Critical Phenomena and Functional Renormalization Group Postprint

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Abstract

This article primarily reviews recent progress in the application of the Functional Renormalization Group (fRG) to the study of phase transitions and critical phenomena in Quantum Chromodynamics (QCD), focusing on the calculation of non-perturbative critical exponents and baryon number fluctuations associated with the QCD Critical End Point (CEP). fRG is a non-perturbative theoretical method for continuous field theory, whose fundamental principle is to continuously integrate quantum and thermal fluctuations across different scales into the theoretical framework through the evolution of the renormalization group scale from high to low energies. The article discusses various solution methods for the non-perturbative effective potential renormalization group flow equation and fixed-point equation, including Taylor expansion, $= 4 - \mathrm{d}$ expansion in spatial dimensions, and a recently proposed direct solution method for non-local potentials. It also systematically examines baryon number fluctuations, which are intimately connected to critical phenomena such as the QCD critical endpoint, and investigates possible origins of the experimentally observed non-monotonic dependence of the kurtosis—the fourth-order moment—of the net-proton number distribution on collision energy.

Full Text

Preamble

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Critical Phenomena and Functional Renormalization Group

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Abstract This article reviews recent progress in studies of quantum chromodynamics (QCD) phase transitions and related critical phenomena within the functional renormalization group (fRG) approach, focusing on nonperturbative critical exponents and baryon number fluctuations pertinent to the critical end point (CEP) in the QCD phase diagram. The fRG is a nonperturbative continuum field theory method in which quantum and thermal fluctuations are successively integrated with the evolution of the renormalization group (RG) scale from high to low energies. Various solution methods for the nonperturbative effective potential flow equations and fixed-point equations are discussed, including Taylor expansion, dimensional expansion in = 4 - d, and recently proposed direct solution methods for non-local potentials. The article also systematically discusses baryon number fluctuations, which are closely related to critical phenomena such as the QCD CEP, and explores possible explanations for the experimentally observed non-monotonic dependence of the kurtosis (fourth-order moment) of net-proton multiplicity distributions on collision energy.

Keywords QCD phase structure, QCD critical end point, Functional renormalization group, Critical phenomena, Relativistic heavy ion collisions

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Introduction

In recent years, the properties of strongly interacting matter under extreme conditions of high temperature and density, the QCD phase diagram, and phase structure have become frontier research topics in medium and high energy nuclear physics, relativistic heavy ion collisions, compact stars, and gravitational waves [1-5]. [Figure 1: see original paper] shows a schematic phase diagram of QCD, where the vertical axis represents temperature and the horizontal axis represents baryon chemical potential; larger chemical potential corresponds to higher density. As seen in [Figure 1: see original paper], the phase boundary in the high-temperature, low-density region is a dashed line representing a continuous crossover rather than a strict phase transition line, which has been confirmed by first-principles lattice QCD simulations [6] and is consistent with experimental measurements [7].

Lattice QCD calculations [8-10] indicate that as density or chemical potential increases, the sign problem makes lattice QCD computations increasingly difficult, with reliability typically limited to the region where _B/T 2-3 in [Figure 1: see original paper], where _B is the baryon chemical potential and T is temperature. Other theoretical methods, such as first-principles functional continuum field theory [11-15] and low-energy effective models [16-20], suggest that the continuous crossover may transform into a first-order phase transition in the high-density region, shown as the red solid line in [Figure 1: see original paper]. The endpoint of this first-order transition line, known as the QCD critical end point (CEP), is represented by the black hollow circle in [Figure 1: see original paper] and connects the first-order line to the crossover region. This reveals the special status of the CEP in the QCD phase diagram: its existence determines whether a first-order phase transition exists in the high-density region. If the CEP does exist in the phase diagram, its location must reflect the properties of nonperturbative QCD at finite temperature and density, providing a new window for studying strongly correlated QCD matter through experimental measurements.

As the endpoint of a first-order transition line, the CEP is a second-order phase transition point. From the perspective of renormalization group scale evolution, the system at the CEP lies on the critical surface of a renormalization group running fixed point [21]. We schematically illustrate this abstract parameter space around the CEP in [Figure 1: see original paper], where several black arrows represent RG flows in parameter space, showing that the CEP indeed sits at a fixed point of the RG evolution. Since Wilson introduced RG concepts and methods to second-order phase transitions and critical phenomena in the early 1970s [22-25], the field has seen breakthrough developments. We now know that critical behavior in second-order phase transitions, such as various critical exponents characterizing critical phenomena, exhibits universality—depending only on spatial dimension, symmetry, etc., but not on interaction details. Specifically, the CEP in the QCD phase diagram belongs to the Z(2) symmetry universality class, i.e., the Ising model universality class.

Currently, many heavy ion experimental facilities are searching for the CEP [7, 26-37]. Near a second-order phase transition point, the correlation length increases significantly, diverging at the CEP in the ideal thermodynamic limit. Basic thermodynamics and statistical physics tell us that certain observables, such as fluctuations of conserved charges, are particularly sensitive to the singularities caused by critical dynamics, with higher-order fluctuations being more sensitive. Over twenty years ago, it was proposed to use the non-monotonic dependence of conserved charge fluctuations on collision energy to search for the CEP in the QCD phase diagram [38-40], with references to [26]. In the first phase of the Beam Energy Scan (BES) program at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory, measurements were made of various moments of net-proton, net-charge, and net-kaon multiplicity distributions and their correlations [41-48], yielding fruitful results. In particular, recent experiments observed with 3.1σ significance the non-monotonic dependence of

net-proton multiplicity kurtosis on collision energy [46].

On the theoretical front, as mentioned earlier, lattice QCD simulations are limited by the sign problem at finite chemical potential [49]. In recent years, first-principles functional continuum field theory methods have developed rapidly [11, 15]. Functional continuum field theory calculations not only reproduce lattice results in the small chemical potential region (_B/T 2-3) but can also be extended to higher chemical potential regions, such as _B/T 3-4. Furthermore, different continuum field theory methods, such as the functional renormalization group (fRG) and Dyson-Schwinger equations, have provided estimates for the CEP location in the phase diagram, yielding consistent results within error margins, both indicating that the CEP corresponds to a chemical potential of about 600 MeV [12-14].

This article reviews recent progress in fRG studies of QCD phase transitions and critical phenomena, including calculations of nonperturbative critical exponents and baryon number fluctuations. The fRG is a nonperturbative continuum field theory method whose basic idea is to continuously integrate quantum and thermal fluctuations at different scales into the theoretical framework through the evolution of the renormalization group scale from high to low energies. fRG has been applied to many nonperturbative fields, with QCD-related fRG reviews available in references [15, 51-58]. Some important recent developments can be found in references [12, 59-61].

Functional Renormalization Group Introduction

First, the generating functional is given by:

$$Z_k[J] = \int \mathcal{D}\hat{\Phi} \exp\{S[\hat{\Phi}] + \Delta S_k[\hat{\Phi}] + J_a \hat{\Phi}_a\} \tag{1}$$

where $\hat{\Phi}$ represents all fields in a field theory and $S[\hat{\Phi}]$ is the classical action. Here we introduce a scale-dependent infrared regulator function $\Delta S_k[\hat{\Phi}]$ that suppresses quantum fluctuations with momentum modes $q < k$ in the generating functional (1) while leaving fluctuations with $q > k$ unaffected. For simplicity, a bilinear field form of the infrared regulator is typically chosen:

$$\Delta S_k[\hat{\Phi}] = \frac{1}{2}\hat{\Phi}_a \mathcal{R}^k_{ab} \hat{\Phi}_b$$

where for bosonic indices $\mathcal{R}^k_{ab} = \mathcal{R}^k_{ba}$, while for fermionic indices $\mathcal{R}^k_{ab} = -\mathcal{R}^k_{ba}$.

Taking a single-component scalar field ϕ as an example, its infrared regulator is:

$$\Delta S_k[\phi] = \frac{1}{2}\int \frac{d^{4q}}{(2\pi)^4}\phi(q)R_k(q^2)\phi(-q) = \frac{1}{2}\int \frac{d^{4q}}{(2\pi)^4}\phi(q)q^2 r_{\text{opt}}(q^2/k^2)\phi(-q) \tag{2}$$

where $r_{\text{opt}}(x) = (\frac{1}{x} - 1)\Theta(1-x)$ and $\Theta(x)$ is the step function.

Using the Fourier transform in momentum space:

$$\phi(x) = \int \frac{d^{4q}}{(2\pi)^4}\phi(q)e^{iqx}$$

From general field theory, the generating functional for connected correlation functions can be written as:

$$W_k[J] = \ln Z_k[J] \tag{3}$$

This is also known as the Schwinger functional. Field expectation values can be obtained by taking first derivatives of W_k with respect to external sources:

$$\Phi_a = \langle\hat{\Phi}_a\rangle = \frac{\delta W_k[J]}{\delta J_a} \tag{4}$$

Taking another derivative yields the propagator:

$$G_{k,ab} \equiv \frac{\delta^2 W_k[J]}{\delta J_a \delta J_b} \tag{5}$$

Performing a Legendre transform on the Schwinger functional gives the one-particle-irreducible (1PI) effective action:

$$\Gamma_k[\Phi] = W_k[J] + J_a\Phi_a - \Delta S_k[\Phi] \tag{6}$$

Introducing the notation:

$$\Gamma_k[\Phi] + \Delta S_k[\Phi] = \gamma_a^b \Phi_a J_b$$

where when a, b are fermionic indices, $\gamma_a^b = \delta_a^b$, and when a, b are bosonic indices, $\gamma_a^b = \delta_a^b$. This leads to:

$$J_b = \gamma_a^b \frac{\delta(\Gamma_k[\Phi] + \Delta S_k[\Phi])}{\delta\Phi_a}$$

and the relationship between propagator and effective action:

$$G_{k,ab} = \gamma_c^a (\Gamma^{(2)}[\Phi] + \Delta S^{(2)}[\Phi])^{-1}_{cb} \tag{7}$$

where $\Gamma^{(2)}[\Phi] + \Delta S^{(2)}[\Phi] = P + F$. Here P is the two-point function matrix without interactions, whose inverse is the propagator matrix, and F is the interaction matrix, i.e., interaction vertices.

Based on the previous requirements, the regulator function should have the following asymptotic behavior:

$$\lim_{k\to\infty} R_k(q) \to \infty \quad \text{(fixed } q\text{)}$$
$$\lim_{k\to 0} R_k(q) \to 0$$

Additionally, to selectively suppress quantum fluctuations in the low-momentum region, we further require:

$$R_k(q) \sim k^2 \quad \text{for } q < k$$
$$R_k(q) \to 0 \quad \text{for } q > k$$

Clearly, there are infinitely many choices satisfying these requirements. Below we introduce two types of regulator functions commonly used in the literature.

The first is the exponential regulator:

$$R_{\text{exp},n}(q) = q^2 r_{\text{exp},n}(q^2/k^2), \quad r_{\text{exp},n}(x) = \frac{x^{n-1}}{e^x - 1} \tag{8}$$

where parameter n controls the steepness of the regulator at $q = k$.

Another regulator more suitable for analytic calculations is called the flat regulator or optimized regulator [66-67]:

$$R_{\text{opt}}(q) = q^2 r_{\text{opt}}(q^2/k^2), \quad r_{\text{opt}}(x) = (\frac{1}{x} - 1)\Theta(1 - x) \tag{9}$$

From equations (1) and (11), we can easily derive the evolution equation for the Schwinger functional with respect to k:

$$\partial_t W_k[J] = -\frac{1}{2}\text{STr}[(\partial_t \mathcal{R}_k) G_k] \tag{10}$$

Here we introduce the RG time $t = \ln(k/\Lambda)$, where Λ is some reference scale, such as the initial UV scale. In equation (10) we also introduce the notation of supertrace:

$$\mathrm{STr}[(\partial_t \mathcal{R}_k) G_k] = (\partial_t \mathcal{R}^k_{ab}) \gamma^b_c G_{k,ca}$$

Using the Legendre transform again:

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \mathrm{STr}[(\partial_t \mathcal{R}_k) G_k] \tag{11}$$

This is the flow equation for the effective action, also known as the Wetterich equation [50].

The effective action flow equation (11) can be rewritten as:

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \mathrm{STr} \left[\partial_t \ln(\Gamma^{(2)}[\Phi] + \mathcal{R}_k) \right] \tag{12}$$

where $\Gamma^{(2)}_t[\Phi]$ represents taking partial derivatives with respect to the fields twice, and the definition of t here is slightly different from before:

$$(\Gamma^{(2)}_t[\Phi])_{ab} \equiv \frac{\delta^2 \Gamma_k[\Phi]}{\delta \Phi_a \delta \Phi_b}$$

We use left and right derivatives here to absorb the extra minus sign from fermionic fields. Making the combination:

$$\Gamma^{(2)}[\Phi] + \mathcal{R}_k = P + F$$

where P is the two-point function matrix without interactions, whose inverse is the propagator matrix, and F is the interaction matrix, i.e., interaction vertices. Expanding the effective action flow equation formally in a Taylor series:

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr} \left[\partial_t \ln(P + F) \right] = \frac{1}{2} \mathrm{STr} \left[\partial_t \ln P \right] + \frac{1}{2} \mathrm{STr} \left[\partial_t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (P^{-1} F)^n \right]$$

By comparing terms with various powers of fields on both sides of the equation, we obtain flow equations for correlation functions at all orders, including propagators and vertices.

In fact, there is a simpler method to conveniently derive flow equations for correlation functions at all orders, which borrows Feynman rules completely identical to perturbation theory. First, we define a general n-point 1PI correlation function, or n-point vertex:

$$V^{(n)}_{k,\Phi_{a_1}\cdots\Phi_{a_n}} \equiv -\Gamma^{(n)}_{k,\Phi_{a_1}\cdots\Phi_{a_n}} = \frac{\delta^n \Gamma_k[\Phi]}{\delta\Phi_{a_1}\cdots\delta\Phi_{a_n}} \tag{13}$$

The flow equation for vertex $V^{(n)}_k$ can then be expressed as:

$$\partial_t V^{(n)}_{k,\Phi_{a_1}\cdots\Phi_{a_n}} = \text{sum of all one-loop correction diagrams of } V^{(n)}_{k,\Phi_{a_1}\cdots\Phi_{a_n}} \tag{14}$$

More detailed discussions can be found in reference [15].

O(N) Scalar Theory

Near the critical point of a continuous phase transition, the system's correlation length diverges, and microscopic details are masked by strong fluctuations near the critical point. The behavior of physical quantities near the critical point can usually be described by critical exponents, which depend only on the spatial dimension and symmetry of the system. The d-dimensional O(N) theory precisely covers different spatial dimensions and symmetries, and can describe second-order phase transitions of most real physical systems near the critical point.

Wilson's renormalization group [22-25] is a powerful theoretical tool for describing phase transitions. Second-order phase transitions are described by stable fixed points on the critical surface determined by RG flow equations. The fRG further includes nonperturbative information in the RG flow, allowing more precise descriptions of phase transitions at smaller spatial dimensions.

First, consider the effective action of O(N) scalar theory:

$$\Gamma_k[\phi] = \int d^{dx}\left[\frac{Z_{\phi,k}}{2}(\partial_\mu\phi)^2 + V_k(\rho) - h\phi_0\right] \tag{15}$$

where h is an external field, $V_k(\rho)$ is a general function of the field, often called the effective potential, with:

$$\rho = \phi^2/2, \quad \phi^2 = \sum_{i=0}^{N-1}\phi_i^2$$

Clearly, $V_k(\rho)$ has O(N) symmetry. In the Wetterich equation, we need to calculate the two-point function in momentum space:

$$\Gamma^{(2)}_{\phi_i\phi_j}(q',q) \equiv \frac{\delta^2\Gamma_k[\phi]}{\delta\phi_j(q)\delta\phi_i(q')} = (2\pi)^d\delta^d(q'+q)\left[(Z_{\phi,k}q^2 + V_k'(\rho))\delta_{ij} + 2\rho V_k''(\rho)\delta_{i0}\delta_{j0}\right] \tag{16}$$

We choose the zero component (longitudinal) of the field to carry the field expectation value while keeping expectation values in other directions (transverse) zero:

$$\Gamma^{(2)}_{i\neq 0,k} = Z_{\phi,k}q^2 + V'_k(\rho)$$
$$\Gamma^{(2)}_{i=0,k} = Z_{\phi,k}q^2 + V'_k(\rho) + 2\rho V''_k(\rho)$$

The regulator function in momentum space is written as:

$$\mathcal{R}_{k,ij}(q',q) = (2\pi)^d \delta^d(q'+q)\delta_{ij} Z_{\phi,k} q^{2r}(q^2/k^2) \tag{17}$$

The derivative of the regulator with respect to the RG scale is:

$$\partial_t \mathcal{R}_{k,ij}(q',q) = (2\pi)^d \delta^d(q'+q)\delta_{ij} Z_{\phi,k} x k^2(\eta r(x) + 2x r'(x)) \tag{18}$$

where $x = q^2/k^2$ and the anomalous dimension is:

$$\eta = -\frac{\partial_t Z_{\phi,k}}{Z_{\phi,k}}$$

Substituting these into the Wetterich equation and performing the following dimensionless rescaling:

$$\tilde{\rho} = Z_{\phi,k} k^{d-2}\rho, \quad u_k(\tilde{\rho}) = k^{-d} V_k(\rho)$$

we obtain the dimensionless effective potential flow equation:

$$\partial_t u(\tilde{\rho}) = -d u(\tilde{\rho}) + (d-2+\eta)\tilde{\rho} u'(\tilde{\rho}) + C\left[(N-1)I(m_T^2) + I(m_L^2)\right] \tag{19}$$

where the function I has a specific form depending on the choice of regulator. In principle, the behavior of physical quantities at a fixed point should not depend on the regulator choice. However, in practical calculations, we typically need to truncate our theory. For example, the commonly used local potential approximation (LPA) considers only the lowest order in the gradient expansion, which introduces some regulator dependence. In this case, we need to choose an appropriate regulator to minimize this dependence and accurately describe the real physics even with low-order truncations. More detailed discussions on regulator selection and optimization can be found in references [66-67].

Here we adopt the regulator introduced in equation (9):

$$r_{\text{opt}}(x) = \left(\frac{1}{x} - 1\right) \Theta(1 - x) \tag{20}$$

For this regulator, we have $I(x) = 1/(1 + x)$. The coefficients in equation (19) are:

$$C = \frac{2}{d} \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \tag{21}$$

Gaussian Fixed Point

Temporarily ignoring the anomalous dimension and setting $\eta = 0$, we expand equation (19) in a Taylor series around $\tilde{\rho} = 0$ [68]:

$$u(\tilde{\rho}) \simeq \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \tilde{\rho}^n \tag{22}$$

where the expansion coefficients λ_n correspond to dimensionless $2n$-point coupling strengths without momentum dependence. In particular, λ_1 is the curvature mass at zero field. Expanding to second order and ignoring the contribution of four-point interactions to the flow of the four-point strength λ_2, we obtain:

$$\partial_t \lambda_1 = 2\lambda_1 - C(N + 2)\lambda_2(1 + \lambda_1) \tag{23}$$

$$\partial_t \lambda_2 = (4 - d)\lambda_2 - 2C(N + 8)(\lambda_2)^2(1 + \lambda_1) \tag{24}$$

This gives the β functions for the corresponding coupling strengths:

$$\partial_t \lambda_i = \beta_i(\lambda) \tag{25}$$

Setting $\beta_1 = \beta_2 = 0$, the above ordinary differential equations become algebraic equations whose solutions are fixed points of the RG equations:

$$\lambda_1^* = 0, \quad \lambda_2^* = 0 \tag{26}$$

If a fixed point of the RG equations satisfies $\lambda_i^* = 0$, we call it a trivial or Gaussian fixed point. Generally, RG flow equations like (23)-(24) cannot be solved analytically. Linearizing equation (25) near the fixed point using small quantities:

$$\delta\lambda_i = \lambda_i - \lambda_i^* \tag{27}$$

we obtain:

$$\partial_t \delta\lambda_i = \sum_j M_{ij}\delta\lambda_j \tag{28}$$

where the matrix is defined as:

$$M_{ij} \equiv \left.\frac{\partial\beta_i}{\partial\lambda_j}\right|_{\lambda=\lambda^*} \tag{29}$$

Denoting the i-th eigenvalue and eigenvector of the matrix as y_i and e_i, the general solution of the RG flow equation can be written as:

$$\delta\lambda_i(t) = \sum_j \alpha_j e_{ij} e^{y_j t} \tag{30}$$

where α_i are small constants. For the Gaussian fixed point discussed here, its eigenvalues and eigenvectors are:

$$y_1 = 2, \quad y_2 = 4 - d \tag{31}$$

$$e_1 = (1,0)^T, \quad e_2 = \left(-\frac{C(N+2)}{d}, 1\right)^T \tag{32}$$

From this we obtain the critical exponent related to correlation length $\nu = 1/y_1 = 0.5$. [Figure 2: see original paper] shows the flow of the RG equations (23)-(24) in the λ_1-λ_2 plane, with arrows indicating the direction of RG flow toward the infrared. The Gaussian fixed point at the origin is the only fixed point. Since the RG scale flows toward the infrared, i.e., $t < 0$ in equation (30), eigenvectors corresponding to positive eigenvalues flow into the fixed point, while those corresponding to negative eigenvalues flow out of it.

Figure 2: see original paper shows the flow diagram for spatial dimension $d > 4$. In this case, $y_1 < 0$ and $y_2 > 0$, so we see RG flow exiting the Gaussian fixed point along eigenvector e_1 and entering along e_2. Thus, the parameter α_1 in equation (30) is a relevant parameter, while α_2 is irrelevant. We call a fixed point with only one relevant parameter a stable fixed point, corresponding to a physical second-order phase transition. The subspace in parameter space where relevant parameters vanish is called the critical surface or critical line. Clearly, eigenvector e_2 in Figure 2: see original paper represents the critical line. As the RG flow runs, irrelevant parameters tend to zero regardless of the starting point, while the only relevant parameter diverges. We conclude that when $d > 4$, the Gaussian fixed point is the unique fixed point and also the physical fixed point. $d = 4$ is the upper critical dimension; above the upper critical dimension, critical exponents coincide with mean-field theory.

Figure 2: see original paper shows the flow diagram for $2 < d < 4$. Here, both eigenvectors e_1 and e_2 flow out of the fixed point. The Gaussian fixed point thus has two relevant parameters and is an unstable fixed point. Therefore, the Gaussian fixed point is not a physical fixed point for $2 < d < 4$, and we need to find new fixed points.

From naive dimensional analysis, the $2n$-vertex coupling strength (before dimensionless rescaling) has dimension $[\tilde{\lambda}_n] = 2n - (n-1)d$. By requiring that λ_{n+1} be a marginal parameter, i.e., $[\tilde{\lambda}_{n+1}] = 0$, we can solve for the expression of critical dimension: $d_n = 2 + 2/n$. When $d_n \leq d < d_{n+1}$, there are n relevant parameters. More complete calculations show that at the Gaussian fixed point, there are n negative eigenvalues and n eigenvectors flowing out of the fixed point. In particular, when $d = 2$, there are infinitely many negative eigenvalues near the Gaussian fixed point, implying infinitely many relevant parameters and infinitely many eigenvectors flowing out of the fixed point.

= 4 - d Expansion and Wilson-Fisher Fixed Point

From the analysis in §3, we know that the Gaussian fixed point cannot describe physical second-order phase transitions for $2 < d < 4$. In this regime, the four-point interaction strength λ_2 is already a relevant parameter, so we must consider its contribution to $\partial_t \lambda_2$ in equation (24). The flow equations then become:

$$\partial_t \lambda_1 = 2\lambda_1 - C(N+2)\lambda_2(1+\lambda_1) \tag{33}$$

$$\partial_t \lambda_2 = (4-d)\lambda_2 - 2C(N+8)(\lambda_2)^2(1+\lambda_1) \tag{34}$$

Besides the Gaussian fixed point $\lambda_1^* = \lambda_2^* = 0$, we can find from the flow equation for λ_2 that when $\epsilon = 4 - d$ is small, there exists a fixed point with $\lambda_2^* \sim \epsilon$. At this fixed point, we can make the approximation $1 + \lambda_1^* \approx 1$, and the fixed point satisfies:

$$\partial_t \lambda_1^* = 0, \quad \partial_t \lambda_2^* = 0 \tag{35}$$

The algebraic equation has solutions:

$$\lambda_1^* \sim -\frac{C(N+2)}{4-d}\epsilon, \quad \lambda_2^* = \frac{4-d}{2C(N+8)} \tag{36}$$

We see they are both of order $\mathcal{O}(\epsilon)$. This non-zero fixed point is called the Wilson-Fisher (W-F) fixed point. Linearizing the flow equations (33)-(34) near the W-F fixed point and calculating strictly to $\mathcal{O}(\epsilon)$ order, we obtain linearized flow equations similar to equation (28). At this point, the M matrix is:

$$M = \begin{pmatrix} 2 - \frac{N+2}{N+8}\epsilon & -C(N+2) \\ 0 & -\epsilon + \frac{2(N+2)}{N+8}\epsilon \end{pmatrix} \tag{37}$$

One eigenvalue and its corresponding eigenvector of this matrix are:

$$y_1 = 2 - \frac{N+2}{N+8}\epsilon, \quad e_1 = (1,0)^T \tag{38}$$

Since $y_1 < 0$, this is a relevant parameter, and the corresponding critical exponent is $\nu = 1/|y_1|$. To $\mathcal{O}(\epsilon)$ order, this is [21]:

$$\nu = \frac{1}{2} + \frac{N+2}{4(N+8)}\epsilon + \mathcal{O}(\epsilon^2) \tag{39}$$

The other eigenvalue and its corresponding eigenvector are $y_2 = \epsilon + \mathcal{O}(\epsilon^2)$ and $e_2 = (B,1)^T$, where B is a constant. Clearly $y_2 > 0$ is an irrelevant parameter.

Figure 3: see original paper shows the flow diagram of RG equations (33)-(34) for $2 < d < 4$. We can clearly see two fixed points in the plane: the Gaussian fixed point at the origin, which is unstable as analyzed previously; and another fixed point marked as μ^*, which is the W-F fixed point—a stable fixed point corresponding to a physical second-order phase transition. Figure 3: see original paper shows the critical exponent ν at the W-F fixed point to $\mathcal{O}(\epsilon)$ order, i.e., equation (39), in the plane of symmetry group parameter N and spatial dimension d. We see that ν increases monotonically as d decreases and N increases. In the large N limit, the exact result from the spherical model is $\nu = 1/(d-2)$. Since equation (39) is only accurate to $\mathcal{O}(\epsilon)$ order, it obviously cannot match the large N limit result when d is small.

At $\mathcal{O}(\epsilon)$ order, the anomalous dimension $\eta = 0$. Using scaling relations, we can calculate other critical exponents to $\mathcal{O}(\epsilon)$ order:

$$\gamma = (2-\eta)\nu = 1 + \frac{N+2}{2(N+8)}\epsilon \tag{40}$$

$$\delta = \frac{d+2-\eta}{d-2+\eta} = 3 + \epsilon \tag{41}$$

Calculation of Critical Exponents

We have calculated critical exponents to $\mathcal{O}(\epsilon)$ order, but this fails at small dimensions. A direct improvement is to avoid expanding $1/y_1$ in ϵ and directly keep the expression for y_1 in equation (38), yielding:

$$\nu = \frac{1}{2 - \frac{N+2}{N+8}\epsilon} \tag{42}$$

In the large N limit, this gives the exact result of the spherical model: $\nu = 1/(d-2)$. However, according to the Mermin-Wagner-Hohenberg (MWH) theorem [69-71], continuous spontaneous symmetry breaking does not exist in two dimensions. Therefore, in the O(N) model, when $d = 2$ and N is finite, the system has no second-order phase transition, and the critical exponent ν diverges. The ν calculated here only diverges when $N \to \infty$, so the calculation still has significant errors when d and N are small.

From naive dimensional analysis, irrelevant parameters require $[\tilde{\lambda}_n] < 0$. Combined with previous dimensional analysis of $2n$-vertex coupling strengths, this requires at least $n > d/(d-2)$. For example, when $d = 3$, $n > 3$; when $d = 2.5$, $n > 5$; when $d = 2.1$, $n > 21$. Clearly, the lower the spatial dimension, the more coupling constants need to be included in the RG flow equations, or equivalently, the higher the order of Taylor expansion needed for the effective potential in equation (22).

The method for calculating critical exponents at the W-F fixed point uses equation (22) to Taylor expand the dimensionless effective potential at $\tilde{\rho} = 0$ to order n_{trunc}, then substitutes equation (22) into the effective potential flow equation (19), yielding n_{trunc} ordinary differential equations. Setting $\partial_t \lambda_n = 0$ gives n_{trunc} algebraic equations. From these coupled algebraic equations, we obtain a series of fixed points. Linearizing these ODEs at each fixed point yields the M matrix in equation (28). Calculating the eigenvalues of the matrix, the W-F fixed point corresponds to an M matrix with exactly one negative eigenvalue, whose absolute value is the inverse of the critical exponent ν we want to calculate.

[Figure 4: see original paper] shows the numerical results for critical exponent ν calculated at different expansion orders in the large N limit and its dependence on spatial dimension. We see that as the expansion order increases, the critical exponent results approach the true value. However, regardless of the expansion order, there is non-monotonic behavior of the critical exponent with d.

[Figure 5: see original paper] shows the behavior of critical exponent ν in the d-N plane at different expansion orders. As the expansion order increases, the critical exponent can indeed be calculated for smaller dimensions, but cannot approach $d = 2$ arbitrarily. When the expansion order becomes higher, the behavior of the critical exponent becomes increasingly singular at small dimensions. This is due to the finite radius of convergence of the effective potential Taylor expansion. As the dimension approaches 2, the position of the effective potential minimum moves further away from zero, and because of the convergence radius issue, below some dimension, even expanding to infinite order cannot extend the effective potential expansion beyond the minimum.

To solve this problem, we can use a recently proposed method [72] that directly solves the non-local potential fixed-point equation without relying on perturbative expansion. This method analytically incorporates the asymptotic behavior of the effective potential at large field values, which is key to satisfying the

MWH theorem in small spatial dimensions. This non-local potential direct solution method has broad application prospects, such as calculating dynamic critical exponents [105], studying multiple fixed points [106], and investigating Yang-Lee edge singularities [107-111]. [Figure 6: see original paper] shows the critical exponent ν calculated using this method. Compared with the results in [Figure 4: see original paper], we see that the non-local solution no longer exhibits singular behavior at small dimensions. The critical exponents obtained by this method only have systematic errors due to truncation. Since the current calculation does not include anomalous dimension effects, at $d = 2$ the critical exponent ν diverges for any finite N, which is a stronger condition than the MWH theorem requirement. If anomalous dimension effects are further included, the systematic errors in numerical calculations of critical exponents will be further reduced and will fully comply with the MWH theorem requirements. Related research is ongoing.

Baryon Number Fluctuations

We discuss fluctuation measurements related to critical phenomena in relativistic heavy ion collision experiments, which are also the main observables for experimental searches for the QCD CEP. To explain these experimental data and connect experimental measurements with QCD phase transitions and the CEP, theoretical calculations and predictions of fluctuations are essential. In recent years, theoretical research and calculations of fluctuations have made many important advances. For example, lattice QCD calculations of fluctuations can be found in references [73-79], fRG studies of low-energy effective theories (LEFT) in references [80-92], relativistic mean-field models in references [93-99], and Dyson-Schwinger equations in references [98-99]. In particular, in recent years within the fRG framework, low-energy effective models have been combined with first-principles QCD to construct QCD-assisted low-energy effective theories (QCD-assisted LEFTs), and baryon number distribution skewness and kurtosis [84-89], baryon-strangeness correlations [88], and ultra-high-order baryon number fluctuations [92] have been studied in this theoretical framework.

QCD-Assisted Low-Energy Effective Model for Baryon Number Fluctuations

We use QCD-assisted low-energy effective theory to calculate baryon number fluctuations and compare the results with experimental data. The Polyakov-Quark-Meson (PQM) model is very suitable for calculating QCD thermodynamic quantities. The PQM model includes quark degrees of freedom and σ and π meson degrees of freedom. Only scalar and pseudoscalar mesons are included here because these mesons have the lightest vacuum masses and play a decisive role in describing chiral symmetry spontaneous breaking and restoration. The contributions of other heavier mesons can be considered as next-to-leading-order corrections. Based on the quark-meson model, we add the contribution of the Polyakov loop to incorporate information about quark confinement and

deconfinement. This gives the effective action of the low-energy effective model:

$$\Gamma_k = \int_0^{1/T} d\tau \int d^{3x} \left\{ Z_{q,k}\bar{q}[\gamma_\mu \partial_\mu - \gamma_0(\mu + igA_0)]q + \frac{Z_{\phi,k}}{2}(\partial_\mu \phi)^2 + V_k(\rho, A_0) + h_k \bar{q}(\tau_0 \sigma + i\tau \cdot \vec{\pi}\gamma_5)q \right\} \tag{43}$$

The first term on the right side is the quark kinetic term, where q and $\bar{q}$ represent quark and antiquark fields, $Z_{q,k}$ is the quark wavefunction renormalization constant, γ_μ are Dirac matrices, μ is the quark chemical potential matrix, and A_0 is the gluon background field. The second term is the meson kinetic term, where $Z_{\phi,k}$ is the meson wavefunction renormalization constant and ϕ is the meson field. The third term is the quark-meson interaction term, where h_k is the Yukawa interaction strength between quarks and mesons, $\tau = (1, i\vec{\tau}\gamma_5)$, and σ and $\vec{\pi}$ are the fields for the two mesons. The last two terms are the meson effective potential and linear symmetry breaking term, respectively, where $\rho = \frac{1}{2}\phi^2$.

In the low-energy effective model calculations described above, we assume the system under study is a grand canonical ensemble. According to the basic definition of the grand canonical ensemble, our system has fixed volume and temperature and can exchange particles and energy with an external heat source. This is why we introduce the fermion chemical potential into our scale-dependent effective action. In the grand canonical ensemble framework, the calculation of thermodynamic quantities can be reduced to the calculation of the grand thermodynamic potential. For a uniform system in thermal equilibrium, the grand thermodynamic potential density can be easily expressed as the sum of the glue potential, meson effective potential, and linear symmetry breaking term in the infrared limit $k \to 0$:

$$\Omega[T, \mu_B] = V_{\text{glue}}(L, \bar{L}) + V_{\text{mat}}(\rho, A_0) + c\sigma \tag{44}$$

When the fRG flow equation is solved to the infrared, we can obtain each part in equation (44) and finally calculate the grand thermodynamic potential density. This grand thermodynamic potential density depends on temperature and baryon chemical potential, meaning all thermodynamic properties of the system are contained within it. Based on this, according to the properties of the grand canonical ensemble, we can directly obtain the pressure of the system under study:

$$P = -\Omega[T, \mu_B] \tag{45}$$

If we want to calculate the equation of state of the QCD system, we can calculate the derivative of pressure with respect to temperature to obtain entropy and other thermodynamic quantities. We will not elaborate further here, but instead focus on the calculation of baryon number fluctuations.

According to the properties of the grand canonical ensemble, particle number fluctuations can be obtained by taking derivatives of pressure with respect to chemical potential at various orders. Ignoring charge and strangeness chemical potentials, the baryon chemical potential can be expressed as three times the quark chemical potential, $\mu_B = 3\mu$. Based on this, we can obtain dimensionless baryon number fluctuations, also known as generalized susceptibilities:

$$\chi_n^B = \frac{\partial^n (P/T^4)}{\partial (\mu_B/T)^n} \tag{46}$$

By solving the fRG flow equation to obtain pressure at different temperatures and chemical potentials, we can then calculate baryon number fluctuations of different orders through numerical differentiation. In principle, as long as we have sufficiently accurate pressure data, we can calculate fluctuations of any order. However, limited by computational precision, we can only calculate fluctuations up to finite order. Current numerical methods can compute up to tenth-order baryon number fluctuations.

The calculated baryon number fluctuations cannot be directly compared with experimental data because experimental data directly give particle number distributions at different collision energies, from which we can calculate cumulants of the distribution. There is still a volume factor V between cumulants and generalized susceptibilities, as shown below:

$$\begin{aligned}
\frac{\langle N_B \rangle}{VT^3} &= \chi_1^B \\
\frac{\langle (\delta N_B)^2 \rangle}{VT^3} &= \chi_2^B \\
\frac{\langle (\delta N_B)^3 \rangle}{VT^3} &= \chi_3^B \\
\frac{\langle (\delta N_B)^4 \rangle - 3\langle (\delta N_B)^2 \rangle^2}{VT^3} &= \chi_4^B
\end{aligned} \tag{47}$$

where $\langle \cdots \rangle$ represents statistical average and $\delta N_B = N_B - \langle N_B \rangle$. Therefore, we cannot directly compare calculated generalized susceptibilities with experimental cumulants. To eliminate the volume factor, we can compare ratios of generalized susceptibilities with ratios of cumulants, such as skewness $S\sigma = \chi_3^B/\chi_2^B$ and kurtosis $\kappa\sigma^2 = \chi_4^B/\chi_2^B$.

Baryon Number Fluctuations at Zero Chemical Potential

[Figure 7: see original paper] shows the temperature dependence of baryon number fluctuation ratios $R_{42}^B = \chi_4^B/\chi_2^B$, $R_{62}^B = \chi_6^B/\chi_2^B$, and $R_{82}^B = \chi_8^B/\chi_2^B$ at zero chemical potential, compared with lattice QCD results. The red bands in [Figure 7: see original paper] show results from fRG low-energy effective

theory calculations, green and blue bands show results from lattice HotQCD Collaboration calculations [78, 100], blue error bars show results from lattice WB Collaboration calculations [79], and magenta dashed lines show results from the hadron resonance gas model [101]. The error bands in the fRG results originate from temperature rescaling. Since there are scale differences in temperature and chemical potential between low-energy effective theory and first-principles QCD calculations, we introduce two linear rescaling coefficients:

$$T_{\mathrm{LEFT}} = c_T T_{\mathrm{QCD}}, \quad \mu_{\mathrm{LEFT}} = c_\mu \mu_{\mathrm{QCD}} \tag{48}$$

To determine these coefficients, we match the effective theory results with lattice QCD results from the WB (Wuppertal-Budapest) Collaboration at the pseudocritical temperature at zero chemical potential. Since lattice QCD calculations have errors, the temperature rescaling coefficient also has corresponding errors. The chemical potential rescaling coefficient can be determined by matching the curvature of the phase boundary at finite chemical potential, i.e., the quadratic coefficient in equation (63):

$$\frac{T_c(\mu_B)}{T_c} = 1 + \kappa \left(\frac{\mu_B}{T_c}\right)^2 + \lambda \left(\frac{\mu_B}{T_c}\right)^4 + \mathcal{O}(\mu_B^6) \tag{49}$$

where $T_c(\mu_B)$ represents the pseudocritical temperature at baryon chemical potential μ_B, with $T_c = T_c(\mu_B = 0)$. Using the phase boundary curvature value $\kappa = 0.0153(18)$ from reference [102] and following the matching procedure described above, we obtain the two rescaling coefficients $c_T = 1.247(12)$ and $c_\mu = 1.110(66)$. The errors in these coefficients are the sources of error in fRG finite-temperature, finite-chemical-potential calculations.

From [Figure 7: see original paper], we see that after rescaling, baryon number fluctuations from fourth to eighth order agree well with lattice QCD results. However, it should be noted that the method of introducing linear rescaling coefficients is the simplest and most preliminary way to connect low-energy effective theory with first-principles QCD. The reason for using low-energy effective theory is that its simple structure can relatively accurately describe the entire process of chiral symmetry breaking and restoration. Of course, the cost of this simple structure is the lack of dynamical effects from gauge field degrees of freedom at high energy scales. Therefore, a more reasonable approach is to calculate observables such as fluctuations within the fRG-QCD framework [12], and related work is in progress. In the next section, we can naturally extend the zero-chemical-potential calculations to finite chemical potential regions.

Baryon Number Fluctuations at Finite Chemical Potential

In relativistic heavy ion collision experiments, decreasing collision energy often signifies increasing system density, corresponding to increasing chemical poten-

tial. Therefore, finite chemical potential results are important for low-collision-energy experiments. [Figure 8: see original paper] shows the temperature dependence of baryon number fluctuations for chemical potentials in the range 0-400 MeV, with baryon chemical potentials taking values of $\mu_B = 0$ MeV, 100 MeV, 160 MeV, 200 MeV, 300 MeV, and 400 MeV. The behavior of fluctuations with increasing chemical potential is clear: fluctuations of all orders produce larger amplitude oscillations at higher chemical potentials. Additionally, as chemical potential increases, fluctuations also produce a new peak in the region above the phase transition temperature. Moreover, the higher the order of fluctuations, the larger (smaller) the peak (valley) becomes at high chemical potentials. For example, R^B_{82} can reach about 2000 at $\mu_B = 400$ MeV. We can note that regardless of chemical potential magnitude, in the low-temperature region we always have $R^B_{n2} = 1$, because in the low-temperature region the net baryon number distribution tends toward a Skellam distribution. In the extreme high-temperature region, due to the asymptotic freedom property of QCD, interactions between quarks and gluons gradually disappear, and fluctuation values gradually approach the Stefan-Boltzmann limit, with $R^B_{n2} = 1$.

We also note that as chemical potential increases, computational errors increase rapidly. In high-order fluctuations, the errors at high chemical potentials almost fill the region where the curves are located. This indicates that small uncertainties at zero chemical potential lead to significant errors at high chemical potential, resulting in large uncertainties in predictions at high chemical potentials. Therefore, an important future goal is to reduce computational errors in the high-chemical-potential region and improve the reliability of theoretical predictions.

Besides errors introduced by temperature and chemical potential rescaling coefficients, truncations in low-energy effective theory also bring corresponding errors. From the effective action of the quark-meson model given earlier, we currently only consider contributions to the phase transition from two light quark flavors and scalar-pseudoscalar mesons. In reality, there are contributions from other interaction channels. Especially as chemical potential increases, effects such as vector mesons and di-quark condensation will have non-negligible impacts on the system's phase transition behavior, which are obviously not considered in the current framework. Therefore, current calculations and conclusions in the high-chemical-potential region need to be treated with caution. For now, calculations at finite chemical potential are limited to the range covered by experimental data, i.e., collision energies in the range $\sqrt{s_{NN}} = 7.7 - 200$ GeV, corresponding to chemical potentials of about $\mu_B = 20 - 400$ MeV. For calculations of fluctuations in even higher chemical potential regions, we need to further improve the truncation of low-energy effective theory and introduce more information from first-principles QCD calculations. Related work is in progress.

The fireball produced in relativistic heavy ion collision experiments undergoes thermalization to form a strongly interacting quark-gluon plasma (sQGP), which exhibits nearly perfect fluid behavior [103]. As the sQGP expands and

evolves, the temperature continuously decreases, and it fragments into various hadrons, achieving evolution from the QGP phase to the hadronic phase. However, at this stage, inelastic interactions still exist between hadrons, and hadron species continue to transform. With further evolution of the system, inelastic interactions disappear and hadron species no longer change. This moment corresponds to a process with a special name: chemical freeze-out. The temperatures and chemical potentials corresponding to chemical freeze-out are usually different for different collision energies. Obviously, measured fluctuation signals are mainly affected by the state of the system at chemical freeze-out. If we further ignore potential effects such as non-equilibrium evolution and overall baryon number conservation, then as long as we calculate corresponding fluctuations at the chemical freeze-out temperatures and baryon chemical potentials corresponding to different collision energies, we can compare with experimental measurements. The curve showing the dependence of chemical freeze-out temperature and baryon chemical potential on collision energy is called the chemical freeze-out curve.

[Figure 9: see original paper] shows some experimental data points for chemical freeze-out temperature and baryon chemical potential obtained from experiments such as LHC and RHIC, as well as several chemical freeze-out curves used. Blue pentagons combine chemical freeze-out experimental data from accelerator experiments in various energy ranges [7], red circles represent chemical freeze-out data from the STAR Collaboration [104], and the small figure in the lower left corner zooms in on the sub-interval of chemical potential 0-450 MeV. The blue dashed line in [Figure 9: see original paper] is a chemical freeze-out curve fitted by Andronic et al. [7]. Its parameterized form can be written as:

$$T_{\rm CF} = T_{\rm CF}^{(0)} \frac{1}{1+\exp[2.60-\ln(\sqrt{s_{NN}})/0.45]}, \quad \mu_B^{\rm CF} = \frac{\alpha}{1+0.288\ln(\sqrt{s_{NN}})} \tag{50}$$

where $\alpha = 1307.5$ MeV, $T_{\rm CF}^{(0)} = 158.4$ MeV, and $\sqrt{s_{NN}}$ represents the nucleon-nucleon center-of-mass energy in GeV. The red solid line in [Figure 9: see original paper] is a curve refitted to STAR data points using the same parameterization form. For the chemical freeze-out data from the STAR Collaboration, considering some general requirements such as chemical freeze-out temperature decreasing with increasing chemical freeze-out chemical potential and the convexity/concavity properties of the freeze-out curve, the first two data points and the last data point clearly have some deviations. Therefore, it is more reasonable to discard these unreasonable data points and refit, yielding the third chemical freeze-out curve in [Figure 9: see original paper]—the green dotted line.

[Figure 10: see original paper] shows the ratio of fourth-order to second-order fluctuations R_{42}^B calculated on the temperature-baryon chemical potential phase diagram. Additionally, [Figure 10: see original paper] shows the third chemical freeze-out curve, named STAR Fit II. We can clearly see that as chemical

potential increases, R^B_{42} gradually produces more violent fluctuations, creating a negative valley in the region slightly above the phase transition temperature, which can also be seen from [Figure 8: see original paper]. Drawing the freeze-out curve on the R^B_{42} phase diagram, we see that the freeze-out curve first approaches the white region at the red-blue boundary and then bends down into the dark red region. This behavior, when converted to a plot of fluctuations versus chemical potential, can show a non-monotonic dependence on chemical potential.

[Figure 11: see original paper] shows the behavior of baryon number fluctuations R^B_{42}, R^B_{62}, and R^B_{82} as functions of collision energy. The results shown by the two error bands on the left correspond to the first two freeze-out curve schemes used in [Figure 9: see original paper]. The gray error band corresponds to the parameterized freeze-out curve, and the red error band corresponds to the freeze-out curve fitted to STAR data. The error bars represent experimental data for net-proton number fluctuations. The results on the right use the third chemical freeze-out curve: STAR Fit II. The lower horizontal axis in the figure gives the collision energy corresponding to experimental data, and the upper horizontal axis gives the corresponding baryon chemical potential.

For the behavior of R^B_{42} with collision energy, for all three chemical freeze-out curves, we find that as collision energy decreases, R^B_{42} first decreases and then increases, showing non-monotonic behavior consistent with experimental results. The origin of this non-monotonicity is that as collision energy decreases, the freeze-out chemical potential increases, making the chiral continuous transition in the QCD phase diagram increasingly violent and the fluctuation amplitude larger. However, as chemical potential further increases, the chemical freeze-out curve begins to deviate from the phase boundary, as shown in [Figure 10: see original paper], causing R^B_{42} to enter a region where its value increases significantly.

For sixth-order fluctuations, theoretical calculations find that R^B_{62} is negative over a large energy range, which is qualitatively consistent with two of the experimental data points, but clearly more statistics are needed. [Figure 11: see original paper] also shows theoretical predictions for eighth-order fluctuations R^B_{82}, but it should be noted that the errors in R^B_{82} are significant, especially at low energies.

Conclusion

This article mainly reviews recent progress in fRG studies of QCD phase transitions and critical phenomena, focusing on nonperturbative critical exponent calculations and baryon number fluctuations related to the QCD CEP. The fRG is a nonperturbative continuum field theory method that provides a systematic theoretical framework well-suited for deriving renormalization group flow equations. The fRG can be used for systematic expansions, such as gradient expansion and $\epsilon = 4 - d$ expansion, as well as for nonperturbative field

theory numerical calculations.

First, we introduced important progress in applying fRG to study critical behavior in d-dimensional O(N) scalar theory. Using fRG, we can easily obtain the nonperturbative effective potential flow equation for O(N) scalar theory. Solving this nonperturbative effective potential flow equation and its fixed-point equation is crucial for studying critical behavior in second-order phase transitions. On one hand, we can use the effective potential Taylor expansion method to solve the flow equations for expansion coefficients order by order, which is essentially consistent with the $\epsilon = 4 - d$ expansion method developed by Wilson and others. Within the fRG framework, we systematically derived and reproduced critical exponent results from ϵ expansion at $\mathcal{O}(\epsilon)$ order. In principle, higher-order calculations can also be performed, which will be an important direction for future development. Corresponding research has important theoretical significance because fRG provides a theoretical framework that can simplify higher-order calculations, and conversely, perturbative studies of ϵ expansion can also promote understanding of the mathematical structure of nonperturbative fixed-point equations.

It is worth noting that a method has recently been developed that does not rely on perturbative expansion but directly solves the nonperturbative fixed-point equation for non-local potentials. This method analytically considers the asymptotic behavior of the effective potential at large field values, which is key to satisfying the MWH theorem in small spatial dimensions. This non-local potential direct solution method has broad application prospects, such as calculating dynamic critical exponents [105], studying multiple fixed points [106], and investigating Yang-Lee edge singularities [107-111].

Additionally, we discussed baryon number fluctuations closely related to critical phenomena such as the QCD CEP, which are the main observables in relativistic heavy ion collision experiments searching for the CEP. Based on fRG theoretical calculations, we explored possible explanations for the experimentally observed non-monotonic dependence of net-proton number distribution kurtosis (fourth-order moment) on collision energy: on one hand, as collision energy decreases, the chemical freeze-out baryon chemical potential continuously increases, making the chiral continuous transition in the QCD phase diagram increasingly violent and fluctuation amplitudes larger; on the other hand, the chemical freeze-out curve deviates from the phase boundary toward lower temperatures in the high-chemical-potential region. To finally determine the location of the CEP in the QCD phase diagram, we need not only to continuously reduce theoretical calculation errors in the high-chemical-potential region, such as theoretical predictions for the CEP location and fluctuations, but also need more experimental data, especially chemical freeze-out data in the high-chemical-potential region.

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Author Contributions

Yin Shi calculated and wrote the baryon number fluctuations section and revised the article; Tan Yangyang calculated and wrote the critical exponents section and formatted the article; Fu Weijie wrote the introduction and theoretical introduction sections and guided the revision of the article.

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Note: Figure translations are in progress. See original paper for figures.