Some Issues in Determining the QCD Phase Boundary from Relativistic Heavy-Ion Collisions (Postprint)

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Abstract

To determine the phase transition critical point and phase boundary predicted by Quantum Chromodynamics (QCD) from relativistic heavy-ion collision experiments, one must consider the effects of non-critical fluctuations, finite system size, and finite evolution time in experimental data. This article reviews the main content, results, and significance of work in these three aspects. Regarding non-critical fluctuations, we mainly discuss the impact of finite event numbers on observable measurements, estimating the number of events required for precise measurement of higher-order moments of conserved charges in the Relativistic Heavy-Ion Collider (RHIC) energy scan region. We propose using the Poisson distribution to describe statistical fluctuations caused by finite final-state particle numbers. Comparing statistical fluctuations with experimental results, we find that the statistical fluctuation contribution dominates, necessitating the subtraction of Poisson-dominant statistical fluctuations. We propose a mixed-event method, defining the dynamical cumulant as the cumulant of the original sample minus that of the mixed sample. Using the default model of A Multi-phase Transport Model (AMPT), we reconstruct a corresponding mixed-event sample. The results demonstrate that dynamical cumulants can effectively subtract Poisson-like statistical fluctuations, particularly the effects of centrality bin width and detector efficiency. Regarding the effects of finite system size, we employ the three-dimensional three-state Potts model to investigate the behavior of higher-order susceptibilities of its magnetization under various finite system sizes in first-order phase transition, critical point, and smooth crossover regions. We find that when crossing the phase boundary at fixed external field, non-monotonic behavior or sign changes appear in susceptibilities from second to sixth order, and this non-monotonic behavior is similar across the three phase transition regions. Therefore, different orders of phase transitions cannot be distinguished based solely on non-monotonic behavior. We further investigate the finite-size scaling behavior of susceptibilities, whose scaling exponents differ across different orders of phase transitions, enabling distinction between them. Based on the finite-size scaling property of observables, we present a quantitative method for determining critical parameters using fixed points, and apply this method to data analysis from three-dimensional three-state Potts model simulations, demonstrating its precision and effectiveness. Regarding the effects of non-equilibrium evolution, we employ the Metropolis algorithm to simulate the evolution from non-equilibrium to equilibrium of the three-dimensional Ising model near the critical point. We find that its order parameter approaches its equilibrium value exponentially during evolution, consistent with results from dynamical Langevin equations. The average relaxation time at critical temperature diverges with system size to the zth power, indicating that it can well represent the relaxation time in dynamical equations. During non-equilibrium evolution, the third and fourth moments of the order parameter exhibit oscillations between positive and negative values, with the sign depending on observation time, consistent with dynamical models in the smooth crossover region. The study also finds that in the smooth crossover region, non-equilibrium evolution duration is very short and its impact on observables is very weak; however, on the first-order phase transition line, non-equilibrium relaxation time is very long and non-equilibrium effects cannot be neglected. These qualitative features provide important guidance for experimentally determining the QCD critical point and phase boundary.

Full Text

Title and Authors

Several Problems in Determining the QCD Phase Boundary by Relativistic Heavy Ion Collisions

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Abstract

The primary scientific goal of relativistic heavy-ion collisions is to determine the phase diagram of quantum chromodynamics (QCD). To achieve this objec-

tive, numerous experiments have been designed and conducted. Currently, the Relativistic Heavy-Ion Collider Beam Energy Scan (RHIC-BES) II program at Brookhaven National Laboratory in the United States, along with future experimental programs including NICA (Nuclotron-based Ion Collider Facility) in Russia, FAIR (Facility for Antiproton and Ion Research) in Germany, and HIAF (High Intensity Heavy-ion Accelerator Facility) in China, all aim to discover signals related to the QCD phase transition through energy scans, thereby enabling experimental determination of the QCD critical point and phase boundary. Consequently, research related to QCD phase transitions will remain at the international forefront of relativistic heavy-ion collisions for the next decade.

QCD predicts that at sufficiently high temperatures and low baryon chemical potentials, matter undergoes a phase transition from hadron gas to quark-gluon plasma. Lattice QCD calculations indicate that near high temperature and zero chemical potential, the transition from hadrons to quark-gluon plasma is not a true phase transition but rather an analytic crossover. QCD-based phenomenological models predict that at low temperatures and high baryon chemical potentials, the hadron-to-quark-gluon-plasma transition is a first-order phase transition. The endpoint of the first-order transition line constitutes the QCD critical point, and the QCD phase diagram is completely determined by the first-order transition line and this critical point.

According to lattice QCD theory and effective potential models, higher-order moments of conserved charges are particularly sensitive to correlation length and may exhibit non-monotonic behavior and sign changes near the critical point. Therefore, sensitive observables for the QCD critical point are the higher-order moments of conserved particles in the final state. At RHIC/BES I, non-monotonic behavior has been observed in the fourth-order moment of net-proton multiplicity as a function of collision energy. However, this non-monotonic behavior is not unique to critical fluctuations. Fluctuations associated with phase transitions also exhibit non-monotonic behavior during first-order transitions and crossovers. Additionally, experiments have observed that the third-order moment of net-proton multiplicity is negative and smaller than the Poisson expectation across various collision energies, which contradicts the positive-definite result expected from equilibrium phase transitions. To determine the critical nature of observed fluctuations, extensive theoretical and experimental data analysis efforts have been devoted to subtracting various possible non-critical effects.

Critical fluctuations arise from correlations among particles within events, characterized by divergent correlation length, whereas non-critical fluctuations lack this divergent correlation feature. Non-critical fluctuations originate from two primary sources: first, conventional conservation mechanisms such as resonance decays and global conservation of energy, momentum, and charge number. These traditional effects have convergent and fixed correlation lengths and are typically small and fixed in their impact on higher-order moments of conserved charges. The second source stems from limitations and characteristics of the observation system itself, such as measurement uncertainties from finite event numbers (i.e., statistics), statistical fluctuations from finite particle numbers, fluctuations in initial system size due to different impact parameters, centrality bin width, detector acceptance efficiency, and artificial cutoffs from limited experimental phase space.

To subtract non-critical fluctuations from traditional correlation mechanisms and systematic effects, various correction schemes have been proposed. For instance: (1) For statistical fluctuations, Poisson distributions with the same average particle number are typically used for simulation; (2) Within a given centrality bin width, events with slightly different impact parameters are actually mixed, causing higher-order moments to vary non-smoothly with centrality bin width. To reduce this effect, the Centrality Bin Width Correction (CBWC) method is widely employed; (3) In relativistic heavy-ion collision experiments, some final-state particles are not observed by detectors, directly affecting the peak of particle number distributions and cumulative moments. To eliminate this detector efficiency effect, correspondence relations between true and measured cumulative moments have been established for correction. However, in real samples, all these non-critical effects are interconnected, making it difficult to eliminate any single effect in isolation. A better approach is to simultaneously address these non-critical effects, and the mixed event method provides a possibility to obtain such a background.

This paper will discuss the influence of these non-critical effects and their subtraction in §1. First, §1.1 examines the impact of finite event numbers on higher-order moments, particularly fourth- and sixth-order moments. Then §1.2 describes statistical fluctuations and their influence on experimental measurements of higher-order moments. Finally, §1.3 presents an effective pooling method for mixed events, defines dynamical higher-order moments as the difference between original and mixed sample moments, and demonstrates the effectiveness of dynamical higher-order moments in subtracting non-critical fluctuations using the default model of A Multiphase Transport Model (AMPT).

In addition to non-critical fluctuations, two other important and non-negligible factors must be considered. First, the system formed in relativistic heavy-ion collisions is not infinite as in the thermodynamic limit but only a few femtometers in size, necessitating consideration of finite system size effects on observables. Second, the system evolution time is only about 10 fm/c, very limited, making it likely that the system cannot fully reach thermodynamic equilibrium, thus requiring consideration of non-equilibrium evolution effects on observables. These two factors are crucial for correctly determining the positions of the critical point and phase boundary in the QCD phase diagram.

Finite system size effects limit the divergence of critical correlation length, causing fluctuation signals to deviate from thermodynamic limit values. For example, in the thermodynamic limit, magnetic susceptibility is a delta function at first-order transitions and diverges to infinity at second-order transitions. However, for finite-size systems, magnetic susceptibility exhibits a finite peak at both first-and second-order transitions, and the peak position in the phase diagram (e.g., critical temperature) deviates from its critical value, becoming a pseudo-critical temperature. Section 2.1 will fully demonstrate these characteristics of finite-size systems near phase boundaries using the three-dimensional three-state Potts model.

For finite-size systems, precise determination of critical point location must utilize finite-size scaling properties of observables. Quantities like magnetic susceptibility can be well described by finite-size scaling, which not only yields precise critical temperature values but also determines phase transition order based on scaling exponent values. The specific form of finite-size scaling for a given system and observable must be determined from its temperature dependence across different system sizes. The conventional approach is to first assume critical parameters (critical temperature and critical scaling exponent ratio), use scaled reduced temperature as the independent variable, plot scaled observables for different system sizes, and judge whether the assumed parameters are correct based on curve overlap at critical reduced temperature. Obviously, this traditional method has limitations: first, scaling curve overlap is mostly judged by eye, preventing quantitative precision assessment; second, near critical reduced temperature, the overlap region is large due to scale factor inclusion, leading to large uncertainties in corresponding phase transition parameters.

If the independent variable is changed from scaled reduced temperature to temperature, scaling curves will intersect at only one point—the phase transition temperature, i.e., a fixed point. The temperature and scaling exponent ratio at the fixed point position correspond to critical temperature and critical scaling exponent ratio. In renormalization group theory, the critical point is an unstable fixed point, while the first-order transition line contains discontinuous fixed points. Using fixed point behavior on phase boundaries allows precise determination of critical parameters. Section 2.2 will detail the quantitative method for determining critical parameters from fixed point behavior of observables on phase boundaries.

Regarding non-equilibrium evolution, no mature non-equilibrium statistical theory currently exists, making quantitative estimation of non-equilibrium effects on phase transitions and critical fluctuations impossible. Non-equilibrium dynamical theories are limited to quantitative calculations in the crossover region. To estimate non-equilibrium effects on the critical point, first-order phase boundary, and near the phase boundary, new effective models and methods for describing non-equilibrium evolution must be sought.

Equilibrium phase transitions are physical phenomena occurring during spontaneous symmetry breaking. Systems with the same symmetry belong to the same universality class and share identical critical exponents. In statistical physics, the Ising model is one of the most effective models for studying phase transition characteristics, well representing dynamical features of real phase transition systems. The QCD critical point belongs to the same Z(2) symmetry group universality class as the three-dimensional Ising model. By selecting appropriate mapping relations, Ising model temperature and external field can be mapped to corresponding QCD system temperature and chemical potential.

Near the critical point, the common feature of non-equilibrium evolution is that relaxation time diverges as a power of the dynamical exponent, which depends on specific system parameters and algorithms. Although the three-dimensional Ising system's dynamical exponent differs quantitatively from QCD, making direct numerical comparison impossible, the qualitative features of non-equilibrium evolution across the entire phase boundary—such as temperature (or phase boundary region) dependence of non-equilibrium effects and whether non-equilibrium effects change the sign of order parameter and its higher-order cumulants—can help estimate non-equilibrium effects on QCD systems. Section 3.1 first uses the Metropolis algorithm to simulate non-equilibrium evolution of the three-dimensional Ising model, analyzing dynamical characteristics near the critical point and providing quantitative description of non-equilibrium states. Then §3.2 presents the time evolution of various moments of magnetization. Finally, §4 provides a summary of all work.

1. Effects of Non-Critical Fluctuations

1.1 Effects of Statistics

As mentioned above, the critical-fluctuation-sensitive observables in relativistic heavy-ion collisions are higher-order moments of conserved charges. The formulas for various order moments are defined as:

$$C_1 = \langle(\delta X)^1\rangle, \quad C_2 = \langle(\delta X)^2\rangle, \quad C_3 = \langle(\delta X)^3\rangle, \quad C_4 = \langle(\delta X)^4\rangle, \quad C_5 = \langle(\delta X)^5\rangle, \quad C_6 = \langle(\delta X)^6\rangle$$

where $\delta X = X - \bar{X}$. C_1 is the mean value of the order parameter; C_2 is the variance of the distribution, i.e., σ^2; C_3 and C_4 relate to the skewness $S = C_3/(C_2)^{3/2}$ and kurtosis $k = C_4/C_2^2$ of the distribution, which quantify non-Gaussianity.

For most observables in relativistic heavy-ion collisions, such as mean values and second-order moments of net-proton multiplicity, we need not consider statistics issues because current experimental statistics are sufficient to guarantee their precision. However, for higher-order moments, especially fourth- or sixth-order moments, the statistics required for precise measurement increase correspondingly in power-law fashion. Therefore, we must first answer whether current and future experimental statistics can guarantee precision for higher-order moment measurements, or how much statistics are needed to ensure reliability.

Additionally, the special measurement method for higher-order moments of conserved charges in relativistic heavy-ion collisions necessitates attention to basic statistical requirements. In estimating higher-order moment values, the corresponding collision centrality parameter is determined by multiplicity intervals.

To eliminate fluctuations in initial volume represented by different multiplicities within this interval, for a given centrality multiplicity interval, one first calculates higher-order moments at each multiplicity N_{ch}, then uses the number of events at each N_{ch} as weighting factors. The higher-order moment for the corresponding centrality is the weighted average of higher-order moments at each N_{ch} bin within the centrality interval. This method is the so-called centrality bin width correction (CBWC) method.

Obviously, this method causes a problem: although total experimental statistics are on the order of millions of events, the statistics allocated to each N_{ch} bin (especially for 0%-5% centrality N_{ch} bins) or even each individual N_{ch} are very limited. Taking Au+Au collisions at $\sqrt{s_{NN}} = 7.7$ GeV as an example, the total number of events currently used to analyze net-proton $\kappa\sigma^2$ (0%-80% centrality) is 3 M. In the 0%-5% centrality range, there are at least 100 N_{ch} bins for centrality determination. Roughly averaging, each N_{ch} has only 1,900 events, so the statistical error for higher-order moments calculated at each N_{ch} is very large.

Therefore, due to the inherent characteristics of higher-order moments of conserved charges and the special nature of their measurement method, we must estimate the basic statistics required for their precise measurement. Below we first discuss the fourth-order moment as an example. We know that only with sufficient statistics does the distribution of experimentally observed random variables become normal, i.e., satisfying the Central Limit Theorem (CLT). Only under CLT conditions does the observed average equal the true value and become independent of statistics, i.e., $\bar{X}_{n_1} = \bar{X}_{n_2} = \bar{X}_{n_3}$, where n_1, n_2, n_3 represent different statistics.

We simulated Au+Au collisions using the UrQMD model at $\sqrt{s_{NN}} = 11.5$ GeV, generating a total of 250 M events, comparable in magnitude to current RHIC relativistic heavy-ion collision experimental statistics. The total sample was then randomly divided into 250, 50, and 10 sub-samples, corresponding to statistics of 1 M, 5 M, and 25 M events respectively. In each sub-sample, we calculated net-proton $\kappa\sigma^2$ using the CBWC method at each N_{ch}. The results showed that $\kappa\sigma^2$ increases with statistics, violating the central limit theorem. The reason is that the number of events at each N_{ch} is too small to meet CLT requirements.

In fact, with existing statistics, we can attempt to improve the estimation method for higher-order moments by finding a slightly wider N_{ch} bin to replace each individual N_{ch}, thereby increasing the number of events in each N_{ch} bin and improving the precision of each $\kappa\sigma^2$ calculation. Of course, the condition for appropriately widening the N_{ch} bin must ensure it does not affect the represented initial volume, i.e., it must have the same resolution as each N_{ch}. To find an appropriate N_{ch} bin width, we divided the 0%-5% centrality into different bin widths, such as 25 bins (δ0.2% centrality), 5 bins (δ1.0% centrality), 2 bins (δ2.5% centrality), each individual N_{ch}, and directly using 0%-5% centrality. Simultaneously, to discuss the statistics dependence of results under different N_{ch} bin widths, we randomly divided the model-generated data into

250, 125, 50, 25, 10, and 5 sub-samples, corresponding to statistics of 1 M, 2 M, 5 M, 10 M, 25 M, and 50 M events respectively.

Following the CBWC method, we first calculated $\kappa\sigma^2$ for each sub-sample, then averaged to obtain $\kappa\sigma^2$. The results for net-proton and net-charge are shown in Figures 1(a) and 1(b) respectively [Figure 1: see original paper].

Figure 1 shows that when total statistics exceed 1 M, $\kappa\sigma^2$ obtained using $\delta1.0\%$, $\delta2.5\%$, and $\delta5.0\%$ becomes independent of statistics. However, for each individual N_{ch}, only when total statistics exceed 25 M does the result become statistics-independent and consistent with the $\delta1.0\%$ result, indicating that $\delta1.0\%$ can accurately represent the initial volume just like each N_{ch}. Meanwhile, $\kappa\sigma^2$ obtained with $\delta2.5\%$ and $\delta5.0\%$ are significantly higher than results calculated from each N_{ch} (or $\delta1.0\%$), indicating that such N_{ch} bin widths are too large to represent the initial volume like each N_{ch}. Therefore, only $\delta1.0\%$ can replace each N_{ch}.

Thus, for the fourth-order moment, the basic statistical requirement is: if calculated using the CBWC method at each N_{ch}, total statistics must exceed 25 M; if calculated using $\delta1.0\%$, total statistics at the 1 M level are sufficient. This conclusion also holds for other centrality ranges. Therefore, when total sample statistics are below 25 M, fourth-order moments calculated from each N_{ch} are likely unreliable, and results using $\delta1.0\%$ centrality as the N_{ch} bin width are more reliable.

For net-proton sixth-order moments, the basic statistical requirement is higher than for the fourth-order moment. We studied the case with transverse momentum cutoff $0.4 < p_T < 0.8$ GeV. When statistics increase from 1 M to 300 M, C_6/C_2 calculated at each N_{ch} in 0%-30% centrality continues to increase with statistics, indicating that even 300 M statistics are insufficient under this phase space cutoff. Whether the $\delta1.0\%$ recommendation is feasible for sixth-order moments requires further investigation.

1.2 Effects of Statistical Fluctuations

The Poisson distribution can describe random fluctuations of independent, uncorrelated particles, i.e., statistical fluctuations. If a random variable X follows a Poisson distribution with mean value λ, then for non-negative integer random variable $k = 0, 1, 2, ...$, the probability distribution is:

$$P(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Particles produced in RHIC experiments include baryons with baryon number 1, hadrons with charge number 1 or 2, hadrons with strangeness 1, 2, or 3, and their corresponding antiparticles. Starting from the simplest case, we assume that each particle type' s multiplicity (e.g., N_B^+ and N_B^-) follows a Poisson distribution. Then net baryon number ($N_B = N_B^+ - N_B^-$) is the cross-correlation of two Poisson distributions:

$$f(x, N_B) = e^{-(\langle N_B^+\rangle + \langle N_B^-\rangle)} \left(\frac{\langle N_B^+\rangle}{\langle N_B^-\rangle}\right)^{N_B/2} I_{N_B}(z)$$

where $z = 2\sqrt{\langle N_B^+\rangle\langle N_B^-\rangle}$ and $I_{N_B}(z)$ is the modified Bessel function of the first kind. Equation (2) is a standard Skellam distribution, which yields the same net baryon probability distribution as the Hadron Resonance Gas (HRG) model. Higher-order moments of net baryon (κ_B^k) can be obtained from the cumulant-generating function (CGF):

$$K_B(t; N_B) = \ln G(e^t; N_B)$$

where $G(t; N_B)$ is the probability generating function (PGF) of the Skellam distribution:

$$G(t; N_B) = \sum_{N_B=0}^{\infty} f(N_B + x, \langle N_B^+\rangle, \langle N_B^-\rangle) t^{N_B} = e^{-(\langle N_B^+\rangle + \langle N_B^-\rangle)} G(t; N_B^+) G(1/t; N_B^-)$$

Thus, even and odd order moments of net baryon are:

$$\kappa_B^{2k} = \langle N_B^+\rangle + \langle N_B^-\rangle, \quad \kappa_B^{2k+1} = \langle N_B^+\rangle - \langle N_B^-\rangle$$

These are uniquely determined by the mean numbers of baryons and antibaryons. For charged particles and strange hadrons, if each particle type’ s multiplicity follows a Poisson distribution, similar derivations yield results consistent with those from the hadron resonance gas model under Boltzmann approximation.

Based on mean proton and antiproton numbers at three RHIC collision energies ($\sqrt{s_{NN}}$ = 19.6 GeV, 27 GeV, 39 GeV, 62.4 GeV, 200 GeV; at 19.6 GeV, only 5 centralities), we calculated standard deviation σ_P, skewness S_P, and kurtosis κ_P of net-proton statistical fluctuations. The results show that all statistical fluctuation moments are very close to experimentally measured moments, with differences an order of magnitude smaller than original values. Therefore, the centrality and energy dependence of net-proton higher-order moments at RHIC are primarily due to statistical fluctuations, which cannot be neglected in measured higher-order moments of conserved charges.

1.3 Mixed Event Method

In real samples, various non-critical effects are interconnected, making it difficult to subtract any single effect in isolation. A better approach is to use a background sample containing identical backgrounds, i.e., a mixed event sample. Typically, particles in a mixed event are randomly selected from different original events while maintaining the same total particle number as the original sample. Thus, the mixed event sample retains the same statistical fluctuations, initial system size fluctuations, centrality bin width, detection efficiency, and experimental cutoffs as the original sample.

The mixed event method has been applied to various observables in relativistic heavy-ion collisions, such as two-particle correlations, particle yields, transverse momentum spectra, and elliptic flow. Different observables require different mixed event construction methods. For two-particle rapidity correlations, the dynamical feature of interest is particle rapidity azimuth within a given event, so the mixed event randomly replaces the true rapidity azimuth of all particles in a given event. For higher-order moments of conserved charges, the dynamical feature of interest is the number of conserved-charge particles. Therefore, the purpose of mixed events is to eliminate correlations among conserved-charge particles and correlations between given conserved-charge particle numbers and events.

We find the best method is the most random pooling method. If the original sample has sufficient statistics (greater than 1,000 events), the total particle number can be approximated as infinite. All particles can be placed in a pool, then N_c charged particles are randomly selected from the pool. These N_c particles can be approximated as selected from different original events, with correlations approximately zero. The probability of randomly selecting a charged particle from the pool is constant, and the average number of conserved-charge particles in the mixed sample remains consistent with the original sample. The resulting mixed sample no longer contains critical fluctuations related to real events but retains overall event and system characteristics. If dynamical cumulative moments are defined as the difference between original and mixed sample cumulative moments, then dynamical cumulative moments represent critically related fluctuations.

Using the default model of A Multiphase Transport Model (AMPT), we simulated 19.6 GeV Au+Au collision samples. For subtraction formulas addressing statistical fluctuations, centrality bin width, and detector efficiency, we compared dynamical cumulative moments with formula-corrected results. The transverse momentum and rapidity cutoffs were $0.4 < p_T < 0.8$ GeV and $|y| < 0.5$, consistent with those used in RHIC/STAR experiments. The net proton number approximates the total net baryon number, i.e., $N_c = N_p$.

Figure 2 [Figure 2: see original paper] shows the centrality dependence of $\kappa\sigma^2$. To observe detector efficiency effects, particles were randomly selected with 80% and 60% probabilities, and corresponding cumulative moments were calculated.

Circles, squares, and triangles mark 100%, 80%, and 60% efficiencies respectively. Figure 2(a) shows centrality dependence of $\kappa\sigma^2$ in original samples under three detector efficiencies. The three point sets are separated, with cumulative moments varying with detector efficiency, and the efficiency effect depends on centrality. Figure 2(b) shows centrality dependence of $\kappa\sigma^2$ in mixed samples under three detector efficiencies. Detector efficiency similarly affects mixed sample moments, similar to its effect on original samples in Figure 2(a).

Using formulas from references [45,46], Figure 2(d) presents efficiency-corrected $\kappa\sigma^2$ that is independent of detector efficiency. In Figure 2(d), squares and triangles coincide with corresponding circles within error bars, indicating that formula-corrected cumulative moments well subtract detector efficiency effects. Figure 2(c) shows dynamical $\kappa\sigma^2$ centrality dependence under three detector efficiencies. The three point sets basically coincide within error ranges, indicating that dynamical $\kappa\sigma^2$ is independent of detector efficiency like formula-corrected results, having subtracted efficiency effects. Moreover, dynamical cumulative moments subtract not only statistical fluctuations but also centrality and detector efficiency effects.

2. Effects of Finite System Size

2.1 Effects of Finite System Size on Phase Transition Fluctuations

At large baryon chemical potentials and near the chiral limit, lattice QCD calculations face severe difficulties. To calculate thermodynamic quantities of finite-size systems near phase boundaries, phenomenological models and effective theories are often employed. Models in the same universality class as QCD phase transitions include three-dimensional Ising, O(4) and O(2) spin models, and the three-dimensional three-state Potts model. The three-dimensional Ising model consists of N particles arranged on an $L \times L \times L$ lattice. Particle spins can point up or down, represented by $+1$ and -1. The system state can be represented by a series of spins $\{s_1, s_2, ..., s_N\}$, abbreviated as $\{s_i\}$. Considering only nearest-neighbor interactions and interactions with external magnetic field H, the total energy is:

$$E\{s_i\} = -J \sum_{\langle ij \rangle} s_i s_j - H \sum_{i=1}^{N} s_i$$

where $\langle i, j \rangle$ indicates summation over all nearest-neighbor spin pairs. The average total magnetization and average magnetization in a microscopic state $\{s_i\}$ are:

$$M = \sum_{i=1}^{N} s_i, \quad m = \frac{M}{N} = \frac{1}{N} \sum_{i=1}^{N} s_i$$

When temperature $k_{BT} \ll J$, spin-spin interactions dominate, all particles tend to align in the same direction, $|m|$ approaches 1, and the system is in an ordered phase. When $k_{BT} \gg J$, spin-spin interactions are weak, each lattice site' s spin randomly points up or down, $|m|$ approaches zero, and the system is completely disordered. Without external field, the system undergoes a continuous phase transition from high-temperature disordered phase to low-temperature ordered phase. The transition point is called the Curie point, which is a critical point in the temperature T and external magnetic field H plane, with critical temperature denoted T_c. Through finite-size scaling analysis, the critical temperature of the three-dimensional Ising model is calculated as $T_c = 4.51$, consistent with renormalization group theory results.

The three-dimensional three-state Potts model is a simple effective model for finite-temperature pure gauge QCD theory. Its critical point and phase boundary can be directly mapped from the three-dimensional Ising model to the QCD phase boundary. Therefore, we use the three-dimensional three-state Potts model and the universality of its magnetization higher-order moment behavior to demonstrate the behavior of magnetization higher-order susceptibilities near first-order transitions, critical points, and crossover regions, as well as their finite-size characteristics.

In the Potts model, each lattice site' s spin can take multiple discrete values, typically represented by integers starting from 1. For a q-state Potts model, spins can take integer values $s = 1, 2, ..., q$. Any two adjacent spins contribute J to the Hamiltonian if they have the same value, and 0 otherwise. If a spin aligns with the external field direction, it contributes H to the Hamiltonian, otherwise 0. The Hamiltonian can be expressed as:

$$H = -J \sum_{\langle ij \rangle} \delta(s_i, s_j) - H \sum_i \delta(s_i, s_g)$$

where J is the interaction between nearest-neighbor spins ij, here set to 1. $\delta(s_i, s_j)$ is 1 when $i = j$ and 0 otherwise. s_g is the so-called ghost spin, whose direction is the external field direction. The partition function of the Potts model is defined as $Z(\beta, h) = \sum_{\{s_i\}} e^{-\beta E}$, where $E = -J \sum_{\langle ij \rangle} \delta(s_i, s_j)$ is the interaction energy between adjacent spins; $M = \sum_i \delta(s_i, s_g)$ is magnetization; $h = \beta H$ is the reduced external field; $\beta = 1/T$ is the inverse temperature.

In the three-dimensional three-state Potts model, spins s_i can take three states (1, 2, 3) on a cubic lattice. It satisfies $Z(3)$ symmetry, whose spontaneous breaking is believed to cause the deconfinement phase transition in quenched QCD. Adding an external field breaks $Z(3)$ symmetry, analogous to how the fermion determinant breaks $Z(3)$ symmetry in QCD. References [49,50] indicate that QCD can be reduced to the three-dimensional three-state Potts model at high temperature and heavy quark mass. The model' s order parameter can be defined using average magnetization $m = \langle M \rangle / N$.

In pure gauge QCD theory, the order parameter is the Polyakov loop, which acts as magnetization, characterizing center symmetry spontaneous breaking in the deconfinement phase. The three-dimensional three-state Potts model undergoes a temperature-driven first-order transition at zero external field. As the external field gradually increases, the first-order transition weakens and terminates at a second-order critical point. Beyond the critical point is the crossover region. This resembles QCD, where the role of baryon chemical potential in QCD is equivalent to external field in the Potts model, with their variations causing temperature-driven transitions to change from first-order to second-order to crossover.

For first-order, second-order, and crossover transitions, we selected three external field values: $h = 0.0005$, 0.000775, and 0.0002. Letting $X = M$, we simulated systems of different sizes with scale L between 40–70 at each external field, calculating second-order susceptibility ($\chi_2 = C_2/V$) and presenting their behavior near phase transition temperatures. The second-order susceptibilities under the three external fields are displayed in three subplots of Figure 3 [Figure 3: see original paper], with different colored lines representing different system scales. The figure shows that near phase transition temperature (T_{pt}), second-order susceptibility exhibits a peak in first-order (Figure 3(a)), second-order (Figure 3(b)), and crossover (Figure 3(c)) transitions. In the thermodynamic limit, second-order susceptibility should be a delta function for first-order transitions and diverge for second-order transitions. Finite systems make it a finite peak in both transition types. As system scale increases, the peak position shifts to higher temperature. Comparing the three subplots in Figure 3, the basic structure of second-order susceptibility near phase transition temperature is similar under different external fields, all showing a peak. This indicates that finite-system susceptibility exhibits non-monotonic behavior near phase transition temperature, occurring not only near critical temperature of second-order transitions but also in first-order transition and crossover regions.

2.2 Determination of Critical Parameters for Finite Systems

The phase diagram gives the location of phase transition boundaries. To precisely determine phase boundary positions, key point locations must first be identified, such as the critical point of second-order transitions. The critical temperature T_c is the parameter of greatest interest. For sufficiently large systems, T_c can be approximated by the peak position of susceptibility distribution. However, as discussed above, for finite-size systems, the peak position varies with system scale. Determining phase transition parameters through fluctuations in finite systems must utilize finite-size scaling laws.

For finite-size systems undergoing second- or first-order transitions, certain observables like magnetic susceptibility from different system scales can be scaled to a universal equation, the so-called finite-size scaling law:

$$Q(T, L) = L^{-\lambda/\nu} f_Q(tL^{1/\nu})$$

where $Q(T, L)$ is the observable, generally a function of temperature and system scale; $t = (T - T_{pt})/T_{pt}$ is reduced temperature, with T_{pt} being the previously mentioned phase transition temperature; f_Q is the scaling function; λ and ν are scaling exponents corresponding to observable Q and correlation length ξ respectively. The scaling exponent ratio $a = \lambda/\nu$ is a fraction between d and 0 for second-order critical points; for first-order transitions, a depends only on system dimension d, being an integer related to dimension.

Figure 4(a) [Figure 4: see original paper] shows finite-size scaling behavior for second-order transitions in the two-dimensional Ising model without external field, while Figure 4(c) shows finite-size scaling for first-order transitions in the three-dimensional three-state Potts model at external field $h = 0.0005$. Figure 4 demonstrates that scaled magnetization curves for different scales overlap well near the critical point and first-order transition region. Conventionally, when using this scaling law to determine critical temperature and critical exponents, one pre-assigns these parameters and checks whether scaling curves overlap, i.e., evaluating whether assumed phase transition parameters are correct by observing overlap of scaled observable curves from different system scales at reduced temperature. Obviously, this method has large uncertainties: first, scaling curve overlap is mostly judged by eye without quantitative description; second, near phase transition reduced temperature, the overlap region is large, as shown in Figures 4(a) and 4(c), and with scale factor inclusion, precise determination of corresponding phase transition parameters is generally impossible.

If the independent variable of scaling curves is changed to temperature, scaling curves separate from each other when deviating from phase transition temperature, as shown in Figures 4(b) and 4(d), intersecting at only one point at phase transition temperature—the fixed point. The temperature and scaling exponent ratio at the fixed point correspond to phase transition temperature and phase transition scaling exponent ratio. Obviously, compared with double scaling curves (Figures 4(a) and 4(c)), single scaling curve fixed points (Figures 4(b) and 4(d)) can completely and precisely give corresponding phase transition parameters, and quantitatively determining fixed points is much simpler and easier than curve overlap.

According to renormalization group theory, the critical point is an unstable fixed point, while the first-order transition line contains discontinuous fixed points. In the crossover transition region, observables are independent of system scale. If finite-size scaling is extended to the crossover region, the scaling exponent ratio is 0 and scaling curves completely overlap.

Figures 4(b) and 4(d) show that as temperature moves further from critical temperature, scaled observables from different scales become increasingly dispersed; conversely, as temperature approaches critical temperature, scaled observables from different scales gradually converge, meeting at the fixed point. Therefore, fixed points can be defined based on the convergence degree (distribution width) of observables from different scales. At given temperature and scaling exponent ratio, the distribution width is defined as the square root of the weighted variance of scaled observable points $Q(T, L)L^a$ from different scales:

$$D(T, a) = \Delta S_Q(T, L)L^a$$

where $D(T, a)$ is a function of temperature T and scaling exponent ratio a; N_L is the number of different system scales; $\Delta S_Q(T, L)L^a$ is the weighted variance of all scaled observables relative to their expected values:

$$\Delta S_Q(T, L)L^a = \sum_{i=1}^{N_L} \omega_i [Q(T, L_i)L_i^a - \langle Q(T, L)L^a \rangle]^2$$

with weight factor $\omega_i = \delta[Q(T, L_i)L_i^a]$ being the measurement error of scaled observable $Q(T, L_i)L_i^a$.

When T and a deviate further from phase transition values, scaling curves from different scales become more dispersed, correspondingly increasing $D(T, a)$. When T and a approach phase transition values, scaling curves converge, decreasing $D(T, a)$. When both T and a reach phase transition values, theoretically scaled points $Q(T, L)L^a$ from different scales take identical values and intersect at the fixed point, where $D(T, a)$ reaches its minimum $D(T, a)_{\min} \approx 1$. As temperature and scaling exponent ratio change, the distribution width of scaled observables shows trends of dispersion and convergence, which are necessary conditions for fixed point formation. Due to measurement errors and other uncertainties in actual measurement processes, the fixed point is not an ideal point, and actual $D(T, a)_{\min}$ will be larger than 1.

If changing T and a yields a minimum $D(T, a)$ of about 1, it means scaled observable $Q(T, L)L^a$ curves intersect at a fixed point. The T and a corresponding to this fixed point are characteristic parameters of the phase boundary. Different phase transition types correspond to different characteristic parameters: at critical points the scaling exponent ratio is a fraction; on first-order transition lines it is an integer related to system dimension; in crossover regions, since observables are independent of system scale, $D(T, a)$ is approximately constant at 1 throughout the phase transition temperature region, with corresponding scaling exponent ratio $a = 0$.

Now, using data generated by the three-state Potts model in three phase transition regions as an example, we illustrate how to apply the above quantitative description of fixed points to determine phase boundary parameters. The three phase transition regions are the second-order critical point, first-order transition line, and crossover region, corresponding to external fields $h = 0.000775$, 0.0005, and 0.002. The observable is average magnetization, i.e., the order parameter. First, on the phase parameter plane (T and a plane), we plot the distribution width $D(T, a)$ of our defined scaled observable as contour lines, shown in Figure 5 [Figure 5: see original paper]. The color bars on the right of each subplot represent width $D(T, a)$ values, with red and blue indicating minimum and maximum respectively. Color distributions within the T and a plane represent the distribution width of scaled curves, with D values ranging from 1.2 to 1.15 at different positions. The a value range is $[0, 3]$ with 0.05 intervals.

When external field is $h = 0.000775$ and 0.0005, corresponding to the second-order critical point in Figure 5(a) and first-order transition line in Figure 5(b), D values show a gradually decreasing red region by changing T and a. The reddest location is where width reaches minimum, where scaled order parameter curves from different scales intersect at the fixed point. The intersection of two perpendicular dotted lines in Figure 5(a) (or 5(b)) marks the minimum width position in the phase parameter plane, whose corresponding phase parameters are critical temperature and scaling exponent ratio (or first-order transition temperature and scaling exponent ratio). Therefore, the width contour map in the phase parameter plane precisely gives the locations of critical region and first-order transition fixed points.

When external field is $h = 0.002$, corresponding to the crossover region in Figure 5(c), D value contour lines are parallel to the T axis, showing band-like distribution with gradual color variation. The band is reddest near the black dotted line $a = 0$, where $D(T, a)$ is close to constant 1. These features are completely different from distributions in Figures 5(a) and 5(b), indicating that in crossover regions, scaling curves are independent of system scale, $D(T, a)$ is independent of phase transition temperature, and its value changes only with a. The scaling exponent ratio for crossover is 0.

Thus, the magnitude of distribution width D is very sensitive to fixed point behavior on phase boundaries, can well describe fixed point behavior, determine phase transition characteristic parameters, and distinguish different phase transition types. Of course, applying the fixed point method to heavy-ion collision experiments like RHIC energy scan programs requires careful discussion of observable selection, as finite-size scaling laws and fixed point behavior only show good finite-size scaling and clear fixed point behavior for certain observables like free energy derivatives with respect to external field (i.e., magnetization, magnetic susceptibility). For energy-like observables such as internal energy and specific heat (free energy derivatives with respect to temperature), finite-size scaling behavior is broken by non-singular parts in critical (phase transition) regions, showing no clear fixed point behavior. Moreover, how to correspond physical quantities like system size and temperature to relevant quantities in relativistic heavy-ion collisions requires careful study.

3. Effects of Finite Evolution Time

Typically, calculations based on lattice QCD and various QCD models assume the system reaches thermodynamic equilibrium. However, the fireball formed in relativistic heavy-ion collisions exists for only about 10 fm/c, very limited, and the system likely has no time to evolve to thermodynamic equilibrium. Possible states are: (1) If collision energy is sufficiently high, the system thermalizes to temperatures required for deconfinement transition, reaching thermal equilibrium and forming quark-gluon plasma (QGP); (2) The system may reach local thermal equilibrium, forming QGP droplets; (3) The system may not reach thermal equilibrium at all, experiencing no phase transition. Thus, equilibrium, local equilibrium, and non-equilibrium are three possible states after relativistic heavy-ion collisions. Additionally, critical slowing down in critical regions makes reaching thermodynamic equilibrium more difficult.

Non-equilibrium effects in QCD critical regions are usually estimated using dynamical evolution equations (e.g., Langevin dynamics, hydrodynamic equations, and various relaxation models). However, current dynamical evolution equations can only be solved exactly at critical points and in crossover regions, not yet at first-order phase boundaries. The Metropolis algorithm can well simulate non-equilibrium evolution of the Ising model, conveniently obtaining numerical results for non-equilibrium relaxation processes across the entire phase boundary. We then present non-equilibrium evolution characteristics of the Ising model, introduce time scales describing non-equilibrium evolution, and finally discuss non-equilibrium evolution effects on observables.

3.1 Temporal Description and Characteristics of Non-Equilibrium Evolution

Starting from an initial configuration, the Ising system can spontaneously evolve to equilibrium. Evolution before reaching equilibrium is called the relaxation process or non-equilibrium evolution. Figures 6(a–d) [Figure 6: see original paper] show the time evolution of absolute magnetization $|m|$ starting from random configurations at system size $L = 60$ and four temperatures: $T/T_c = 0.93$, 0.99, 1.00, and 1.03. The first two temperatures are on the first-order transition line, $T/T_c = 1.00$ is the critical point, and $T/T_c = 1.03$ is crossover. The horizontal axis is time t in sweep numbers (one sweep completes judgment for all lattice sites ($N = L \times L \times L$), updating the system configuration once). Two curves show two randomly selected evolution processes from the sample.

In Figure 6(a), $|m|$ initially increases gradually, then approaches a stable value μ. If the difference between the observable and equilibrium expectation exceeds half the equilibrium root-mean-square $\sigma = \sqrt{\langle (x-\mu)^2 \rangle}$, the system is far from equilibrium. The time from non-equilibrium evolution to equilibrium is called relaxation time. In simulations, the relaxation time τ_i^{eq} for the i-th evolution process is the number of steps after $|m|$ enters the region $(\mu-\sigma, \mu+\sigma)$. In Figure 6(a), two curves have the same stable value, but one curve' s relaxation time is much longer than the other' s. At low temperatures, different curves show significant relaxation time differences (Figures 6(a) and 6(b)), while differences decrease noticeably at high temperatures (Figures 6(c) and 6(d)).

To quantify relaxation characteristics of different system evolutions, we define the average relaxation time for a given system as:

$$\bar{\tau}^{\mathrm{eq}} = \frac{1}{n}\sum_{i=1}^{n} \tau_i^{\mathrm{eq}}$$

where n is the total number of evolution processes and τ_i^{eq} is the relaxation time for the i-th evolution process. Obviously, at $\tau = \bar{\tau}^{\mathrm{eq}}$, not all systems have reached equilibrium—some may still be in non-equilibrium states.

Typically, relaxation time depends on system evolution dynamics, system size, temperature, and initial configuration. Figure 7 [Figure 7: see original paper] shows average relaxation time varying with temperature, system size, and initial configuration. Near T_c, the peak of average relaxation time increases with system size (Figure 7(a)). Figure 7(b) shows that average relaxation time peaks at T_c for both random and polarized initial configurations due to critical slowing down.

Theoretically, for infinite systems near critical temperature T_c, dynamical relaxation time diverges as:

$$\tau_{\mathrm{dyn}} \propto |T - T_c|^{-z\nu}$$

where ν is the critical exponent of correlation length and z is the dynamical exponent related to dynamical universality class. For finite system sizes, relaxation time shows power-law behavior:

$$\tau_{\mathrm{dyn}} \propto L^z$$

Correspondingly, we examined the double-logarithmic plot of our defined average relaxation time $\bar{\tau}^{\mathrm{eq}}$ versus system size at critical temperature, with linear fit slope 2.06 ± 0.03, consistent with literature [69,70]. Thus, average relaxation time $\bar{\tau}^{\mathrm{eq}}$ also diverges as L^z at critical temperature, same as τ_{dyn}. This indicates that the simulated system' s average relaxation time can well quantify time evolution characteristics of given systems, like dynamical relaxation time.

For $T > T_c$, average relaxation time is basically independent of system size and initial configuration (Figures 7(a) and 7(b)). Regardless of system size and initial configuration, average relaxation time is very short and approaches zero. This is because the acceptance probability $A(u \to v)$ is an increasing function of T; higher temperature means higher acceptance probability, shorter required relaxation time, and easier evolution between configurations.

Conversely, for $T < T_c$, average relaxation time strongly depends on system size and initial configuration. First, Figure 7(a) shows larger system size leads to longer average relaxation time. Additionally, Figure 7(b) shows random configuration average relaxation time is much longer than polarized configuration because equilibrium state at $T < T_c$ is near ordered phase. Polarized configuration itself is ordered, requiring little evolution time to reach ordered phase equilibrium. However, random configuration is near disordered phase, requiring long evolution to reach ordered phase equilibrium. Therefore, polarized initial configuration evolves to ordered state faster than random initial configuration. Obviously, the further the initial configuration deviates from equilibrium, the longer the relaxation time. Thus, left of T_c, random configuration average relaxation time is longer, while right of T_c, polarized configuration average relaxation time is longer (Figure 7(b)). To observe maximum non-equilibrium evolution effects, we use random initial configuration for $T < T_c$ and polarized initial configuration for $T > T_c$ below.

3.2 Effects of Non-Equilibrium Evolution on Higher-Order Moments of Magnetization

Non-equilibrium evolution effects on observables can be viewed from the observation time perspective. We know relaxation time distribution varies with temperature. For fixed temperature and size systems, two simulations may have different relaxation times, as shown by curves in Figure 6(a). Therefore, if measurement time is not sufficiently long, some systems may not have reached equilibrium, and a certain proportion of non-equilibrium processes will exist among all observed processes—very similar to relativistic heavy-ion collisions.

As mentioned, the critical-fluctuation-sensitive observable in relativistic heavy-ion collisions is higher-order moments of the order parameter. For the Ising model, setting $X = |m|$ and $\delta X = |m| - \langle |m| \rangle$ in equation (1), Figure 8 [Figure 8: see original paper] shows time evolution of various moments at two fixed temperatures near critical temperature. At $t = 0$, $C_{n=1,2,3,4}$ in Figures 8(a–d) are all zero due to random initial configuration. For polarized initial configuration, C_1 is 1 and $C_{2,3,4}$ are zero (Figures 8(e–h)).

In Figures 8(a) and 8(e), C_1 changes monotonically with time, finally approaching equilibrium value exponentially, consistent with Langevin equation dynamical evolution behavior. This indicates the Metropolis algorithm well simulates non-equilibrium evolution of the Ising model. In Figures 8(a-d) at $T/T_c = 0.99$, C_2 in Figure 8(b) first increases then decreases, forming a peak during evolution. In Figures 8(c) and 8(d), C_3 and C_4 both undergo oscillations before approaching stable values, causing sign changes during evolution. The signs of C_3 and C_4 can be positive or negative, depending on evolution time.

In Figures 8(e–h) at $T/T_c = 1.01$, the time evolution trends of various moments are similar to Figures 8(a–d). C_2 undergoes non-monotonic changes before approaching stable values. C_3 also shows sign changes similar to Figure 8(c), but Figure 8(g) shows opposite sign behavior—first negative then positive. C_4 in Figure 8(h) shows similar sign changes as Figure 8(d), first negative then positive. The signs of C_3 and C_4 can be positive or negative, depending on observation time. These sign changes are consistent with solutions of dynamical equations.

However, note that quantitative differences between moments on both sides of T_c are very large. First, at $T/T_c = 0.99$, the time needed to approach C_3 equilibrium expectation is about 4,000 sweeps, while at $T/T_c = 1.01$ it is less than 2,000 sweeps—a difference of more than twofold. That is, systems at $T < T_c$ require longer time to reach equilibrium within the same time period. Additionally, oscillation amplitudes of C_3 and C_4 at $T < T_c$ are about two orders of magnitude larger than at $T > T_c$. This means non-equilibrium effects are very small and negligible above critical temperature, but very large and non-negligible below critical temperature.

4. Summary

We systematically studied the effects of non-critical fluctuations, finite system size, and finite evolution time on observational results when determining critical fluctuations through the observable—higher-order moments of conserved charges—in relativistic heavy-ion collisions.

1) For non-critical fluctuations, we discussed the basic statistics required for measuring higher-order moments of conserved charges. For the fourth-order moment, if calculated using the CBWC method at each N_{ch}, total statistics must exceed 25 M. We recommend using 0.1% centrality as the N_{ch} bin width, which not only can precisely represent initial volume fluctuations like each N_{ch}, but also requires only 1 M-level total statistics to ensure reliable fourth-order moment measurements. For the sixth-order moment, even increasing total statistics by an order of magnitude (300 M) is insufficient; whether the 0.1% centrality recommendation is feasible for sixth-order moments requires further study.

For statistical fluctuations caused by finite final-state particle numbers, we recommend starting from the assumption that all particles and antiparticles are emitted independently and follow Poisson distributions. We derived distributions and higher-order moments for three conserved charges, all consistent with results from the hadron resonance gas model under Boltzmann approximation. Comparing Poisson-estimated statistical fluctuations with net-proton higher-order moments from RHIC, we found that the centrality and energy dependence of measured net-proton higher-order moments are mainly due to statistical fluctuations, emphasizing that statistical fluctuations cannot be ignored.

Finally, we presented a pooling mixed event sample method to estimate the effects of global and systematic non-critical fluctuations on higher-order moments of conserved charges, defining dynamical higher-order moments as measured higher-order moments from experimental data samples minus those from corresponding mixed samples. Using the AMPT default model to simulate 19.6 GeV Au+Au collision samples, we demonstrated the effectiveness of the pooling mixed event sample method. Results show that dynamical higher-order moments can not only effectively subtract statistical fluctuations but also subtract effects of centrality bin width and detector efficiency on higher-order moment measurements, just like correction formulas. Therefore, for non-critical fluctuations from global and systematic effects, mixed samples provide an excellent background signal, and dynamical higher-order moments can effectively observe criticality-related fluctuations in RHIC BES experiments.

2) For finite system size effects on critical observables, we first used the three-dimensional three-state Potts model to calculate magnetization higher-order moment fluctuations at different phase boundary regions and system scales. We found that the second-order moment of magnetic susceptibility shows similar peak structures in first-order, second-order critical point, and crossover regions, with peak positions varying with system scale. The fourth-order moment of magnetic susceptibility also shows sign changes and oscillations in all three regions. This indicates that non-monotonic or sign-changing behavior appears in higher-order susceptibilities of finite systems near phase transition temperature, occurring not only near critical temperature of second-order transitions but also in first-order transition and crossover regions. Moreover, for finite-size systems, critical parameters cannot be simply determined by peak positions of observables but must be determined by finite-size scaling laws.

Therefore, we propose that fixed point behavior exhibited by finite-size scaling laws can more precisely determine phase transition parameters. We provide a quantitative description of fixed point behavior by defining the distribution width of scaled observables from different scales near phase transition parameters. The minimum width corresponds to the fixed point. Conversely, from contour maps of such defined width in the phase transition parameter (temperature and phase transition exponent ratio) plane, phase transition parameters can be precisely determined.

Using average magnetization of the three-state Potts model as an example, we present contour maps of scaling curves in three different phase boundary regions on the phase transition parameter plane. From the extremal positions and distribution characteristics of the maps, phase transition parameters can be precisely determined and phase transition order can be distinguished.

Of course, many issues remain to be solved when applying the fixed point method to relativistic heavy-ion collisions. First, observables similar to magnetic susceptibility must be selected—higher-order moments of conserved charges may be an excellent choice. Then, how to correspond collision energy and centrality in experiments to system temperature and scale requires careful research.

3) For effects of finite evolution time leading some systems to not reach thermodynamic equilibrium (non-equilibrium evolution), we propose using the Metropolis algorithm to simulate non-equilibrium evolution of the three-dimensional Ising model. We define the system’ s average relaxation time.

> Simulations of non-equilibrium evolution in three-dimensional Ising model with zero external magnetic field show that non-equilibrium evolution of the order parameter indeed approaches its stable value exponentially, same as Langevin equation solutions. Average relaxation time at critical temperature diverges as L^z, same as relaxation time in dynamical equations, indicating our simulation method is as effective as solving dynamical equations and our defined average relaxation time correctly describes system dynamical time evolution characteristics.

Through simulation methods, we studied non-equilibrium evolution characteristics and found average relaxation time depends on system size and temperature. For $T > T_c$, average relaxation time is very short and almost independent of initial configuration. For $T < T_c$, when initial configuration is far from equilibrium, average relaxation time is much longer than for $T > T_c$, and systems have more difficulty reaching equilibrium on the first-order transition line. Therefore, non-equilibrium effects cannot be ignored in first-order transition regions, but are negligible in crossover regions.

By studying non-equilibrium evolution effects on observables, we present time evolution results for various moments of the order parameter. Results show non-equilibrium processes affect higher-order moments more than lower-order moments, with C_3 and C_4 showing oscillations whose values can be positive or negative depending on observation time, consistent with dynamical theory calculations.

These qualitative features of non-equilibrium evolution are system-independent and can directly serve as references for relativistic heavy-ion collision experiments. Non-equilibrium evolution effects on determining the QCD phase boundary are negligible in crossover regions but cannot be ignored on the first-order transition line. Additionally, due to non-equilibrium effects, the sign of the third-order moment of the order parameter may be negative, consistent with RHIC/STAR measurements.

This review summarizes our major work over the past 15 years on determining the QCD phase boundary through relativistic heavy-ion collisions. Clearly, final determination of the QCD phase boundary remains a long-term task with many issues requiring further study. Related work on intermittency theory and experiments can be found in review articles [71].

Author Contributions

Wu Yuanfang is primarily responsible for writing this paper, overall design, and guidance of most specific work. Li Xiaobing is the main contributor to §3 work and primarily responsible for editing and reviewing the paper. Chen Lizhu is the main contributor to §1.1 work, completed part of §1.2 work, participated in discussions of §1.3 and §2 work, and completed foundational work. Li Zhiming participated in discussions of some work and provided suggestions. Xu Mingmei is the main author and guide for §3 work, participated in discussions of other work and provided suggestions. Pan Xue is the main contributor to §1.2 and §2.1 work, and completed foundational work for §2.2. Zhang Fan is the main contributor to §1.3 work, participated in discussions of §1.2 work and completed partial work. Zhang Yanhua is the main contributor to §2.2 work, participated in discussions of §3 work and provided suggestions. Zhong Yuming is primarily responsible for formula input and auxiliary editing of this paper.

References

1. STAR Note 0598. BES-II whitepaper[M]. 2014.
2. Adamczyk L, Adkins J K, Agakishiev G, et al. Beam energy dependence of moments of the net-charge multiplicity distributions in Au+Au collisions at RHIC[J]. Physical Review Letters, 2014, 113(9): 092301. DOI: 10.1088/1674-1137/44/1/014002.
3. Aggarwal M M, Ahammed Z, Alakhverdyants A V, et al. Higher moments of net proton multiplicity distributions at RHIC[J]. Physical Review Letters, 2010, 105(2): 022302. DOI: 10.1103/PhysRevLett.105.022302.
4. Aoki Y, Endrodi G, Fodor Z, et al. The order of the quantum chromodynamics transition predicted by the standard model of particle physics[J]. Nature, 2006, 443(7112): 675–678. DOI: 10.1038/nature05120.
5. Ding Hengtong, Li Shengtai, Liu Junhong. Progress on QCD properties in strong magnetic fields from lattice QCD[J]. Nuclear Techniques, 2023, 46(4): 040008. DOI: 10.11889/j.0253-3219.2023.hjs.46.040008.
6. Masayuki A, Koichi Y. Chiral restoration at finite density and temperature[J]. Nuclear Physics A, 1989, 504(4): 668–684. DOI: 10.1016/0375-9474(89)90002-X.
7. Blume C. Search for the critical point and the onset of deconfinement[J]. Central European Journal of Physics, 2012, 10(6): 1245–1253. DOI: 10.2478/s11534-012-0088-x.
8. Stephanov M, Rajagopal K, Shuryak E. Signatures of the tricritical point in QCD[J]. Physical Review Letters, 1998, 81(22): 4816–4819. DOI: 10.1103/physrevlett.81.4816.
9. Stephanov M A. Sign of kurtosis near the QCD critical point[J]. Physical Review Letters, 2011, 107(5): 052301. DOI: 10.1103/PhysRevLett.107.052301.
10. Friman B, Karsch F, Redlich K, et al. Fluctuations as probe of the QCD phase transition and freeze-out in heavy ion collisions at LHC and RHIC[J]. The European Physical Journal C, 2011, 71(7): 1694. DOI: 10.1140/epjc/s10052-011-1694-2.
11. Adamczyk L, Adkins J K, Agakishiev G, et al. Energy dependence of moments of net-proton multiplicity distributions at RHIC[J]. Physical Review Letters, 2014, 112(3): 032302. DOI: 10.1103/PhysRevLett.112.032302.
12. Luo X F, Xu N. Search for the QCD critical point with fluctuations of conserved quantities in relativistic heavy-ion collisions at RHIC: an overview[J]. Nuclear Science and Techniques, 2017, 28(8): 112. DOI: 10.1007/s41365-017-0254-0.

13. Luo X. Exploring the QCD phase structure with beam energy scan in heavy-ion collisions[J]. Nuclear Physics A, 2016, 956: 75–82. DOI: 10.1016/j.nuclphysa.2016.01.011.
14. Yin Y. QCD phase structure at high baryon density region[R]. Wuhan: CCNU, 2019.
15. Liu Yunpeng, Zhuang Pengfei. Heavy flavor production in relativistic heavy ion collisions[J]. Nuclear Physics Review, 2012, 29(1): 1-13. DOI: 10.11804/NuclPhysRev.29.01.001.
16. Chen L Z, Pan X, Chen X S, et al. Critical behavior of higher cumulants of order parameter in the 3D-Ising universality class[J]. Chinese Physics C, 2012, 36(8): 727–732. DOI: 10.1088/1674-1137/36/8/008.
17. Chen L Z, Zhao Y Y, Pan X, et al. High cumulants of conserved charges and their statistical uncertainties[J]. Chinese Physics C, 2017, 41(10): 104103. DOI: 10.1088/1674-1137/41/10/104103.
18. Chen L Z, Li Z, Cui F, et al. Measurement of the ratio of the sixth order to the second order cumulant of net-proton multiplicity distributions in relativistic heavy-ion collisions[J]. Nuclear Physics A, 2017, 957: 60–70. DOI: 10.1016/j.nuclphysa.2016.07.007.
19. Chen L Z, Pan X, Xiong F B, et al. Statistical and dynamical fluctuations in the ratios of higher net-proton cumulants in relativistic heavy-ion collisions[J]. Journal of Physics G: Nuclear and Particle Physics, 2011, 38(11): 115004. DOI: 10.1088/0954-3899/38/11/115004.
20. Pan X, Chen L Z, Chen X S, et al. Statistical and dynamical parts of the cumulants of conserved charges in relativistic heavy ion collisions[J]. Physical Review C, 2014, 89(1): 014904. DOI: 10.1103/physrevc.89.014904.
21. Bzdak A, Koch V, Skokov V. Baryon number conservation and the cumulants of the net proton distribution[J]. Physical Review C, 2013, 87(1): 014901. DOI: 10.1103/physrevc.87.014901.
22. Luo X F. Probing the QCD critical point with higher moments of net-proton multiplicity distributions[J]. Journal of Physics: Conference Series, 2011, 316(1): 012003. DOI: 10.1088/1742-6596/316/1/012003.
23. Zhang F, Li Z M, Chen L Z, et al. The method of mixed events for higher cumulants of conserved charges[EB/OL]. 2019: arXiv:1908.05465. DOI: 10.48550/arXiv.1908.05465.
24. Zhang F, Li Z M, Chen L Z, et al. Subtracting non-critical fluctuations in higher cumulants of conserved charges[EB/OL]. 2019: arXiv:1908.05470. DOI: 10.48550/arXiv.1908.05470.
25. Chen L Z, Li Z M, Zhong X, et al. Influence of statistics on the measured moments of conserved quantities in relativistic heavy ion collisions[J]. Journal of Physics G: Nuclear and Particle Physics, 2015, 42(6): 065103. DOI: 10.1088/0954-3899/42/6/065103.
26. Chen L Z, Li Z M, Wu Y F. Influences of statistics and initial size fluctuation on high-order cumulants of conserved quantities in relativistic heavy ion collisions[J]. Journal of Physics G: Nuclear and Particle Physics, 2014, 41(10): 105107. DOI: 10.1088/0954-3899/41/10/105107.
27. Pan X, Zhang F, Li Z M, et al. Statistical and dynamical parts of the cumu-

lants of conserved charges in relativistic heavy ion collisions[J]. Physical Review C, 2014, 89(1): 014904. DOI: 10.1103/physrevc.89.014904.
28. Pan X, Chen L Z, Wu Y F. Behavior and finite-size effects of the sixth order cumulant in the three-dimensional Ising universality class[J]. Chinese Physics C, 2016, 40(9): 093104. DOI: 10.1088/1674-1137/40/9/093104.
29. Pan X, Zhang Y H, Chen L Z, et al. Finite-size behaviour of generalized susceptibilities in the whole phase plane of the Potts model[J]. Chinese Physics C, 2018, 42(2): 023110. DOI: 10.1088/1674-1137/42/2/023110.
30. Bzdak A, Koch V. Local efficiency corrections to higher order cumulants[J]. Physical Review C, 2015, 91(2): 027901. DOI: 10.1103/physrevc.91.027901.
31. Luo X F. Unified description of efficiency correction and error estimation for moments of conserved quantities in heavy-ion collisions[J]. Physical Review C, 2015, 91(3): 034907. DOI: 10.1103/physrevc.91.034907.
32. Skellam J G. The frequency distribution of the difference between two Poisson variates belonging to different populations[J]. Journal of the Royal Statistical Society Series A (General), 1946, 109(Pt 3): 296–296. DOI: 10.2307/2981372.
33. Lacey R A. Indications for a critical end point in the phase diagram for hot and dense nuclear matter[J]. Physical Review Letters, 2015, 114(14): 142301. DOI: 10.1103/PhysRevLett.114.142301.
34. Fraga E S, Palhares L F, Sorensen P. Finite-size scaling as a tool in the search for the QCD critical point in heavy ion data[J]. Physical Review C, 2011, 84(1): 011903. DOI: 10.1103/physrevc.84.011903.
35. Pan X, Chen L Z, Chen X S, et al. High-order cumulants from the 3-dimensional O(1, 2, 4) spin models[J]. Nuclear Physics A, 2013, 913: 206–216. DOI: 10.1016/j.nuclphysa.2013.06.010.
36. Pan X, Xu M M, Wu Y F. Generalized susceptibilities along the phase boundary of the three-dimensional, three-state Potts model[J]. Journal of Physics G: Nuclear and Particle Physics, 2015, 42(1): 015104. DOI: 10.1088/0954-3899/42/1/015104.
37. Nahrgang M, Bluhm M, Schäfer T, et al. Diffusive dynamics of critical fluctuations near the QCD critical point[J]. Physical Review D, 2019, 99(11): 116015. DOI: 10.1103/PhysRevD.99.116015.
38. Wu S J, Wu Z M, Song H C. Universal scaling of the σ field and net-protons from Langevin dynamics of model A[J]. Physical Review C, 2019, 99(6): 064902. DOI: 10.1103/physrevc.99.064902.
39. Berdnikov B, Rajagopal K. Slowing out of equilibrium near the QCD critical point[J]. Physical Review D, 2000, 61(10): 105017. DOI: 10.1103/physrevd.61.105017.
40. Pisarski R D, Wilczek F. Remarks on the chiral phase transition in chromodynamics[J]. Physical Review D, 1984, 29(2): 338-341. DOI: 10.1103/physrevd.29.338.
41. de Forcrand P, Philipsen O. Constraining the QCD phase diagram by tricritical lines at imaginary chemical potential[J]. Physical Review Letters, 2010, 105(15): 152001. DOI: 10.1103/PhysRevLett.105.152001.

42. Parotto P. Parametrized equation of state for QCD from 3D Ising model[J]. Pos, 2018, CPOD2017: 036. DOI: 10.22323/1.311.0036.
43. Imry Y. Finite-size rounding of a first-order phase transition[J]. Physical Review B, 1980, 21(5): 2042–2043. DOI: 10.1103/physrevb.21.2042.
44. Challa M S S, Landau D P, Binder K. Finite-size effects at temperature-driven first-order transitions[J]. Physical Review B, 1986, 34(3): 1841. DOI: 10.1103/physrevb.34.1841.
45. Wu Y F, Chen L Z, Pan X, et al. Finite-size behaviour of a critical related observable[J]. Central European Journal of Physics, 2012, 10(6): 1341–1344. DOI: 10.2478/s11534-012-0046-7.
46. Lacey R A, Liu P, Magdy N, et al. Finite-size scaling of non-Gaussian fluctuations near the QCD critical point[EB/OL]. arXiv preprint arXiv:1606.08071, 2016. DOI: 10.48550/arXiv.1606.08071.
47. Wilson K G, Kogut J. The renormalization group and the expansion[J]. Physics Reports, 1974, 12(2): 75–199. DOI: 10.1016/0370-1573(74)90023-4.
48. Goldenfeld N. Lectures on phase transitions and the renormalization group[M]. Addison-Wesley, 1992.
49. Fisher M E, Berker A N. Scaling for first-order phase transitions in thermodynamic and finite systems[J]. Physical Review B, 1982, 26(5): 2507–2513. DOI: 10.1103/physrevb.26.2507.
50. Zhang Y H, Zhao Y Y, Chen L Z, et al. Locating fixed points in the phase plane[J]. Physical Review E, 2019, 100(5–1): 052146. DOI: 10.1103/PhysRevE.100.052146.
51. Braun-Munzinger P, Stachel J. The quest for the quark-gluon plasma[J]. Nature, 2007, 448(7151): 302–309. DOI: 10.1038/nature06080.
52. Shen C, Yan L. Recent development of hydrodynamic modeling in heavy-ion collisions[J]. Nuclear Science and Techniques, 2020, 31(12): 122. DOI: 10.1007/s41365-020-00829-z.
53. Wu S J, Shen C, Song H C. Dynamically exploring the QCD matter at finite temperatures and densities: a short review[J]. Chinese Physics Letters, 2021, 38(8): 081201. DOI: 10.1088/0256-307x/38/8/081201.
54. Wu Shanjin, Song Huichao. Critical dynamical fluctuations near the QCD critical point[J]. Nuclear Techniques, 2023, 46(4): 040004. DOI: 10.11889/j.0253-3219.2023.hjs.46.040004.
55. Wu J, Lin Y F, Wu Y F, et al. Probing QCD critical fluctuations from intermittency analysis in relativistic heavy-ion collisions[J]. Physics Letters B, 2020, 801: 135147. DOI: 10.1016/j.physletb.2019.135147.
56. Li Z M. Overview of intermittency analysis in heavy-ion collisions[J]. Modern Physics Letters A, 2022, 37(13): 2230009. DOI: 10.1142/s0217732322300099.
57. Acharyya M. Nonequilibrium phase transition in the kinetic Ising model: critical slowing down and the specific-heat singularity[J]. Physical Review E, 1997, 56(3): 2407–2411. DOI: 10.1103/physreve.56.2407.
58. Newman M E J, Barkema G T. Monte Carlo methods in statistical physics[M]. Oxford: Clarendon Press, 1999.

59. Hasenbusch M. Dynamic critical exponent z of the three-dimensional Ising universality class: Monte Carlo simulations of the improved Blume-Capel model[J]. Physical Review E, 2020, 101(2–1): 022126. DOI: 10.1103/PhysRevE.101.022126.
60. Alexandrou C, Boriçi A, Feo A, et al. Deconfinement phase transition in one-flavor QCD[J]. Physical Review D, 1999, 60(3): 034504. DOI: 10.1103/physrevd.60.034504.
61. DeGrand T A, DeTar C E. Phase structure of QCD at high temperature with massive quarks and finite quark density: a Z(3) paradigm[J]. Nuclear Physics B, 1983, 225(4): 590–620. DOI: 10.1016/0550-3213(83)90536-9.
62. Wu F Y. The potts model[J]. Reviews of Modern Physics, 1982, 54(1): 235–268. DOI: 10.1103/revmodphys.54.235.
63. Talapov A L, Blöte H W J. The magnetization of the 3D Ising model[J]. Journal of Physics A: Mathematical and General, 1996, 29(17): 5727–5733. DOI: 10.1088/0305-4470/29/17/042.
64. Pawley G S, Swendsen R H, Wallace D J, et al. Monte Carlo renormalization-group calculations of critical behavior in the simple-cubic Ising model[J]. Physical Review B, 1984, 29(7): 4030–4040. DOI: 10.1103/physrevb.29.4030.
65. Blume C. Search for the critical point and the onset of deconfinement[J]. Central European Journal of Physics, 2012, 10(6): 1245–1253. DOI: 10.2478/s11534-012-0088-x.
66. Stephanov M, Rajagopal K, Shuryak E. Signatures of the tricritical point in QCD[J]. Physical Review Letters, 1998, 81(22): 4816–4819. DOI: 10.1103/physrevlett.81.4816.
67. Stephanov M A. Sign of kurtosis near the QCD critical point[J]. Physical Review Letters, 2011, 107(5): 052301. DOI: 10.1103/PhysRevLett.107.052301.
68. Friman B, Karsch F, Redlich K, et al. Fluctuations as probe of the QCD phase transition and freeze-out in heavy ion collisions at LHC and RHIC[J]. The European Physical Journal C, 2011, 71(7): 1694. DOI: 10.1140/epjc/s10052-011-1694-2.
69. Aoki Y, Endrodi G, Fodor Z, et al. The order of the quantum chromodynamics transition predicted by the standard model of particle physics[J]. Nature, 2006, 443(7112): 675–678. DOI: 10.1038/nature05120.
70. Ding Hengtong, Li Shengtai, Liu Junhong. Progress on QCD properties in strong magnetic fields from lattice QCD[J]. Nuclear Techniques, 2023, 46(4): 040008. DOI: 10.11889/j.0253-3219.2023.hjs.46.040008.

Note: Figure translations are in progress. See original paper for figures.