 b [?] as an example, Kepler-1656 b is a high-eccentricity sub-Saturn with eccentricity 0.84 ± 0.1.

Initial orbital elements are $a = 0.197$ AU, $e = 0.5$, $i = 0°$, $\Omega = 0°$, $\omega = 0°$, $M = 0°$, $\theta_1 = \theta_2 = 0°$, with initial orbital period $T = 0.0868$ yr. Step size h is 1/2700 of the orbital period, and integration time $t = 10^6 T$. Figure 3 [Figure 3: see original paper] presents energy error comparisons. IM4, A4, and S4 show lower precision than RKF8(9), with similar accuracy among the three. A4 demonstrates slightly higher precision than S4 and IM4 along with better stability. S4 exhibits large fluctuations in energy error without long-term stability. IM4's error curve shifts upward during later integration stages.

In summary, high-order Runge-Kutta RKF8(9) demonstrates the best numerical precision across all three orbit types but suffers from poor stability. S4 shows relatively poor precision overall, though it exhibits good stability for low-eccentricity and high-inclination orbits while producing large fluctuations for high-eccentricity orbits. IM4 demonstrates relatively poor precision, with error offsets appearing in later stages for low-eccentricity and high-inclination orbits due to excessive iterations and increased round-off error. A4 shows the best precision and stability across all three orbit types.

Table 4 lists CPU times required for the four algorithms to solve the three orbit types. A4's computational efficiency significantly exceeds other algorithms. S4 requires approximately twice A4's computation time due to its double permutation requirement. IM4 and RKF8(9) require substantially more time, with RKF8(9) needing about three times A4's duration. Overall, A4 performs best in terms of precision, stability, and computational efficiency.

3.2.4 Relationship Between Planetary Orbital Inclination/Eccentricity and Algorithm Energy Error

To clarify the relationship between planetary orbital parameters and algorithm energy errors, we scanned and obtained relationship diagrams between energy errors and orbital inclination/eccentricity for A4, S4, IM4, and RKF8(9).

For the energy error versus inclination relationship, initial orbital elements are set as $a = 0.04539$ AU, $e = 0.05$, $\Omega = 0°$, $\omega = 0°$, $M = 0°$, $\theta_1 = 90°$, $\theta_2 = 95°$, with initial period $T = 0.0081$ yr, step size $h = T/600$, and integration time $t = 10^4 T$. Inclination i varies from 0° to 180° in increments of 8°. Figure 4 [Figure 4: see original paper] shows the relationship between energy errors and planetary orbital inclination through mean energy error E. RKF8(9) accuracy remains nearly unaffected by inclination, maintaining the highest precision. For the other three algorithms, accuracy improves by approximately two orders of magnitude when inclination ranges from 0° to 28°. Subsequently, for prograde non-near-coplanar orbits, algorithm precision stabilizes with increasing initial inclination. The midpoint permutation extended phase space symplectic-like algorithm A4 shows precision one order of magnitude lower than RKF8(9) for prograde non-near-coplanar orbits, while achieving comparable precision to RKF8(9) for retrograde orbits.

For the energy error versus eccentricity relationship, initial elements are $a = 0.04539$ AU, $i = 0^\circ$, $\Omega = 0^\circ$, $\omega = 0^\circ$, $M = 0^\circ$, $\theta_1 = 90^\circ$, $\theta_2 = 95^\circ$, with $T = 0.0081$ yr, $h = T/600$, and $t = 10^4 T$. Eccentricity e varies from 0 to 1 in increments of 0.1. Figure 5 [Figure 5: see original paper] illustrates the relationship between energy errors and planetary eccentricity. RKF8(9) maintains the highest overall precision. Among the other three algorithms, A4 shows the best precision for $e < 0.9$. At $e = 0.9$, A4's precision degrades, ranking S4, IM4, A4 from highest to lowest precision. For $0 \leq e \leq 0.1$, IM4 outperforms S4, but as eccentricity increases, S4 and IM4 achieve similar precision. For high-eccentricity orbits, S4, IM4, and A4 show significantly lower precision than RKF8(9) because these fixed-step algorithms introduce substantial error when derivatives change rapidly, whereas RKF8(9) adapts its step size to reduce error accumulation.

3.2.5 Stellar Radial Velocity from Each Algorithm

We simulated stellar radial velocities using A4, S4, IM4, and RKF8(9) across the three orbit types (high-inclination, low-eccentricity, and high-eccentricity). Figure 6 [Figure 6: see original paper] compares results over 10 orbital periods, with the vertical axis showing stellar radial velocity along the x-axis. All algorithms produce overlapping radial velocity curves for the three orbit types.

For longer integrations, using RKF8(9) results as the reference solution, we examined relative errors in maximum x-axis radial velocity during the first period after extended integration ($10^6 T$ for high-inclination, $10^7 T$ for low-eccentricity, $10^5 T$ for high-eccentricity orbits). Results appear in Table 5 . All three algorithms produce extremely small relative errors in maximum radial velocity without significant differences.

4 Summary and Outlook

This work investigates explicit symplectic-like algorithms in post-Newtonian orbits of double-rotating exoplanet systems. We examined algorithm precision and computational efficiency of fourth-order implicit midpoint method, fourth-order coordinate-momentum permutation extended phase space symplectic-like algorithm, fourth-order midpoint permutation extended phase space symplectic-like algorithm, and RKF8(9) in first-order post-Newtonian double-rotating two-body problems, and presented relationships between orbital inclination/eccentricity and energy errors.

Numerical results show that across the three orbit types, the fourth-order implicit midpoint method incurs high computational costs due to multiple iterations. It outperforms the fourth-order coordinate-momentum permutation algorithm only in low-eccentricity orbits, showing relatively low precision for high-eccentricity and high-inclination orbits, with energy error offsets appearing in later integration stages due to excessive round-off errors. The fourth-order coordinate-momentum permutation algorithm demonstrates moderate precision and stability across all three orbit types, but produces large energy error fluctuations in high-eccentricity orbits and incurs increased computational costs from double permutations. The fourth-order midpoint permutation algorithm exhibits excellent precision and stability overall with the highest computational efficiency. Additionally, it performs exceptionally well in retrograde orbits, achieving precision comparable to RKF8(9), and shows higher precision than same-order comparison algorithms for orbits with eccentricity below 0.9.

The fourth-order midpoint permutation extended phase space symplectic-like algorithm demonstrates the best applicability for non-near-coplanar, retrograde, and near-circular orbits. Future work will employ this algorithm to investigate the formation of high-inclination hot Jupiters. With diverse orbital configurations in exoplanet systems, the algorithm can be applied to solve long-term integration problems for more exoplanet orbits, providing clearer understanding of the dynamical formation and evolution of exoplanet systems.

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Note: Figure translations are in progress. See original paper for figures.