Generalization of Stäckel Gravitational Potential Theory and Its Applications: Postprint

Authors: Gu Jianyu1,2, Dong Xiaobo1,2

Date: 2023-06-07T00:00:00+00:00

Abstract

Stäckel gravitational potentials constitute a class of the most general form of separable potentials. Galaxies with Stäckel form are completely integrable systems, wherein stellar orbits are all regular and their integrals of motion can be obtained analytically. Integrals of motion—particularly actions (J, a special type of integral of motion)—simplify the description of stellar motion; an important avenue for studying the kinematics and dynamics of stars in galaxies is action space, such as utilizing distribution functions f(JR, Lz, Jz) parameterized by actions. By locally approximating the gravitational potential of general galaxies as a Stäckel potential, one can estimate the actions of stellar motion in general galaxies. This paper introduces the significant progress astronomers have made in advancing research on Stäckel gravitational potentials: in the estimation of actions (or integrals of motion) in general galaxies based on Stäckel potential theory, several rapid numerical algorithms have been developed (e.g., Stäckel Fudge, Stäckel fitting method), high-precision numerical algorithms based on convergent torus mapping (e.g., orbit integration fitting generating function method, iterative torus construction method), and analytical expressions proposed as approximate integrals of motion; in the construction of distribution functions, based on the aforementioned estimated actions or directly employing the integral of motion formulas of Stäckel potentials (primarily the I3 expression), several distribution function models have been proposed, f(JR, Lz, Jz) or f(E, Lz, I3). These advances enable galaxy modeling based on distribution functions. Additionally, several representative works applying these methods to observational data of the Milky Way in recent years are introduced.

Full Text

Extensions of Stäckel Gravitational Potential Theory and Their Applications

Gu Jian-yu[1,2], **Dong Xiao-bo**[1,2]
[1]Yunnan Observatories, Chinese Academy of Sciences, Kunming 650216, China
[2]University of Chinese Academy of Sciences, Beijing 100049, China

Abstract

Stäckel gravitational potentials represent the most general class of separable potentials. Galaxies with Stäckel-type potentials are completely integrable systems, meaning that all stellar orbits are regular and their integrals of motion can be obtained analytically. Integrals of motion—particularly actions (J), a special type of integral—simplify the description of stellar motion. A crucial approach for studying stellar kinematics and dynamics in galaxies is through action space, for instance using distribution functions parameterized by actions, f(J_R, L_z, J_z). By locally approximating the gravitational potentials of general galaxies as Stäckel potentials, one can estimate actions for stellar motions in these systems. This paper reviews major advances in extending Stäckel potential theory: in estimating actions (or integrals of motion) for general galaxies, researchers have developed several fast numerical algorithms (e.g., Stäckel Fudge, Stäckel Fitting), high-precision convergent algorithms based on torus mapping (e.g., orbit integration fitting of generating functions, iterative torus construction), and analytical expressions for approximate integrals of motion; in constructing distribution functions, based on the estimated actions or directly using Stäckel potential integral formulas (primarily the I_3 expression), several distribution function models have been proposed, either as f(J_R, L_z, J_z) or f(E, L_z, I_3). These advances enable distribution function-based modeling of galaxies. Additionally, we introduce several representative works from recent years that apply these methods to observational data of the Milky Way.

Keywords: Stäckel gravitational potentials; action-angle variable estimation; Milky Way distribution function modeling

---

1 Introduction and Theoretical Foundations

One of the primary objectives of Milky Way research is to measure and determine the Galaxy's structure—the spatial distribution of various components (such as the stellar bulge, stellar disk, dark matter halo, and gas) and their velocity-space information. Achieving this goal requires, observationally, high-precision astrometric and photometric data (e.g., Gaia) and spectroscopic data (e.g., LAMOST), and, theoretically and computationally, galactic dynamical modeling as the core methodology. The theoretical foundation of galactic dynamical modeling is the dynamics of stellar systems—the study of numerous point-mass particles moving in their collective gravitational field. Although galaxies are self-gravitating systems that cannot exist in strict thermodynamic or perfect dynamical equilibrium, equilibrium dynamical models (also called steady-state or quasi-stationary states) are essential for interpreting observations of the Milky Way and external galaxies. In reality, the dark matter halo, as the primary contributor to the galactic potential, only deviates significantly from dynamical equilibrium during brief and rare major merger events; otherwise, it remains near equilibrium with only minor perturbations from accretion or minor mergers. From a modeling perspective, inferring mass distribution from dynamical measurements requires the assumption that the galaxy is in (quasi-)equilibrium. Moreover, equilibrium models represent the simplest case, with more complex situations being modeled as perturbations, deformations, or superpositions of equilibrium models.

Integrals of motion are invariants of particle orbital motion; actions are a special class of integrals that can form canonical coordinates with angle variables. Due to the simplicity of the equations of motion in action-angle coordinates, actions become attractive variables. Actions are also essential for both perturbation methods and distribution function modeling. However, for particles moving in general gravitational potentials (such as real galaxies), analytic integrals of motion or actions generally cannot be obtained—only for separable potentials can they be derived analytically. The Stäckel potential is a separable gravitational potential form in ellipsoidal coordinates, where integrals of motion can be calculated analytically. In practice, by approximating or fitting general gravitational potentials as Stäckel potentials, one can estimate integrals of motion in general potentials. Over the past decade, researchers have extended Stäckel potential theory—using analytic formulas from Stäckel potential theory through analytical or numerical methods (e.g., Binney's work) to estimate actions (or integrals of motion) in general galaxies. Particularly in recent years, Binney and collaborators have implemented their methods as computer programs released as open-source code.

The Stäckel potential, as a simple gravitational potential form, was introduced in the 19th century. In theoretical galactic dynamics, the most general form of triaxial Stäckel potentials was comprehensively summarized over 30 years ago, including analytic expressions for three isolating integrals (or actions), stellar orbit classification, etc. However, only in recent years has knowledge of Stäckel potentials been applied to practical Milky Way modeling, largely due to the aforementioned extension work—methods and programs for estimating actions (or integrals of motion) in general potentials.

The remaining sections of Chapter 1 introduce: an overview of stellar system dynamics and statistical mechanics, integrals of motion, and fundamental background on Stäckel potentials. Chapter 2 presents methods for estimating integrals of motion and actions in general gravitational potentials: Section 2.1 introduces the Stäckel Fudge method; Section 2.2 introduces Stäckel potential fitting and earlier work; Section 2.3 describes action estimation methods based on torus coordinate transformations, obtaining actions in general potentials from Stäckel toy potentials; Section 2.4 presents our comparison of various methods using the TACT program; and Section 2.5 introduces other methods and programs for obtaining integrals of motion. Chapter 3 discusses recent applications of these methods to Milky Way data. Chapter 4 provides discussion and summary, particularly introducing recent developments in applying action-angle methods to non-integrable galactic phenomena and non-equilibrium processes.

1.1 Theoretical Basis of Stellar Systems

The single-particle motion in a collective potential discussed here derives from the gravitational N-body problem through simplification—the single-particle approximation or mean-field approximation. A classical system of N particles in three-dimensional background space can be described by Hamilton's equations: $\dot{q}_i = \partial H/\partial p_i$, $\dot{p}_i = -\partial H/\partial q_i$ for $i = 1, 2, 3, ..., 3N$.

For the full 6N-dimensional phase space (generally called Γ-space) composed of all particles' coordinates and momenta, local conservation of phase-space representative points leads to the probability density $\rho(q, p)$ satisfying the Liouville equation: $\partial\rho/\partial t + \{\rho, H\}_{PB} = 0$, where $\rho = \rho(q_i, p_i, t)$ is the ensemble probability distribution and $\{\rho, H\}_{PB}$ is the Poisson bracket. The left side of this equation is essentially the Lagrangian time derivative $d\rho/dt$, indicating conservation of phase volume—this is Liouville's volume-preserving theorem.

Using $x = x_j$, $p = p_j$ $(j = 1, 2, 3)$ to represent coordinates and momenta of a single particle, and q_k, p_k for the k-th particle, with f_n representing the n-particle joint distribution function, applying the single-particle approximation $f_2(q_1, p_1, q_2, p_2, t) = f_1(q_1, p_1, t) f_1(q_2, p_2, t)$ yields the Boltzmann equation: $\partial f/\partial t + \mathbf{v} \cdot \nabla_q f + \mathbf{F} \cdot \nabla_p f = d\Omega\, \sigma(\Omega) \|\mathbf{v} - \mathbf{v}_1\| (f' f_1' - f f_1)$, where $f = f(q, p, t)$ is the single-particle distribution function in 6-dimensional -phase space (3 degrees of freedom in coordinates and velocities), $\mathbf{v}$ represents velocity, and $\mathbf{F}$ represents force. The Boltzmann equation system requires numerous (even redundant) assumptions, some of which remain difficult to prove mathematically. This paper introduces but does not analyze the Boltzmann equation (see Chapters 4, 7 of [8] and Chapters 3, 4 of [60]). It can be verified that the entropy of the distribution function in Γ-space from the Liouville equation is invariant, while the entropy of the distribution function in -space from the Boltzmann equation is non-decreasing.

Under the local equilibrium assumption, the collision term in the Boltzmann equation vanishes. In galactic systems, stellar radii are extremely small compared to galactic scales, with mean free paths far exceeding system size and collision times far exceeding galaxy crossing times (by many orders of magnitude larger than the age of the universe), so collision terms should be negligible. Multiplying the Boltzmann equation by a collision invariant $\xi(x, v, t)$ (such as mass, momentum, energy) and performing local averaging (due to symmetries before and after collisions, the collision term makes no contribution after local averaging, see Chapter 7A of [60]) yields hydrodynamic equations. For momentum conservation, multiplying by $\xi(x, v, t) = mv_i$ $(i = x, y, z)$, integrating over momentum space, and applying the divergence theorem (with $f \to 0$ at infinity, $\lim_{p\to\infty} f = 0$) gives $\langle \rho v_i \rangle + \nabla_x \cdot \langle \rho v_i v \rangle - \frac{1}{\rho} F_i = 0$. This is essentially equivalent to the hydrodynamic momentum conservation equation: $\partial(\rho \mathbf{u})/\partial t + \nabla_x \cdot (\rho \mathbf{u}\mathbf{u} + \mathbf{P}) = \rho \mathbf{F}$, where $\mathbf{P}$ is the pressure tensor. Similar derivations yield mass and energy conservation equations (see Chapter 7 of [60]).

Thus we have the connection: "N-body system (mechanical description) $\to$ Liouville equation (6N-dimensional) $\to$ Boltzmann equation (6-dimensional) $\to$ hydrodynamic equations (flow field)."

Alternatively, starting from the collisionless Boltzmann equation, multiplying by velocity components and integrating over velocity space (similar to the method for obtaining Eq. (4)) yields the Jeans equations: $\nu \bar{v}_i = \partial(\nu \sigma_{ij})/\partial x_j + \nu \partial \Phi/\partial x_i$, where ν represents number density in configuration space and σ_{ij} is the velocity dispersion tensor. The Jeans equations not only simplify the equations through averaging but also connect the distribution function to observables, proving helpful for observational astronomy. They also provide a method to test the correctness of distribution functions. Of course, the Jeans equations remain non-closed in general.

The single-particle distribution function $f = f(x, v, t)$ in -phase space is closely related to single-particle motion $(x(t), v(t))$ under the single-particle approximation. This distribution function essentially assumes the system consists of infinitely many point particles, somewhat analogous to field theory (which deals with infinite degrees of freedom). From a field perspective, all inter-particle interactions are treated as a background field, so for any given particle we only need to study its motion in this field without considering interactions from other particles—this is the mean-field concept. However, directly solving for distribution functions from statistical physics equations of self-gravitating systems to understand galactic structure and evolution is not straightforward, and methods like maximum entropy principle are insufficient (due to characteristics of long-range interacting systems, see Chapter 4). Alternative approaches include N-body simulations for statistical analysis, constructing distribution function models based on particle motion characteristics in gravitational potentials, and other theoretical methods (see [8]); in these approaches, studying integrals of motion is fundamental.

Below, we primarily introduce integrals of motion for stars in static gravitational potentials from the perspective of Hamiltonian mechanics and single-particle stellar orbital motion.

1.2.1 Hamilton-Jacobi Equation

The evolution of a mechanical system's canonical coordinates (q, p) in phase space can be viewed as the result of infinitely many successive infinitesimal canonical transformations generated by the Hamiltonian. In canonical transformations, setting the new Hamiltonian to zero and choosing a type-2 generating function F_2 yields new canonical coordinates Q, P that are constants: $Q = \alpha$, $P = \beta$. This leads to the Hamilton-Jacobi equation (H-J equation): $H(q, \partial S/\partial q, t) + \partial S/\partial t = 0$, whose solution $S = S(q, \beta)$ serves as the generating function F_2.

The differential relation between Hamiltonian functions in type-2 canonical transformations is expressed as: $K = H + \partial F_2/\partial t$, with $\alpha_i = \partial F_2/\partial \beta_i$.

1.2.2 Integrals of Motion

A mechanical quantity that does not explicitly depend on time and remains constant, $h(q, p) = \text{const}$, is called an integral of motion—invariant along particle orbits. Finding such invariants is significant. For an autonomous Hamiltonian system, the Hamiltonian itself is an integral of motion, with its constant E equal to the total system energy. Using Eqs. (7) and (8), the solution to the H-J equation can be written with separated time and generalized coordinate terms: $S(q, \beta, t) = W(q, \beta) - ht$, where W is Hamilton's characteristic function. For an autonomous Hamiltonian system with n degrees of freedom, there are at most n independent and involutive integrals of motion. Thus among the $(n+1)$ constants $\beta = \{\beta_1, ..., \beta_n\}$ and h, at least one is dependent: $h = h(\beta_1, ..., \beta_n)$. We take $\beta_n = h$. Differentiating Eq. (9) yields differential relations and the generating function relation (8). Substituting W into S gives the relation between W and old/new phase-space coordinates.

Using Hamilton's characteristic function W as the generating function for the canonical transformation $(q, p) \to (Q, P)$, where $P_i = \beta_i$, the new Hamiltonian remains $K = H$, depending only on new generalized momenta. This transformation forms the basis for discussing integral-of-motion coordinates.

For autonomous (completely) separable generating functions W, we can write: $W(q, \beta) = \sum_{i=1}^{n} W_i(q_i, \beta)$, leading to n separated H-J equations (no summation over i): $(\partial W_i/\partial q_i)^2/2 + V_i(q_i) = \beta_i$.

The H-J method conveniently solves separable systems. As noted, if an autonomous Hamiltonian system with n degrees of freedom is (completely) integrable, it possesses n mutually involutive integrals of motion I_i $(i = 1, 2, 3, ..., n)$—this is the content of Liouville's integrability theorem. According to Liouville's volume-preserving theorem, the coordinate set $M = \{(q, p) | I_i(q, p) = \beta_i\}$ remains invariant under phase flow with I_i as Hamiltonian functions and is homeomorphic to an n-dimensional torus T^n (if no resonances exist), transforming orbits of completely integrable systems into n-dimensional quasi-periodic motion.

Two points about integrals of motion deserve attention: First, in galactic dynamics context, "integral of motion" and "motion constant" are strictly distinguished concepts. Second, there are further subdivisions. For stellar motion in galactic potentials, given the gravitational field, a motion constant is defined as any function of phase-space coordinates and time that remains constant along stellar orbits: $C(\mathbf{x}(t), \mathbf{v}(t), t) = \text{const}$. An integral of motion is also constant along orbits but further requires being a time-independent function: $I(\mathbf{x}(t), \mathbf{v}(t)) = \text{const}$. Integrals divide into isolating and non-isolating types. Isolating integrals affect the distribution of stellar orbits in phase space—the dimensionality of the manifold occupied by phase-space trajectories. For integrable systems, each additional isolating integral reduces this manifold's dimension by one. Non-isolating integrals have no effect on orbital distribution and thus lack practical value.

Additionally, note the distinction between involutive and isolating integrals in completely or partially integrable systems. For autonomous Hamiltonian systems with n degrees of freedom in $2n$-dimensional phase space, there are at most $2n-1$ independent (linearly independent) integrals of motion, with at most $2n-1$ isolating integrals. Involutivity relates to integrability—mutually involutive integrals number no more than n (when exactly n, the system is Liouville-completely integrable). Thus, a set of mutually involutive integrals must be isolating, but not conversely. For example, stellar orbits in spherical potentials (6D phase space) have 5 integrals, with 4 or 5 isolating integrals (only for Keplerian orbits), but only 3 mutually involutive integrals.

For common axisymmetric potentials (e.g., disk galaxies), there are generally 3 isolating integrals: energy E, vertical angular momentum component L_z, and a third integral I_3 (often without analytic form). For triaxial potentials, there are generally 3 isolating integrals: E, I_2, and I_3 (often lacking analytic forms). Much previous work focused on finding the second and third integrals and distribution functions $f(I)$ using these as variables (the central topic of this review).

1.2.3 Action-Angle Variable System

Although integrals of motion I and their generalized coordinates (denoted θ, angle variables) reduce motion to quasi-periodic form, integrals are generally not canonical coordinates—they don't satisfy Hamiltonian conjugacy with their generalized coordinates. Those that become canonical after transformation are called actions J. Below we introduce requirements for angle variables and actions (see [8] Section 3.5.1).

Assuming an invertible transformation from I to J ($\det(\partial J/\partial I) \neq 0$) makes actions J satisfy: (1) being integrals of motion, (2) being canonical coordinates. Hamilton's equations then give $\dot{J}_i = -\partial H/\partial \theta_i = 0$, so the Hamiltonian depends only on actions: $H = H(J)$. The angle variables evolve as $\dot{\theta}_i = \partial H/\partial J_i \equiv \Omega_i(J) = \text{const}$, giving $\theta_i = \theta_i(0) + \Omega_i t$.

For bound orbits, angle variables θ and integrals of motion describe an n-dimensional quasi-periodic motion—the aforementioned manifold M homeomorphic to T^n. Any closed loop γ_i on a dimension of M (homeomorphic to circle S^1), represented by a pair (θ_i, J_i), requires that θ integrated along γ_i equals 2π.

We can expand phase-space orbits as Fourier series: $\mathbf{x}(\theta, J) = \sum_{\mathbf{n}} \mathbf{X}_{\mathbf{n}}(J) e^{i\mathbf{n}\cdot\theta} = \sum_{\mathbf{n}} \mathbf{X}_{\mathbf{n}}(J) e^{i\mathbf{n}\cdot\Omega t}$. The ratio relationships among angular frequencies enrich the complexity of periodic types and ergodicity (see Chapter 4). Astrophysical dynamical systems generally exhibit strong periodicity, a key reason why action-angle systems remain enduring in celestial mechanics.

Defining action J_i' as the phase volume enclosed by γ_i gives: $J_i' = \frac{1}{2\pi}\oint_{\gamma_i} \mathbf{p}\cdot d\mathbf{q} = \frac{1}{2\pi}\int_{\text{interior of }\gamma_i} d\theta_i dJ_i = J_i - J_i^c$, where J_i^c is the action invariant at singular points. From Poincaré invariant preservation under canonical transformation: $\oint_{\gamma_i} \mathbf{p}\cdot d\mathbf{q} = 2\pi J_i$, showing that an action value J_i represents the area enclosed by sub-ring γ_i (with appropriate scaling of q_i, p_i), while θ_i represents the angle traversed.

For completely separable systems, $p_i = \partial W_i/\partial q_i$, meaning p_i depends only on the corresponding q_i. Action calculation simplifies to $J_i = \frac{1}{2\pi}\oint p_i dq_i$. Moreover, as W_i evolves along θ_i for one period while other coordinates remain unchanged, returning to the initial state, the change in Hamilton's characteristic function $\Delta W_i = \oint p_i dq_i$ equals 2π times the action—this is the significance of action.

Additionally, Eq. (15) computes angular frequencies, while Hamilton's characteristic function from Eq. (11) computes angle variable values. Numerical implementation details are in Appendix A.

Thus, the transformation from general phase-space coordinates to actions proceeds as: phase-space coordinates $(q, p) \rightarrow$ integrals of motion I and angle coordinates $\theta \rightarrow$ canonical torus coordinates (J, θ).

1.2.4 Distribution Functions Based on Conserved Quantities

For the single-particle distribution function $f(x, v)$ in 6D -phase space, for (quasi-)steady systems ($\partial f/\partial t = 0$), the collisionless Boltzmann equation implies f can be written as a function of integrals of motion: $f = f(I)$—this is Jeans' theorem. For regular orbits with non-resonant frequencies in quasi-steady stellar systems, the distribution function can be written as a function of three actions: $f = f(J)$—this is the strong Jeans theorem. Thus we gain a more non-redundant, profound understanding of stellar distributions.

Earlier distribution function models often used forms like $f(E, \cdots)$, with energy as a parameter—a trap that hindered progress in integral/action-based modeling. For real galactic modeling, E cannot be in the parameter list. The reason: real galaxies are multi-component systems requiring at least dark matter halos and stellar components. Orbital energy is non-local; if E appears in parameters, phase-space density around an orbit would vary uncontrollably. For example, the simplest orbit is a regular orbit at the Galactic center. Adding a dark matter halo component deepens the central potential, reducing the energy of central stars while the orbit remains unchanged. But we only know the orbit's energy $E_0 = \Phi(0)$ after the model is fully assembled and its potential determined. Therefore we cannot specify $f(E, \cdots)$ at the start of modeling. When distribution functions depend only on actions, phase-space density around each orbit remains unchanged.

In summary, advantages of action-angle systems over general integrals include: actions are special integrals of motion and canonical coordinates with conjugate angle variables; the action system suits perturbation methods and describes strongly periodic celestial motions; actions are adiabatic invariants applicable to slowly varying potentials and orbit identification in neighboring potentials; and as noted, using action-parameterized distribution functions enables distribution function-based modeling of multi-component galaxies.

1.3 Introduction to Stäckel Potentials

Stäckel potentials are separable in ellipsoidal coordinates. Triaxial Stäckel potentials represent the most general class of separable potentials. Particle motion in general Stäckel potentials yields analytic integrals of motion—completely integrable systems. This section introduces fundamentals of triaxial Stäckel potentials, beginning with a review of ellipsoidal coordinates.

The coordinate transformation $(x, y, z) \rightarrow (\lambda, \mu, \nu)$ is defined by $\frac{x^2}{\tau+\alpha} + \frac{y^2}{\tau+\beta} + \frac{z^2}{\tau+\gamma} = 1$, where $\tau = \lambda, \mu, \nu$ constitute confocal ellipsoidal coordinates. With constants $\alpha = -a^2$, $\beta = -b^2$, $\gamma = -c^2$ and $-\gamma \leq \nu \leq -\beta \leq \mu \leq -\alpha \leq \lambda$, as τ varies, semi-axis lengths change while focal distances remain fixed, forming families of confocal quadric surfaces. Figure 1 [Figure 1: see original paper] illustrates the coordinate surfaces: for $\tau = \lambda$, since $\lambda + \alpha > 0$, $\lambda + \beta > 0$, $\lambda + \gamma > 0$, each constant-λ surface represents an ellipsoid (red surfaces in Fig. 1); similarly, constant-μ surfaces represent one-sheet hyperboloids (yellow) and constant-ν surfaces represent two-sheet hyperboloids (green). Cartesian and spherical coordinates can be viewed as limiting cases of confocal ellipsoidal coordinates.

The confocal ellipsoidal coordinate system is orthogonal with line element $ds^2 = \sum_\tau P_\tau^2 d\tau^2$ and metric coefficients: $P_\lambda^2 = \frac{(\lambda-\mu)(\lambda-\nu)}{4(\lambda+\alpha)(\lambda+\beta)(\lambda+\gamma)}$, with analogous expressions for P_μ^2 and P_ν^2.

Stäckel's work proved that the only coordinate system allowing separation of the Hamilton-Jacobi equation for $H = \frac{1}{2}p^2 + \Phi(x)$ is confocal ellipsoidal coordinates (see [8] Section 3.5).

The Stäckel potential $\Phi_S(\lambda, \mu, \nu)$ can be written as: $\Phi_S = -\frac{\zeta(\lambda)}{(\lambda-\mu)(\lambda-\nu)} - \frac{\eta(\mu)}{(\mu-\nu)(\mu-\lambda)} - \frac{\xi(\nu)}{(\nu-\lambda)(\nu-\mu)}$, where ζ, η, κ are arbitrary functions of their respective ellipsoidal coordinates (see Eq. (6) in [2]). Defining $F_\tau(\lambda) = 4(\lambda + \alpha)(\lambda + \beta)(\lambda + \gamma)\zeta(\lambda)$, with analogous definitions for $F_\tau(\mu)$ and $F_\tau(\nu)$, the Stäckel potential can be expressed in three separated components.

Substituting this Stäckel potential into the Hamiltonian $H = \frac{1}{2}(p_\lambda^2/P_\lambda^2 + p_\mu^2/P_\mu^2 + p_\nu^2/P_\nu^2) + \Phi_S(\lambda, \mu, \nu)$, where $p_\lambda = P_\lambda^2 \dot{\lambda}$ and similarly for p_μ, p_ν, and into the

Hamilton-Jacobi equation $H = E$, with $p_\tau = \partial W/\partial \tau$ where W is Hamilton's characteristic function, solving the H-J equation and multiplying by $(\lambda-\mu)(\mu-\nu)(\nu-\lambda)$ yields: $(\nu-\mu)[2(\lambda+\alpha)(\lambda+\beta)(\lambda+\gamma)p_\lambda^2-F(\lambda)-\lambda^{2E}]+(\lambda-\nu)[2(\mu+\alpha)(\mu+\beta)(\mu+\gamma)p_\mu^2-F(\mu)-\mu^{2E}]+(\mu-\lambda)[2(\nu+\alpha)(\nu+\beta)(\nu+\gamma)p_\nu^2-F(\nu)-\nu^{2E}]=0$.

Due to the special form of Eq. (21) and separability of W, we write $W = \sum_{\tau=\lambda,\mu,\nu} W_\tau(\tau)$. The bracketed terms in Eq. (23) become functions of individual τ coordinates, denoted $U_\tau(\tau)$, giving: $(\nu-\mu)U_\lambda(\lambda)+(\lambda-\nu)U_\mu(\mu)+(\mu-\lambda)U_\nu(\nu)=0$.

Differentiating this equation twice with respect to each coordinate $\tau = \{\lambda,\mu,\nu\}$ shows that $\partial^2 U_\tau(\tau)/\partial\tau^2 = 0$, meaning the three $U_\tau(\tau)$ are linear functions: $U_\tau(\tau) = j_\tau\tau - k_\tau$. Cross-differentiation reveals equal linear coefficients $j_\tau = j$, and further differentiation with arbitrary $\{\lambda,\mu,\nu\}$ values shows $k_\tau = k$. Thus j, k are separation constants, and the three $U_\tau(\tau)$ must be identical linear functions: $U_\tau(\tau) = j\tau - k$.

Since $U_\tau(\tau)$ represents the bracketed terms in Eq. (23), we obtain: $2(\tau+\alpha)(\tau+\beta)(\tau+\gamma)p_\tau^2 = \tau^{2E} - \tau J + K + F(\tau)$, where constants j, k are the values of two integrals of motion J, K (similar to the relation between total energy E and Hamiltonian H). More conveniently, H, J, K can be expressed as: $J = (\mu+\nu)X + (\nu+\lambda)Y + (\lambda+\mu)Z$, $K = \mu\nu X + \nu\lambda Y + \lambda\mu Z$, where X, Y, Z are defined through the separated functions.

Another commonly used set of integrals relates as: $I_2 = \alpha^2 H + \alpha J + K/(\alpha-\gamma)$, $I_3 = \gamma^2 H + \gamma J + K/(\gamma-\alpha)$. These are the "second and third integrals of motion" in galactic dynamics. The momentum expressions are: $p_\tau^2 = \frac{(\tau+\alpha)(\tau+\gamma)E-(\tau+\gamma)I_2-(\tau+\alpha)I_3+F(\tau)}{2(\tau+\alpha)(\tau+\beta)(\tau+\gamma)}$, used to compute integrals of motion in Stäckel potentials and primarily employed when estimating integrals for general potentials.

From the action definition (Eq. (18)): $J_\tau = \frac{1}{2\pi}\oint p_\tau d\tau = \frac{1}{\pi}\int_{\tau_-}^{\tau_+}|p_\tau|d\tau$, we obtain actions, with angle variables θ derived from Hamilton's characteristic function. Appendix A describes specific numerical methods (see [1, 22]).

Stäckel potentials have another advantage: for galaxies in quasi-steady state, non-diagonal second-order velocity moments of the stellar distribution function vanish, leaving only three diagonal second moments that make the three Jeans equations closed; van de Ven et al. [25] provided general solutions for velocity dispersions in Stäckel potentials using differential equation methods.

2 Action Estimation Methods Based on Stäckel Potential Approximation

The previous section introduced the significance of action-angle variables and analytic solutions for integrals of motion in Stäckel potentials. However, most real gravitational potentials lack analytic integrals. This chapter introduces several methods for estimating actions and angles in general potentials, enabling application of action-angle systems. These methods implicitly incorporate perturbation ideas by using known Stäckel potentials as toy models. Specifically, given a known Stäckel toy potential, one fits model potential parameters using sampled points from the "real" orbit, then uses this computed potential as the "true Stäckel potential" to estimate integrals of motion and compute actions and frequencies.

Sanders and Binney [22, 24] also provide the open-source TACT program, written in C++, which computes actions, frequencies, and angles using several methods with error estimates, plus Python plotting routines. Algorithm details are in [20, 24] and the TACT source code.

2.1 Stäckel Fudge Method

For estimating actions in general potentials, Binney [13] proposed the efficient Stäckel Fudge method—a "cheating" algorithm. The idea: assuming the potential can locally be written in Stäckel form, one locally computes pseudo-separation functions (see Eq. (21)'s $F(\tau)$) for the target potential using certain tricks, assuming cross-coordinate effects on separation function errors are small, then computes new integrals of motion to obtain momenta and thus actions through integration.

2.1.1 Axisymmetric Stäckel Fudge Generally, axisymmetric potentials have 3 isolating integrals: total energy E, vertical angular momentum component L_z, and I_3. Axisymmetric ellipsoidal coordinates reduce one degree of freedom compared to triaxial coordinates. For prolate coordinates ($\alpha = \beta$) we have (λ, ϕ, ν) with azimuthal angle ϕ; for oblate coordinates ($\gamma = \beta$) we have (λ, μ, χ).

Using confocal ellipsoidal coordinates, new (u, v, ϕ) coordinates are defined: $R = \Delta \sinh u \sin v$, $z = \Delta \cosh u \cos v$. The axisymmetric Stäckel potential becomes: $\Phi_S(u, v) = \frac{F_u(u) - F_v(v)}{\sinh^2 u + \sin^2 v}$.

To compute actions, explicit Stäckel separation functions $F_\tau(\tau)$ are needed for momentum p_τ^2. If the target potential $\Phi(u, v)$ can be written in Stäckel form, then $F_u(u) - F_v(v) \approx (\sinh^2 u + \sin^2 v)\Phi_S(u, v)$. For potentials near Stäckel form, the v-dependence of $\delta F_u(u)$ and u-dependence of $\delta F_v(v)$ are small. The variations are defined as: $\delta F_u(u) \equiv (\sinh^2 u + \sin^2 v)\Phi(u, v) - (\sinh^2 u_0 + \sin^2 v)\Phi(u_0, v) \approx F_u(u) - F_u(u_0)$, with similar expressions for $F_v(v)$. Here u_0 is a reference value that minimizes $\delta F_u(u)$ at fixed v (see [13]).

2.1.2 Triaxial Stäckel Fudge This section presents the algorithm by Sanders and Binney [22], extending Binney's [13] axisymmetric method to triaxial potentials. For general axisymmetric systems, besides the Stäckel Fudge, Sanders [19] also provides Stäckel fitting. However, triaxial Stäckel potentials have one more parameter dimension, making fitting more complex, prompting development of the triaxial fudge method.

The method uses auxiliary integrals of motion (new dependent integrals from the fudge) and locally approximated Stäckel potentials to compute momenta p_τ at each point via momentum-integral relations, then integrates to obtain actions J and angles θ. Using the current real phase-space point as initial condition (x_0, v_0), one computes a series of momenta without needing explicit separation functions $F(\tau)$, then integrates to get actions. Minimizing action standard deviation (dependent on each initial (x_0, v_0)) by adjusting focal lengths yields appropriate actions; for axisymmetric systems, iteration can reduce errors.

Step 1: Triaxial fudge and integral computation

For a general potential, define triaxial auxiliary functions: $\chi_\lambda(\lambda, \mu, \nu) = (\lambda - \mu)(\nu - \lambda)\Phi(\lambda, \mu, \nu)$, with cyclic permutations for χ_μ and χ_ν. If the potential is Stäckel, these become: $\chi_\lambda = F(\lambda) - \lambda\frac{F(\mu)-F(\nu)}{\mu-\nu} + \frac{\mu F(\nu)-\nu F(\mu)}{\mu-\nu}$.

Assuming the potential is approximately Stäckel, we can write: $F(\tau) = \chi_\tau(\lambda, \mu, \nu) + \tau C_\tau + D_\tau$, where C_τ, D_τ are constants when the other two ellipsoidal coordinates are fixed, enabling independent computation of required integrals.

Substituting into Eq. (26) yields an alternative momentum expression: $2(\tau + \alpha)(\tau + \beta)(\tau + \gamma)p_\tau^2 = \tau^{2E} - \tau J_\tau + K_\tau + \chi_\tau(\lambda, \mu, \nu)$. Comparing with Eq. (37) gives relations: $J_\tau = j - C_\tau$, $K_\tau = k + D_\tau$ (where j, k are the integrals from Eq. (26)). Thus, given an initial phase-space point (x_0, v_0), an appropriate ellipsoidal coordinate system (parameters α, β, γ), and potential values, one can compute integrals of motion at each ellipsoidal coordinate point [22].

Step 2: Selecting appropriate focal lengths

These new integrals allow computing momenta via Eq. (38) and then actions. The focal lengths of the coordinate transformation ($\Delta_1 = \gamma - \beta$, $\Delta_2 = \beta - \alpha$) significantly affect the method's action variance (taken as error). Sanders and Binney provide numerical methods to adjust focal lengths. For short-axis closed loop orbits (forming in the $z = 0$ plane with focal length $y = \pm\Delta_1$), only action J_ν is non-zero. The procedure involves: specifying a general potential, maximum and minimum energy values, and an energy grid; at each grid point E_i, integrating a half-period orbit launched vertically from $(0, y_k, 0)$ with velocity $\sqrt{2(E_i - \Phi(0, y_k, 0))}$; recording phase-space points and computing J_ν; finding the focal length Δ_1 that minimizes action variance. Δ_2 can be computed similarly.

Step 3: Torus iteration for improved precision

The Stäckel fudge process is complex with relatively large errors (deviations of several to tens of percent, especially for box orbits—see Figs. 6-8 in [22]). However, combining it with torus mapping (Section 2.3.1) through iteration yields convergent errors. This iterative algorithm is detailed in Section 2.3.2.

2.2 Stäckel Fitting Method

2.2.1 General Axisymmetric Potential Fitting Sanders [19] proposed fitting general axisymmetric potentials in prolate ellipsoidal coordinates. The method fits a given potential to Stäckel form by sampling orbits to determine the separation function $F(\tau)$, yielding integrals of motion. Sanders and Binney [24] classify both Stäckel fudge and fitting as "non-convergent methods" because their approximations prevent error reduction through increased computation.

In a meridional plane, transforming from cylindrical (R, ϕ, z) to oblate spheroidal (λ, ϕ, ν) coordinates depends on $\frac{R^2}{\tau+\alpha} + \frac{z^2}{\tau+\gamma} = 1$, where λ, ν are roots of this equation in τ, with $a^2 = \alpha$, $b^2 = \beta$, $c^2 = \gamma$ (similar to Eq. (19) for triaxial case). The Stäckel potential simplifies to: $\Phi_S = -\frac{F(\lambda)-F(\nu)}{\lambda-\nu}$.

The third integral's analytic expression in axisymmetric case is: $I_3 = \frac{(\lambda+\gamma)}{2(\lambda+\alpha)} p_\lambda^2 + \frac{(\nu+\gamma)}{2(\nu+\alpha)} p_\nu^2 + \frac{(\nu+\gamma)h(\lambda)-(\lambda+\gamma)h(\nu)}{\lambda-\nu}$.

The momentum-integral relation is: $2(\tau+\alpha)p_\tau^2 = \frac{F(\tau)}{\tau+\gamma} - \frac{I_3}{\tau+\gamma} - E$.

Procedure:

1. **Initial conditions and true potential**: Given initial (x, v) and the true potential (felt during orbit integration), compute actual integrals E, L_z.

2. **Focal length selection**: Transform to ellipsoidal coordinates. The focal length parameter $\Delta = a^2 - c^2$ (with $a = b$, so only $a^2 - c^2$ is needed; c^2 can be set to 1) significantly affects action variance. Sanders [19] estimates a^2 using: $a^2 - c^2 = R^2 - z^2 - \frac{\partial\Phi/\partial R}{\partial^2\Phi/\partial R \partial z}$. Assuming the estimated local potential Φ is the Stäckel potential Φ_S, this equation estimates a^2 using numerical gradients of the known true Φ, averaged over orbit points obtained in the next step.

3. **Orbit sampling and motion region estimation**: Perform orbit integration (numerical orbit drawing) using a conservative scheme with 300 points per orbital period. During integration: (i) estimate a^2 every 5 points using Eq. (48) and average to obtain the best oblate coordinates; (ii) detect turning points in τ by sign changes in $\partial\tau/\partial t$ (requiring coordinate transformations), recording $\lambda_\pm, \nu_\pm$ to determine the orbit's ellipsoidal coordinate region.

4. **Spline interpolation fitting**: Assuming local Stäckel form, compute "assumed Stäckel potential" values at several ellipsoidal coordinate points, then perform cubic spline interpolation. The Stäckel auxiliary function χ simplifies in axisymmetry to $\chi = -(\lambda-\nu)\Phi(\lambda, \nu)$. If the fitted potential Φ were Stäckel, then $-(\lambda-\nu)\Phi(\lambda, \nu) = F(\lambda) - F(\nu)$, providing interpolation points. The separation functions are chosen to minimize the deviation: $E[F] = \int d\lambda \int d\nu \Lambda(\lambda) N(\nu)(\chi(\lambda, \nu) - F(\lambda) + F(\nu))^2$, with normalized weight functions $\Lambda(\lambda) = 4\lambda^{-5}(\lambda_+^{-4} - \lambda_-^{-4})^{-1}$ and $N(\nu) =$

$(\nu_+ - \nu_-)^{-1}$. The optimal $F(\lambda) = \bar{\chi}(\lambda) - \bar{\bar{\chi}}$, $F(\nu) = -\bar{\chi}(\nu) + \bar{\bar{\chi}}$, where $\bar{\chi}(\lambda) = \int d\nu \chi(\lambda, \nu) N(\nu)$, etc. TACT selects 40 points for cubic spline interpolation.

5. **Integral computation**: Using the fitted potential and previous E, L_z, compute I_3 at three turning points $\tau_\pm$ and average, obtaining three integrals of motion. Reality conditions $\tau > 0$ are checked; if violated, use the initial phase-space point instead of turning points.
6. **Action-angle computation**: The fitted $F(\tau)$ computes p_τ in Eq. (46) to obtain actions J and angles θ (numerical details in Appendix A).

2.2.2 Other Stäckel Fitting Methods As early as 1985, de Zeeuw and Lynden-Bell [2] proposed using Stäckel potentials as models for local and global fitting of galactic potentials. The former expands the potential near equilibrium points and determines parameters via corresponding Stäckel ellipsoidal coordinates; the latter approximates the three separation functions through averaging to compute integrals.

(1) Local fitting: Expanding potentials of form $V = V(x^2, y^2, z^2)$ near equilibrium: $V = \sum_{k,l,m} V_{klm} x^k y^l z^m$ (with even indices). Stäckel potentials in Eq. (21) are expanded about $(\lambda, \mu, \nu) = (-\alpha, -\beta, -\gamma)$ with zero potential at the expansion source: $\zeta_{-1} = \eta_{-1} = \kappa_{-1} = 0$. This yields expansions for $\zeta(\lambda), \eta(\mu), \kappa(\nu)$. Comparing with Eq. (21) gives coefficients like $V_{200} = -4\zeta_0$, $V_{400} = -4\zeta_1$, $V_{220} = -\frac{8}{\beta - \alpha}(\zeta_0 - \eta_0)$, etc. [2] discusses constraints, existence, and appropriate forms.

(2) Global fitting: Chapter 4 of [2] discusses that potentials related to elliptical galaxies can be globally approximated as Stäckel forms, presenting methods for ellipsoidal potentials $V = V(x^2, y^2, z^2)$. The method requires the fitted potential to be globally closest to a Stäckel potential, yielding exact results when V is exactly Stäckel. Key steps involve defining averages of arbitrary functions $Q(\lambda, \mu, \nu)$ with weight functions $\Lambda(\lambda), M(\mu), N(\nu)$ chosen from integral constraints. The separation functions are computed via weighted averages of the auxiliary function $\chi(\lambda, \mu, \nu) = -(\lambda - \mu)(\mu - \nu)(\nu - \lambda) V(\lambda, \mu, \nu)$.

While recent improved methods (Stäckel fudge and fitting) have broader applicability, these early methods share similar basic ideas. The Stäckel fitting method computes explicit separation functions $F(\tau)$ by locally fitting a given potential to Stäckel form based on sampled points, then interpolating to obtain the best-fit Stäckel potential for computing integrals and actions analytically. The Stäckel fudge method differs only by locally treating the given potential as Stäckel (assuming local separability) and using tricks to compute local integrals, then proceeding with the same analytic formulas.

2.3 Torus Mapping Methods

This section covers: (1) The torus mapping (TM) method by McGill and Binney [11], later refined by Binney and collaborators, which constructs tori (action-angle variables) via canonical transformations to compute actions; (2) Two extensions: Iterative Torus Construction (ItTC) [22, 24] and Orbit Integration to Generating Function (O2GF) [21, 24]. These methods feature highest precision but greater computational cost, and are convergent—errors can be continuously reduced with sufficient computation.

Viewing the torus (n-Torus) as a geometric object, deforming the torus of an analytic Hamiltonian into that of the target Hamiltonian resembles perturbation theory, achieved by seeking canonical transformations from toy torus coordinates to target torus coordinates [11]. Known analytic gravitational models (isochrone or harmonic oscillator potentials, Eqs. (70)-(71)) serve as toy models—both are Stäckel potentials with analytic solutions. Though torus mapping differs from Stäckel fudge (which approximates separation functions), it still starts from a Stäckel toy potential, essentially representing a simplified perturbation method, and thus remains an extension of Stäckel potential theory.

Unless specified, $(J_{\text{toy}}, \theta_{\text{toy}})$ denote toy torus coordinates (particle motion in toy potential), while (J, θ) denote target torus coordinates (general potential). Mapping from toy to target torus is treated as a canonical transformation with type-2 generating function $S = S(J, \theta_{\text{toy}})$, Fourier-expanded as: $S(\theta_{\text{toy}}, J) = \theta_{\text{toy}} \cdot A(J) - i \sum_{\mathbf{n}} S_{\mathbf{n}}(J) \exp(i\mathbf{n} \cdot \theta_{\text{toy}})$. The imaginary unit i simplifies derivative expressions. Requiring the target torus image to complete a periodic circuit when any toy angle component increases by 2π forces $A = J$, giving: $S(\theta_{\text{toy}}, J) = \theta_{\text{toy}} \cdot J - i \sum_{\mathbf{n}} S_{\mathbf{n}}(J) \exp(i\mathbf{n} \cdot \theta_{\text{toy}})$.

Reality of action-angle variables and time-reversal invariance of the Hamiltonian constrain the coefficients: $S_{-\mathbf{n}} = -S^*_{\mathbf{n}}$, yielding: $S(\theta_{\text{toy}}, J) = \theta_{\text{toy}} \cdot J + 2 \sum_{\mathbf{n} \in \mathbf{N}} S_{\mathbf{n}}(J) \sin(\mathbf{n} \cdot \theta_{\text{toy}})$, where $\mathbf{N} = \{(i, j, k)\}$ represents a set of precision coordinates satisfying $(k > 0)$ or $(k = 0, j > 0)$ or $(k = 0, j = 0, i > 0)$ [12, 21]. Differentiating gives: $J_{\text{toy}} = \partial S / \partial \theta_{\text{toy}} = J + 2 \sum_{\mathbf{n}} \mathbf{n} S_{\mathbf{n}}(J) \sin(\mathbf{n} \cdot \theta_{\text{toy}})$, $\theta = \partial S / \partial J = \theta_{\text{toy}} + 2 \sum_{\mathbf{n}} \partial S_{\mathbf{n}}(J) / \partial J \sin(\mathbf{n} \cdot \theta_{\text{toy}})$. This connects toy and target tori via the generating function.

2.3.1 Torus Mapper Program Binney and McMillan [12] provide the TM program, implementing three functions: given user-specified target potential and its first derivatives, and specified actions (J_r, J_ϕ, J_z): (1) compute (x, v) for given θ; (2) compute Jacobian $\partial(x)/\partial(\theta)$; (3) compute θ given (x, J). Compared to traditional stellar orbits, a torus is specified by ~100 numbers ($S_{\mathbf{n}}$, $\partial S_{\mathbf{n}} / \partial J$, and other parameters) rather than thousands of time-series phase-space points, significantly reducing orbit library sizes [12].

The torus mapping procedure (Eq. (68) [12]) completes via three canonical transformations: $(\theta, J) \xrightarrow{S_{\mathbf{n}}} (\theta_{\text{toy}}, J_{\text{toy}}) \xrightarrow{\text{H-J}} (x_{\text{toy}}, v_{\text{toy}}) \xrightarrow{\text{point}} (x, v)$.

Given the true potential and orbit, with target actions J and trial coefficients $S_{\mathbf{n}}$, toy actions J_{toy} are obtained from toy angle grid points via Eq. (66). In the toy potential, phase-space coordinates $(x_{\text{toy}}, v_{\text{toy}})$ are obtained analytically from torus coordinates. Energy at sample points is computed, then the Levenberg-Marquardt algorithm iteratively minimizes Hamiltonian deviation δH to obtain final $S_{\mathbf{n}}$. Similarly, $\partial S_{\mathbf{n}}/\partial J$ are obtained, then target angles θ via Eq. (67) [24].

Thus, using real torus coordinates (target torus as known), appropriate coefficients $S_{\mathbf{n}}(J)$ and $\partial S_{\mathbf{n}}/\partial J$ transform to toy torus coordinates; analyticity in the toy potential yields $(x_{\text{toy}}, v_{\text{toy}})$, then point transformations give (x, v). Point transformations are canonical; in the program, they are used only when $J_z/J_r > 0.05$ and torus fitting fails without them [12]. Appendix Fig. A1 in [12] illustrates that for good toy models, higher-order generating function Fourier coefficients improve orbit fitting.

2.3.2 Iterative Torus Construction As described, the TM program [11, 12] provides conversion between phase-space and torus coordinates, primarily computing (x, v) from (J, θ). For computing torus coordinates, besides repeated torus construction until phase-space coordinates pass through given points, TACT improves this via Iterative Torus Construction (ItTC) [22, 24]—an iterative "Stäckel fudge + TM" algorithm. It uses Stäckel fudge to compute trial torus coordinates, finds the nearest phase-space point on the torus, then adds residuals and iterates [22].

Given an initial phase-space point (x, v), first compute a torus coordinate (J_S, θ_S) via Stäckel fudge. Then use torus mapping to obtain a torus with this action, find the phase-space point (x_S, v_S) on this torus closest to the given point by minimizing $\eta = |\Omega|^2(x - x_S)^2 + (v - v_S)^2$. Compute the torus coordinates $J_P(x_S, v_S), \theta_P(x_S, v_S)$ at this point, obtaining error $\Delta J = J_P - J_S$. Assuming this error varies slowly, add it back as residual to get improved action $J_2 = J_S + \Delta J$.

Construct a new torus with action J_2, find the closest point again, and iterate. This process converges to more precise actions.

2.3.3 Orbit Integration Fitting of Generating Function Sanders and Binney [21, 24] present the O2GF method: orbit time-series sampling $(x, v)(t_i) \rightarrow$ compute toy torus coordinates $(J_{\text{toy}}, \theta_{\text{toy}})(t_i) \rightarrow$ solve generating function coefficient matrix. Unlike TM (based on Hamiltonian deviation δH minimization at a single time), O2GF applies torus mapping to a motion process (time series), computing actions, angles, and frequencies along the orbit.

Procedure: Given a real phase-space initial point, integrate an orbit for time $N_T T$ (where T is the period of a circular orbit with same energy), sampling N_{samp} phase-space points. At each point, analytically compute toy potential $(\theta_{\text{toy}}, J_{\text{toy}})$. Substitute these into Eqs. (66) and (67) to solve for unknowns:

target actions J, generating function Fourier coefficients $S_{\mathbf{n}}$ and derivatives $\partial S_{\mathbf{n}}/\partial J$, target frequencies Ω, and constant $\theta(0)$.

(1) Orbit integration and classification: In non-rotating general triaxial potentials, two basic non-resonant orbit types exist: loop and box orbits [1, 21]. Different initial conditions produce different orbit types, so TACT matches appropriate toy potentials to given orbit classes with matching torus geometry.

(2) Toy potential fitting: For loop orbits, the toy potential is isochrone: $\Phi_{\rm iso} = -\frac{GM}{b+\sqrt{b^2+r^2}}$. For box orbits, it's harmonic oscillator: $\Phi_{\rm har} = \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$. Orbit type determines the choice (based on L_x and L_z direction changes). Before using toy potentials, sample the orbit to fit mass and scale length parameters.

(3) Generating function: At each sample time t_i, compute toy action-angle variables and generate Eq. (66) with unknown J and $S_{\mathbf{n}}$. The implementation technique: "We cannot solve these equations exactly because they have infinitely many unknowns. We can include only finite terms on the right side, so exact equality fails. The correct approach is: square the difference between both sides of each equation, sum these squared differences across all equations (i.e., orbit integration), and minimize this total sum" [21].

The total squared sum is: $E_1 = \sum_{i,k} \left[J_{{\rm toy},k}(t_i) - J_k - 2\sum_{\mathbf{n}} n_k S_{\mathbf{n}}(J)\cos(\mathbf{n}\cdot\theta(t_i))\right]^2$, where $k = 1, 2, 3$ indexes dimensions, $\mathbf{n}$ is limited to finite precision $|\mathbf{n}| \leq N_{\rm max} \approx 6$, and the outer sum over i traverses all sample points—this is the "orbit integration" meaning.

Minimizing E_1 by setting partial derivatives to zero yields a matrix equation $\mathbf{A}_{J_{\rm toy}}\mathbf{x}_{J_{\rm toy}} = \mathbf{b}_{J_{\rm toy}}$, where $\mathbf{x}_{J_{\rm toy}} \equiv (J, S_{\mathbf{n}})$. Solving gives target actions J and generating function coefficients $S_{\mathbf{n}}$.

Similarly, minimizing $E_2 = \sum_{i,k}\left[\theta_k(0) + \Omega_k t_i - \theta_k(t_i) - 2\sum_{\mathbf{n}} \partial S_{\mathbf{n}}/\partial J_k \sin(\mathbf{n}\cdot\theta_{\rm toy}(t_i))\right]^2$ yields frequencies Ω and $\partial S_{\mathbf{n}}/\partial J$.

Averaging toy actions over toy angle space provides a simpler method—Average Generating Function (AvGF) [24, 29]: $\langle J_{\rm toy}\rangle = J$.

2.4 Comparison of TACT Methods

Running TACT [23, 24] with several methods to estimate actions yields Fig. 2 [Figure 2: see original paper]. The potential used is an axisymmetric Milky Way mass model from McMillan [57] (thin/thick stellar disks, HI and H_2 gas disks, bulge, dark matter halo); parameters are provided on the author's GitHub. The stellar orbit is a typical thin-disk orbit with initial conditions $(x, v) = (8.178, 0.1, 0.225; 20.13, 187.01, 14.95)$ in kpc and km/s. Methods shown: Fudge v1 and v2 (Stäckel fudge, with v1 using Eq. (48) for focal length, v2 using shell orbits); Fit (Stäckel fitting); ItTC (iterative torus); O2GF (orbit integration fitting); AvGF (average generating function); CAA and SAA (cylindrical and spherical adiabatic approximations, introduced next section). Each orbit point independently computes actions, causing temporal fluctuations.

Sanders and Binney [24] provide detailed performance comparisons across orbit types, precision, and timing (see their Figs. 2-6). Overall, convergent methods achieve higher precision but are 1-3 orders of magnitude slower. O2GF has highest precision but longer runtime; Stäckel fudge is fastest and most economical for large calculations, but less accurate, especially for box orbits. For axisymmetric potentials, torus iteration (ItTC) improves Stäckel fudge precision at increased cost. TACT's O2GF and fudge methods also work for triaxial potentials.

2.5 Other Methods and Programs

Beyond Binney's recent work, other Stäckel-based methods exist, all publicly available.

2.5.1 Analytic "Approximate Integrals of Motion" Bienaymé [54] provides a method to compute gravitational potentials at all positions using representative points on ellipsoidal axes, fitting general potentials to Stäckel form. Then, using Stäckel integral expressions, "approximate integrals of motion" (quasi-integrals, particularly quasi-I_3) can be expressed analytically. This suits potentials near Stäckel form.

From Eq. (21), substituting P_τ^2 gives: $\Phi = -\frac{1}{4}\left[\frac{\zeta(\lambda)(\lambda+\alpha)(\lambda+\beta)(\lambda+\gamma)}{(\lambda-\mu)(\lambda-\nu)} + \text{cyclic}\right]$. Continuity at $(-\alpha,-\beta,-\gamma)$ requires $\Phi(-\alpha,-\beta,-\gamma) = 0$. Evaluating at $(\lambda,-\beta,-\gamma)$ yields $\zeta(\lambda) = -\Phi(\lambda,-\beta,-\gamma)/[4(\lambda+\alpha)]$, with similar expressions for $\eta(\mu), \kappa(\nu)$. Substituting back gives $\Phi(\lambda,\mu,\nu)$ expressed solely by potentials on "ellipsoidal coordinate axes."

Similarly, integrals J, K can be expressed analytically. Using these in Eq. (15) gives I_2, I_3 expressions: $I_2 = \psi(\lambda,\mu,\nu) + \frac{1}{2}(\alpha-\beta)v_x^2 + \frac{1}{2}(\alpha-\gamma)v_z^2 + \frac{1}{2}L_y^2/(\alpha-\beta)$, with similar forms for I_3 [54]. This method computes "per-point" integrals requiring current position, velocity, and angular momentum. Bienaymé [54] uses high-precision numerical orbits to study errors and performs Jeans equation tests.

2.5.2 Adiabatic Approximations For axisymmetric potentials, approximate action formulas exist by assuming separability in cylindrical or spherical coordinates.

(1) Cylindrical adiabatic approximation (CAA) [45]: For disk orbits, vertical frequency $\Omega_z \gg$ radial frequency Ω_r, so the vertical oscillation potential varies slowly during radial oscillation, making J_z adiabatically invariant. This yields approximate formulas: $J_z = \frac{1}{\pi}\int_{z_-}^{z_+} dz\, p_z$, $J_R = \frac{1}{\pi}\int_{R_{\rm peri}}^{R_{\rm apo}} dR\, p_R$, requiring vertical energy and effective potential.

(2) Spherical adiabatic approximation (SAA): Assumes adiabatic invariance along prolate spheroidal coordinate spheres. Approximate actions are:

$J_\nu = \frac{1}{\pi}\int_{\nu_-}^{\nu_+} d\nu\, p_\nu$, $J_\lambda = \frac{1}{\pi}\int_{\lambda_-}^{\lambda_+} d\lambda\, p_\lambda$, with potentials $\Phi_\nu(\nu) = \frac{L^2}{2R^2(\lambda,\nu)} + \Phi(\lambda,\nu) - \Phi(\lambda, c^2)$, etc.

2.5.3 Related Programs Beyond Sanders & Binney's TACT and TM programs, other galactic dynamics software is publicly available:

1. **Gadget** and its successor **Arepo**: N-body + hydrodynamics simulation codes for collisionless N-body simulations (especially dark matter halo dynamics).
2. **galpy**: Bovy's [30] Python library for galactic dynamics, providing numerical orbit integration, action-angle variables, distribution functions, etc.
3. **AGAMA**: Vasiliev's [31] action-based galaxy modeling architecture, offering potential computation from analytic density profiles or N-body models, orbit integration, position-velocity to action-angle conversion, distribution function computation, and iterative construction of self-consistent multi-component galaxy models.

2.5.4 Deep Learning for Action Computation A recent development: Ibata et al. [32] apply deep learning to action computation. Their AI algorithm maps phase-space (x, v) to action-angle variables (θ, J) and obtains acceleration fields in static potentials. The core principle resembles generating function methods: fit a toy potential from data, then iterate canonical transformation generating functions G and P to obtain the target torus (θ'', J''). The networks are: $G = \mathrm{net}_G(\cos\theta, \sin\theta, J')$, $P = \mathrm{net}_P(\cos\theta, \sin\theta)$.

Their program **ACTIONFINDER** (Python/PyTorch) achieves higher precision than Stäckel fudge, with advantages: less sensitive to initial conditions than early TM; doesn't require fair samples of system orbits; needs no prior Hamiltonian or potential; deep neural networks generate more flexible nonlinear functions than Fourier coefficients, potentially fitting more general dynamical systems. Future development may make AI approaches more widely applicable and performant than traditional methods.

3 Applications to Milky Way Data

A key application of actions is in distribution functions—using action-parameterized DFs for galactic modeling [15]. Direct DF forms are generally unknown. In practice, researchers adopt prior DF forms for components (disk, bulge, halo) and fit parameters to data.

Until ~10 years ago, DF-based modeling was difficult (see reviews by Read [61] and Binney [15]); Jeans equation-based "moment methods" were common. Assuming steady state ($\partial f/\partial t = 0$), multiplying the collisionless Boltzmann equation by velocity components and integrating yields Jeans equations. For the Milky Way, axisymmetry is often assumed ($\partial f/\partial \phi = 0$, $\partial f/\partial v_\phi = 0$), giving cylindrical coordinate Jeans equations that relate observables like number density ν (zeroth moment), mean velocities v_i (first moment), and velocity dispersion tensor σ_{ij} (second moment). Moment methods are fast and explore large parameter spaces without assuming DF forms, but have disadvantages: data must be binned for second moments (though velocities aren't always Gaussian); DF forms aren't used (losing full distribution information); Jeans equations are generally non-closed and may yield non-physical solutions.

Earlier work even simplified further to "1D approximation" [61] or "Kz force" method [39], using only vertical Jeans equation and Poisson equation $4\pi G\rho = \partial^2\Phi/\partial z^2$, or $K_z = 2\pi G\Sigma$ using surface density Σ.

The following subsections review recent representative Milky Way modeling using 3-integral (or action) DFs. For observed stellar disk components, these works typically use action-based quasi-isothermal DF models (or variants). Dark matter halos use mass distribution models (e.g., NFW) rather than DF models, though Piffl et al. [51] and Binney & Piffl [49] began attempting direct halo DF modeling with limited success due to data and software limitations [15]. We also introduce alternative disk DF models.

These works primarily used RAVE and SDSS data. With Gaia astrometry/photometry and large spectroscopic surveys (e.g., China's LAMOST), DF-based methods are expected to flourish.

3.1.1 Bovy & Rix Model

Bovy & Rix [44] modeled SEGUE G-dwarf samples ($5 < R_{\rm GC} < 12$ kpc, $0.3 < |z| < 3$ kpc) to infer stellar and dark matter mass distributions. Data were divided into 43 mono-abundance populations (MAPs). Each MAP's phase-space data independently constrained a parameterized Milky Way potential and an action-based quasi-isothermal stellar DF, assuming each MAP's data constrain vertical force K_z at different radii. Results include total surface density $\Sigma_{1.1}(R)$ within $|z| < 1.1$ kpc and vertical force $K_{z,1.1}(R)$ for $4.5 < R_{\rm GC} < 9$ kpc, plus dark matter halo contributions.

(1) Stellar DF model: Quasi-isothermal DF (qDF) [45] is common, modeling tracer distributions as isothermal components with constant velocity dispersions (Gaussian distributions) [39]. Each MAP in [44] uses qDF from [46]: $f_{\sigma_R}(J_R, L_z) \times \frac{\nu}{2\pi\sigma_z^2} \exp\left(-\frac{\nu J_z}{\sigma_z^2}\right) \times \exp\left(-\frac{\kappa J_R}{\sigma_R^2}\right) \times [1+\tanh(L_z/L_0)] \exp(-\Omega/\sigma_R^2)$, where κ, Ω, ν are epicyclic, circular, and vertical frequencies, R_c is the radius of a circular orbit with angular momentum L_z. Tracer density radial profiles and velocity dispersions are assumed exponential: $n(R_c) \propto \exp(-R_c/h_R)$, $\sigma_R(R_c) = \sigma_{R,0} \exp(-(R_c - R_0)/h_{\sigma_R})$, etc. Actions are computed using Binney's [13] Stäckel fudge.

(2) Milky Way potential model: A complete 3D model includes "bulge + stellar disk + gas disk + dark matter halo" with many free parameters. Disk density uses a double-exponential model: $\rho_{\rm disk}(R, z) = \rho_d \exp(-R/R_d - |z|/z_h)$.

Bulge uses Hernquist ($\alpha = 1, \beta = 4$) or Jaffe ($\alpha = 2, \beta = 4$) models. Dark matter halo uses a power-law model: $\rho_{\rm DM}(R, z) \propto 1/r^{\alpha}$.

In fitting each MAP's qDF, most parameters are fixed (disk scale height z_h, circular velocity $v_c(R_0)$ and its flatness), leaving disk scale length R_d and halo's radial force contribution at R_0 as free.

(3) Fitting procedure: Maximum likelihood estimation is used. The observed distribution is: $\lambda(l, b, D, v, r, g-r, [\mathrm{Fe/H}]) = \rho(r, g-r, [\mathrm{Fe/H}]|R, z, \phi) \times \mathrm{qDF}(x, v) \times |J(R, z, \rho; l, b, D)| \times S(\mathrm{plate}, r, g-r)$, where D is distance, l, b are Galactic coordinates, $g-r$ is color, [Fe/H] metallicity, v observed velocity, and S is selection function. Normalized likelihood per data point: $\ln L_{i,\rm DF} = \ln \mathrm{qDF}(J(x_i, v_i)|p_\Phi, p_{\rm DF}) - \ln \int d[\mathrm{observables}]\lambda(\cdots|p_\Phi, p_{\rm DF})$. Total likelihood includes outlier treatment: $\ln L = \sum_i \ln[(1-p_{\rm out})L_{i,\rm DF} + p_{\rm out} L_{i,\rm out}]$.

(4) Results: Best-fit disk scale length $R_d = 2.15 \pm 0.14$ kpc, consistent with photometric stellar mass distributions. Combining 43 MAPs yields $\Sigma_{1.1}(R) = 70 M_\odot \mathrm{pc}^{-2} \exp(-(R-R_0)/2.5\ \mathrm{kpc})$ and $K_{z,1.1}(R) = 68 M_\odot \mathrm{pc}^{-2} \exp(-(R-R_0)/2.7\ \mathrm{kpc})$. They compare full Jeans equation and vertical force methods, finding dark matter halo radial profile constraints: at 95% confidence, $\rho_{\rm DM}(r; U_{R_0}) \propto 1/r^{\alpha}$ with $\alpha < 1.53$.

3.1.2 Piffl et al. Model

Similar to Bovy & Rix [44], Piffl et al. [50] used ~200,000 giant stars within $|z| \sim 1.5$ kpc to measure vertical mass density distribution and local dark matter contribution. Their model includes five components: gas disk, thin disk, thick disk, bulge, and dark matter halo.

(1) Mass model: Disk density: $\rho_1(R, z) = \Sigma_0 \exp(-R/R_d) \times \mathrm{sech}^2(z/2z_d)$, with non-zero $R_{\rm hole}$ creating a central cavity for gas disk. Bulge and halo use: $\rho_2(R, z) = \rho_0 m^{-\gamma}(1+m)^{\gamma-\beta} \exp[-(m r_0/r_{\rm cut})^2]$, where $m(R, z) = \sqrt{R^2 + (z/q)^2}/r_0$. The mass model has 8 free parameters (3 each for thin/thick disks, 2 for halo), with disk radial scale lengths equal. Table 1 in [50] gives fixed parameters.

(2) Distribution function: $f(J_r, J_z, L_z) = f_{\rm disk} + F_{\rm halo} f_{\rm halo}$, with $f_{\rm disk} = f_{\rm thin} + F_{\rm thick} f_{\rm thick}$. Disk DFs are action-based, estimated via Stäckel fudge [13], using quasi-isothermal components: $f_{\sigma_r}(J_r, L_z) = \frac{\Omega}{2\pi\sigma_r^2}[1 + \tanh(L_z/L_0)] \exp(-\kappa J_r/\sigma_r^2)$, etc. Thin disk velocity dispersions depend on age τ: $\sigma_r(L_z, \tau) = \sigma_{r,0}(\tau + \tau_1)/(\tau_m + \tau_1) \exp[(R_0 - R_c)/R_{\sigma_r}]$, etc. Assuming exponentially declining star formation with timescale t_0, the thin disk DF integrates over age: $f_{\rm thin} = \frac{1}{t_0(e^{\tau_m/t_0} - 1)} \int_0^{\tau_m} d\tau e^{\tau/t_0} f_{\sigma_r} f_{\sigma_z}$. A stellar halo component $f_{\rm halo}$ from Posti et al. [27] prevents thick disk distortion.

(3) Fitting: Least-squares fitting yields best parameters and density profiles for each component, plus local contributions to circular velocity and midplane surface density from baryons and dark matter.

3.2 Milky Way Modeling Based on $f(E, L_z, I_3)$

Besides action-based models, earlier DFs used three integrals E, L_z, I_3 directly, with I_3 approximated using Stäckel forms.

3.2.1 Bienaymé et al. Model Bienaymé et al. [52] used RAVE red clump stars to study radial forces and local dark matter contribution, assuming a Stäckel potential: $\Phi(\lambda,\nu) = -\frac{h(\lambda)-h(\nu)}{\lambda-\nu}$, where $h(\tau) = GM/(\tau+q)$ (Kuzmin-Kutuzov-like). The third integral is: $I_3 = \frac{(\nu+\gamma)[G(\lambda)-E]-(\lambda+\gamma)[G(\nu)-E]}{(\gamma-\alpha)(\lambda-\nu)} - \frac{L_z^2}{2(\gamma-\alpha)}$, where (r, z, ϕ) are cylindrical coordinates and $z_0 = \pm\sqrt{\gamma-\alpha}$ are focal points on the z-axis.

The stellar disk DF assumes a three-integral model: $f(E, L_z, I_3) = \frac{2\Omega(R_c)\Sigma(L_z)}{2\pi\kappa\sigma_r^2(L_z)} \times [1+\tanh(L_z/L_0)]\exp\left(-\frac{E-E_{\rm circ}}{\sigma_r^2}\right) \times \left(\frac{I_3-I_3^-}{I_3^+-I_3^-}\right)^{-1/2}$. They fit parameters via χ^2 minimization of model vs. observed moments (density $\nu(z)$ and vertical dispersion $\sigma_{zz}(z)$) for three metallicity samples, plus a term for circular velocity difference between 7.5 and 9.5 kpc.

Results: local disk surface density $\Sigma_{\rm disc}(R_0) = 44.4 \pm 4.1 M_\odot {\rm pc}^{-2}$, local dark matter density $\rho_{\rm DM}(z=0) = 0.0143 \pm 0.0011 M_\odot {\rm pc}^{-3}$, Oort limit baryonic and total densities 0.077 ± 0.007 and $0.091 \pm 0.0056 M_\odot {\rm pc}^{-3}$.

Bienaymé et al. [53] also approximate general axisymmetric potentials as locally Stäckel: $V(R,z) = \tilde{V}(\lambda,\nu)$ at position $(R_1, z_1) = (\lambda_1, \nu_1)$, with third integral: $I_3 = \Psi(R,z) - \frac{L^2-L_z^2}{2(\gamma-\alpha)}$, where $\Psi(R,z) = -\frac{V(R,z)-V(\lambda,0)}{\lambda+z^2/(\gamma-\alpha)}$. Bienaymé [54] also gives approximate analytic expressions for I_2, I_3 in general triaxial potentials (Section 2.5.1), discussing DF models $f = \exp(bI_2 + cI_3)$ and Jeans equation tests.

3.2.2 A More Parameterized DF Model Famaey et al. [42] construct a complex $f(E, L_z, I_3)$ DF to fit real galaxy disks via linear combinations. They: (1) compute I_3 using Stäckel expressions; (2) extend earlier 2-integral DFs [40] with more parameters.

The DF form is: $F(E, L_z, I_3) = f(L_z)\left(\frac{E-S_{z0}(L_z)}{S_z(L_z)-S_{z0}(L_z)}\right)^{p+qE+rL_z^2+sI_3}$, where $f(L_z) = [1+e^{-aL_z}]^{-1}\exp(-L_z^2/2\sigma_{Lz}^2)$. The (E, L_z^2) plane has boundary families $E > \Phi - L_z^2/(2R^2)$ and envelope $E = S_{z0}(L_z)$.

Moments are computed by transforming velocity-space integrals to integral-of-motion space: $\mu_{l,m,n}(\lambda,\nu) = \int dE \int dL_z^2 \int dI_3\, \omega(\lambda-\nu) \times (\cdots) f(E, L_z, I_3)$, where the integration region is bounded by lines where the integrand vanishes. The region is parameterized by linear combinations of boundary lines, converting to new integration variables and solving for combination coefficients, enabling 1D numerical integration (see [42] Sections 4.2-4.3). This allows fitting real disks via linear combinations of DF components on a configuration-space grid $(\bar{\omega}_i, z_i)$, minimizing variance $\chi^2 = \sum_i \left[\sum_\Lambda c_\Lambda \mu_{0,0,0}^{(\Lambda)}(\bar{\omega}_i, z_i) - \rho_0(\bar{\omega}_i, z_i)\right]^2$ until

χ^2 minima stabilize.

4 Discussion and Summary

The preceding focused on quasi-steady-state galactic DFs and modeling. However, self-gravitating systems differ fundamentally from common thermodynamic systems, potentially limiting these methods' applicability. For non-steady structures and evolution, or few-body systems, whether action-based methods remain effective—and how to adapt them—are important research questions.

This chapter examines: (4.1) self-gravitating system dynamics from dynamical systems theory; (4.2) recent progress applying action-angle methods to non-integrable phenomena (resonances, bars); (4.3) applications to non-equilibrium processes—tidal streams, including formation, characterization in action/angle space, and using streams to constrain host galaxy potentials; and (4.4) concluding remarks.

4.1 Dynamical Systems Theory Perspective

Gravitational N-body systems are long-range interacting systems. Unlike short-range systems, interactions between any particles cannot be ignored (potential $\psi \propto r^{-\alpha}$ with $\alpha < d$, space dimension), and macroscopic quantities related to potential lack additivity. Self-gravitating systems can develop core-halo structures preventing maximum entropy, thus lacking true equilibrium. Relaxation can be faster (e.g., violent relaxation [55, 58]). These features make self-gravitating systems distinct from statistical physics' short-range systems.

From dynamics, action-angle coordinates transform $2n$-dimensional phase space of integrable systems into n-dimensional tori T^n. If non-zero integers k_i exist such that $\sum_i k_i \Omega_i = 0$, resonance occurs. Poincaré sections reduce dimensionality for studying motion. For 3D integrable systems (e.g., Stäckel potentials), phase space shows 3-tori; rational frequency ratios produce periodic motion (discrete fixed points in sections), while irrational ratios yield quasi-periodic motion (closed curves in sections).

Under non-integrable perturbations (near-integrable systems), KAM theorem gives conditions for persistent invariant tori. For $n > 2$ degrees of freedom, n-dimensional KAM tori cannot partition the $(2n-1)$-dimensional energy surface, allowing Arnold diffusion—highly complex behavior with unstable orbits threading between stable ones, generating chaos. Nonlinear resonances destroy rational tori and cause chaotic motion [63]. Modern dynamical systems theory offers more rigorous definitions and awaits broader application to self-gravitating systems.

4.2 Applications to "Non-Integrable" Galactic Phenomena

Most galaxies are non-integrable; chaos prevents precise long-term computation of individual stellar orbits. Nevertheless, for quasi-steady stellar ensembles (galaxies), Binney's work shows statistical properties like distributions can be reliably obtained via action-angle methods. Recent interest focuses on applying these methods to resonances and chaos.

Binney [16, 17] studied applying torus mapping to bar resonances in the Milky Way. For bar perturbation strength, he examined applicability to outer Lindblad resonance (OLR), corotation resonance (CR), and inner Lindblad resonance (ILR). Torus mapping fits orbits trapped at OLR and CR well, but performs poorly for ILR orbits deeper inside the bar. Applying this to solar neighborhood velocity data, Binney [17] found CR fits observations, while OLR is not supported—demonstrating practical utility for resonance trapping in barred galaxies.

4.3 Applications to Non-Equilibrium Processes: Tidal Streams

When a star cluster or dwarf galaxy (progenitor) orbits a larger galaxy (host), tidal forces stretch it, forming long filamentary structures—tidal streams [33]. Action-angle space provides the most convenient framework for characterizing stream formation and properties [34, 35].

While progenitor stars remain bound, their actions in the host potential are not conserved. Once tidal stripping occurs, neglecting progenitor and stream self-gravity, the stripped star's action remains constant at its 剥离 value. Following Eyre & Binney [36], let the progenitor orbit have action J_0 in fixed background potential $H(J)$. For a stream star with action J (near J_0), expand Hamiltonian: $H(J) \approx H(J_0) + \Omega_0 \cdot \delta J + \frac{1}{2}\delta J^T \cdot \mathbf{D} \cdot \delta J$, where $\delta J = J - J_0$, $\Omega_0 = \partial H/\partial J|_{J_0}$, and $\mathbf{D}$ is the Hessian matrix. The star's frequency is $\Omega(J) = \Omega_0 + \mathbf{D} \cdot \delta J$.

Stream members have distributions in action and angle with dispersions ΔJ and $\Delta\theta_0$. After formation, ΔJ remains constant while angle dispersion grows linearly: $\Delta\theta(t) = t\Delta\Omega + \Delta\theta_0 \approx t\Delta\Omega$, with $\Delta\Omega = \mathbf{D} \cdot \Delta J$. The symmetric matrix $\mathbf{D}$ maps action-space distribution to angle-space distribution linearly. Its eigenvectors $\hat{e}_n$ and eigenvalues λ_n determine stream stretching: $\Delta\Omega \approx \hat{e}_1(\lambda_1 \hat{e}_1 \cdot \Delta J)$ along the maximum eigenvalue direction [20].

Technical details on necessary potential conditions for stream formation are in Sanders' PhD thesis [20], particularly Chapter 6 on using stream data to constrain host galaxy potentials.

Note: When progenitors are small, stream tracks approximate progenitor orbits to first order. Past "orbit-fitting" methods (e.g., [38]) simply fit host potentials assuming streams are orbits. However, stream tracks need not coincide with any orbit [36], causing significant potential biases [20].

4.4 Concluding Remarks

This review aims to introduce new developments in galactic dynamical modeling—action computation methods and recent applications of action/integral-based DFs to Milky Way data—plus extensions to non-integrable phenomena (resonances, bars) and non-equilibrium processes (tidal streams), hoping to contribute to China's "Gaia + LAMOST" era. A second goal is clarifying theoretical foundations. Stäckel potentials are the only exactly integrable galactic systems; thus nearly all analytic modeling work builds on Stäckel theory. Chapter 1 therefore derives from N-particle classical mechanics (Hamilton's equations, Liouville's equation) through mean-field approximation to collisionless Boltzmann equation, then introduces integrals of motion, canonical torus coordinates (actions and angles), and finally Stäckel theory's analytic formulas. Understanding this lineage may enable Chinese researchers not only to apply Binney's methods but also to innovate theoretically and methodologically.

References: (preserved as in original)

Appendix A: Numerical Steps for Computing Actions, Frequencies, and Angles

(Details of numerical implementation from Sanders & Binney's TACT program, including action integral formulas, frequency computation via Jacobian relations, and angle variable calculations with handling of singularities.)

Appendix B: Moment Computation for Famaey et al. DF Model

(Details of transforming velocity-space integrals to integral-of-motion space, parameterizing integration domains via linear combinations of boundary lines, and numerical evaluation.)

Note: Figure translations are in progress. See original paper for figures.